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SUBSERIES AND SIGNED SERIES

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Abstract. For any positive decreasing to zero sequence a_n such that $\sum a_n$ diverges we consider the related series $\sum k_n a_n$ and $\sum j_n a_n$. Here, k_n and j_n are real sequences such that $k_n \in \{0,1\}$ and $j_n \in \{-1,1\}$. We study their convergence and characterize it in terms of the density of 1's in the sequences k_n and j_n. We extend our results to series $\sum m_n a_n$, with $m_n \in \{-1,0,1\}$ and apply them to study some associated random series.

1. Introduction and main results.

Given a divergent series $\sum a_n$, with $a_n > 0$, decreasing and with limit zero, we study properties of its subsums, $\sum k_n a_n$, where $k_n \in \{0,1\}$ and of its signed sums, $\sum j_n a_n$, where $j_n \in \{-1,1\}$. As we will see, both questions are related, and moreover can be treated simultaneously studying the series $\sum m_n a_n$, with $m_n \in \{-1,0,1\}$.

For a sequence of real numbers c_n we will say that the sequence f_n, given by the quotient between the number of A's in the list $c_1, c_2, \ldots, c_n$ and n, is the *sequence of densities of* A's associated to c_n. If $\lim f_n = f \in [0,1]$ exists we will say that f is the density of A's of the sequence c_n.

We characterize the convergence of the series in terms of properties of the sequences of densities of 1's in k_n and j_n. As usual, when $\lim a_n/b_n = 1$ we will write $a_n \sim b_n$. We split our main results in Theorems A and B, the first one concerning with subsums and the second one with signed sums. As we will see in Theorem C, some points can be treated together.

A key tool in many of our proofs will be a restricted version of the celebrated Toeplitz Theorem about the summability of weighted sequences, see for instance [6]. For completeness, in Section 2 we present a simple proof in the restrictive case of non-negative weights.

Theorem A. Let a_n be a positive monotonous sequence such that $\lim a_n = 0$, $\sum a_n = \infty$ and set $U_n = \sum_{i=1}^n a_i$. Let k_n be a sequence with $k_n \in \{0,1\}$, $f_n = \frac{\sum_{i=1}^n k_i}{n}$ be the associate sequence of densities of 1's and $S_n = \sum_{i=1}^n k_i a_i$ be its associated sequence of partial sums. Then the following assertions hold:

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