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On the period of the limit cycles appearing in one-parameter bifurcations

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Abstract

The generic isolated bifurcations for one-parameter families of smooth planar vector fields $\{X_\mu\}$ which give rise to periodic orbits are: the Andronov–Hopf bifurcation, the bifurcation from a semi-stable periodic orbit, the saddle-node loop bifurcation and the saddle loop bifurcation. In this paper we obtain the dominant term of the asymptotic behaviour of the period of the limit cycles appearing in each of these bifurcations in terms of μ when we are near the bifurcation. The method used to study the first two bifurcations is also used to solve the same problem in another two situations: a generalization of the Andronov–Hopf bifurcation to vector fields starting with a special monodromic jet; and the Hopf bifurcation at infinity for families of polynomial vector fields.

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1. Introduction

In this paper we consider one-parameter families of analytic vector fields and study the dependence, with respect the parameter, of the period of the limit cycles appearing in the most elementary bifurcations. To fix the concepts, let us introduce some definitions.

For any $m \in \{1, 2, 3, \ldots, \infty, \omega\}$, let $\mathcal{C}^m(K)$ be the space of planar vector fields with the corresponding regularity and defined on a given compact set K. As usual, we can endow $\mathcal{C}^m(K)$ with the topology of the uniform convergence, taking into account the vector field and all its derivatives up to order m. In this setting, for $m \geqslant 3$, it is said that a given $X_0 \in \mathcal{C}^m(K)$ has *first degree of structural instability in* K if it is structurally unstable in K whereas any vector field in $\mathcal{C}^m(K)$ sufficiently close to X_0 is either structurally stable or topologically conjugated to X_0 (see [1]).

Take $m \geqslant 3$ and consider a one-parameter $\mathcal{C}^m$-family of vector fields in $\mathcal{C}^m(K)$, $\{X_\mu\}_{\mu\in\Lambda}$, such that X_0 has first degree of structural instability. All the possible bifurcations appearing in the family for $\mu \approx 0$ are listed in [1,7,13]. Among them there are isolated and non-isolated bifurcations (see [1,14] for details). In this paper we will only study the isolated ones and among them we are just interested in the ones giving rise to periodic orbits. From now one we will refer to them as *elementary bifurcations*. They are: (i) The Andronov–Hopf bifurcation, (ii) The bifurcation from a semi-stable periodic orbit, (iii) The saddle-node loop bifurcation, (iv) The saddle loop bifurcation.

Although, as we have said, the above list does not include the non isolated bifurcations, from the local viewpoint the limit cycles appearing from them are not different from the ones appearing from semi-stable periodic orbit bifurcations.

From now on we will assume that our family of vector fields is in $\mathcal{C}^\omega(K)$ and that the dependence on μ is also analytic. It is worth to notice that for some of the results given in this paper less regularity is needed. For instance, in the study of the saddle-loop bifurcation only the $\mathcal{C}^\infty$ dependence of the vector field with respect to μ is needed, or in the study of the Andronov–Hopf bifurcation only derivatives up to order three of the return map are used (so the result proved in this case also follows for $\mathcal{C}^4$-families of vector fields).

In what follows we denote by $T(\mu)$ the period of the periodic orbit arising from an elementary bifurcation and recall that we are interested in its behaviour as $\mu \longrightarrow 0$. It is clear that $T(\mu)$ tends to constant in the first two cases and to infinity in the last two. Consequently, in cases (i) and (ii) we can expect some kind of Taylor expansion for $T(\mu)$, and in cases (iii) and (iv) an asymptotic development. We will only study the dominant terms of $T(\mu)$. These terms constitute what we call the *principal term* of the asymptotic expansion. As usual we use the notation $T(\mu) \sim a + f(\mu)$ as $\mu \longrightarrow 0$ meaning that $\lim_{\mu\to 0}(T(\mu) - a)/f(\mu) = 1$.

Until now we have said nothing about the concrete one-parameter families that we consider. It may happen for instance that the family $\{X_\mu\}$ does not present any bifurcation although X_0 has first degree of structural instability. So in each case we need a condition on μ that forces the family to present one of the four bifurcations listed above. This condition will be given in detail in the statement of the corresponding result. Let us advance however that roughly speaking the condition is that when μ changes sign then, in the corresponding case,

(i) the origin reverses its stability,
(ii) the solution starting at a given point of the semi-stable limit cycle goes, after a complete turn, from inside the limit cycle to outside the limit cycle,
(iii) the saddle-node presents the well-known saddle-node bifurcation of the critical point,
(iv) the loop breaks and the separatrices forming the loop change their relative position.

Furthermore we will also assume that the above bifurcations occur in the "most generic way". We will say in this case that the above one-parameter families present *generic elementary bifurcations*.

The results of this paper show that, essentially, the principal term of the period of the periodic orbit arising from generic elementary bifurcations characterizes the bifurcation. More concretely, the principal term of the period is given in the following list:

(i) Andronov–Hopf bifurcation: $T(\mu) \sim T_0 + T_1\mu$ (see Theorem 7).
(ii) Bifurcation from a semi-stable periodic orbit:[1] $T(\mu) \sim T_0 + T_1\sqrt{\mu}$ (see Theorem 11).
(iii) Saddle-node loop bifurcation: $T(\mu) \sim T_0/\sqrt{\mu}$ (see Theorem 14).
(iv) Saddle loop bifurcation: $T(\mu) \sim T_0 \ln \mu$ (see Theorem 16).

Let us point out that $T_0 \neq 0$ in all the expressions above and that, although T_1 is generically nonzero, it may be zero (see Examples 8 and 12). It is also to be mentioned that the results in (i) and (iii) are more or less common knowledge. The proof of (iv) is the most difficult part of the paper and it strongly relies on the techniques introduced in [12].

The proofs of cases (i), (ii) and (iv) follow a similar scheme. Firstly we translate the problem of the existence of the periodic orbits to a problem of solving an equation. Afterwards, some variant of the Implicit Function Theorem is used to locate the limit cycles and to obtain the dependence with respect to μ of the distance of the limit cycle to the limit set at which the bifurcation occurs. The last step consists in computing the period of the located limit cycle. The first two steps can be avoided to study the case (iii) because, curiously enough, the principal term of the period of the limit cycle in this bifurcation does not depend on its exact location. It is also worth to mention that the study of cases (iii) and (iv) is based on the knowledge of a good normal form of the family $\{X_\mu\}$ near the singularity of the loop that exists for X_0 (see expression (11) in proof of Theorem 14 and Lemma 18, respectively).

From the applied point of view, this kind of information concerning $T(\mu)$ can be useful to estimate parameters associated to a system. Suppose that a vector field X_μ is a good model for some *realistic* phenomenon, being μ an experimentally controllable parameter, and assume that there exist other parameters gathered in $\lambda \in \mathbb{R}^p$ that need to be estimated. This occurs, for instance, when studying neuron activities in the brain with the aim of determining the synaptic conductances λ that it receives. In the experiments, by injecting different external currents (which would correspond here to μ), people is

[1] In this case there appear two periodic orbits and their periods have similar principal terms, only the sign of T_1 changes from one orbit to the other.

able to extract information about the period of the oscillations of the voltage of the cell. So one has $T(\mu_i, \lambda)$ for $i = 1, \dots, q$ (where generally $q > p$) and some kind of regression is needed to estimate λ. Then the (analytical) knowledge of $T(\mu_i, \lambda)$ is determinant to do this regression/estimation properly.

Let us conclude this introduction by noticing that the tools developed to study the Andronov–Hopf bifurcation are also useful to study another two bifurcations. The first one is a generalization of the Andronov–Hopf bifurcation that occurs in one-parameter families of vector fields whose first non zero jet is of order $2p+1$. For these bifurcations it follows that $T(\mu) \sim T_0/\mu^p$ (see Theorem 7). The second one is the so-called Hopf bifurcation at infinity (see [15]). This bifurcation occurs in one-parameter families of planar polynomial vector fields of degree $2p+1$. It consists essentially in the creation of a periodic orbit from infinity due to a change of its stability. In this case (see Theorem 10) we have that $T(\mu) \sim T_0\mu^p$. Both bifurcations are studied in the same section that the usual Andronov–Hopf bifurcation.

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2. Preliminary results

In this paper the one-parameter family of analytic vector fields will be denoted by $\{X_\mu\}_{\mu\in\Lambda}$ where $X_\mu(x, y) = P(x, y; \mu)\,\partial_x + Q(x, y; \mu)\,\partial_y$ with $(x, y) \in \mathbb{R}^2$ and $\mu \in \Lambda$, being $\Lambda \subset \mathbb{R}$ an open interval containing zero. It defines the analytic planar differential system

$$\begin{cases} \dot{x} = P(x, y; \mu) = \sum_{n\geqslant k} P_n(x, y; \mu), \\ \dot{y} = Q(x, y; \mu) = \sum_{n\geqslant k} Q_n(x, y; \mu), \end{cases} \tag{1}$$

where P_n and Q_n are homogeneous polynomials of degree n in x and y and $k \in \mathbb{N}\cup\{0\}$. In the sequel we include several results used along the paper. The following two lemmas are corollaries of the Implicit Function Theorem.

Lemma 1. *Let $D(x, \mu)$ be an analytic function in a neighbourhood of $(0, 0) \in \mathbb{R}^2$ verifying*

$$D(0, 0) = D_x(0, 0) = 0 \quad \textit{and} \quad D_{xx}(0, 0)D_\mu(0, 0) \neq 0.$$

Then there exists a neighbourhood U of $(0, 0)$ and an analytic function φ, defined for $|\mu|$ small enough, satisfying

$$\varphi(0) = 0 \quad \textit{and} \quad \varphi'(0) = \sqrt{2\left|\frac{D_\mu(0, 0)}{D_{xx}(0, 0)}\right|},$$

and such that:

(a) *In case that* $D_{xx}(0,0)D_\mu(0,0) < 0$ *then* $D(x_0,\mu_0) = 0$ *with* $(x_0,\mu_0) \in U$ *if and only if* $\mu_0 \geqslant 0$ *and either* $x_0 = \varphi\left(\sqrt{\mu_0}\right)$ *or* $x_0 = \varphi\left(-\sqrt{\mu_0}\right)$.

(b) *In case that* $D_{xx}(0,0)D_\mu(0,0) > 0$ *then* $D(x_0,\mu_0) = 0$ *with* $(x_0,\mu_0) \in U$ *if and only if* $\mu_0 \leqslant 0$ *and either* $x_0 = \varphi\left(\sqrt{-\mu_0}\right)$ *or* $x_0 = \varphi\left(-\sqrt{-\mu_0}\right)$.

Proof. Let us prove first (a). Since $D(0,0) = 0$ and $D_\mu(0,0) \neq 0$, by the Implicit Function Theorem, there exists an analytic function ψ, with $\psi(0) = 0$, such that $D\left(x,\psi(x)\right) = 0$ for all x. Taking $D_x(0,0) = 0$ into account, one can easily verify that $\psi'(0) = 0$. Then a straightforward computation shows that $\psi''(0) = a$, where

$$a := -\frac{D_{xx}(0,0)}{D_\mu(0,0)} > 0.$$

Consequently $\psi(x) = \frac{a}{2}x^2 + o(x^2)$. Note in addition that, for $(x_0,\mu_0) \approx (0,0)$, $D(x_0,\mu_0) = 0$ if and only if $\mu_0 = \psi(x_0)$. This shows, due to $a > 0$, that $\mu_0 \geqslant 0$. On the other hand, it is clear that the function

$$f(x) := x\sqrt{\frac{\psi(x)}{x^2}},$$

which is analytic for $x \approx 0$, verifies $\psi(x) = f(x)^2$, $f(0) = 0$ and $f'(0) = \sqrt{a/2}$. Accordingly $\mu_0 = f(x_0)^2$. Therefore $x_0 = f^{-1}\left(\sqrt{\mu_0}\right)$ in case that $x_0 \geqslant 0$ and $x_0 = f^{-1}\left(-\sqrt{\mu_0}\right)$ otherwise. This, setting $\varphi := f^{-1}$, shows (a).

Part (b) follows from applying (a) to the function $\widetilde{D}(x,\mu) := D(x,-\mu)$. $\square$

Lemma 2. *Let* $D(x,\mu)$ *be an analytic function in a neighbourhood of* $(0,0) \in \mathbb{R}^2$ *verifying*

$$D(0,\mu) = D_x(0,0) = D_{xx}(0,0) = 0 \quad \textit{and} \quad D_{xxx}(0,0)D_{x\mu}(0,0) \neq 0.$$

Then there exists a neighbourhood U *of* $(0,0)$ *and an analytic function* φ, *defined for* $|\mu|$ *small enough, satisfying*

$$\varphi(0) = 0 \quad \textit{and} \quad \varphi'(0) = \sqrt{6\left|\frac{D_{x\mu}(0,0)}{D_{xxx}(0,0)}\right|},$$

and such that:

(a) *In case that* $D_{xxx}(0,0)D_{x\mu}(0,0) < 0$ *then* $D(x_0,\mu_0) = 0$ *with* $(x_0,\mu_0) \in U$ *and* $x_0 \neq 0$ *if and only if* $\mu_0 \geqslant 0$ *and either* $x_0 = \varphi\left(\sqrt{\mu_0}\right)$ *or* $x_0 = \varphi\left(-\sqrt{\mu_0}\right)$.

(b) *In case that* $D_{xxx}(0,0)D_{x\mu}(0,0) > 0$ *then* $D(x_0,\mu_0) = 0$ *with* $(x_0,\mu_0) \in U$ *and* $x_0 \neq 0$ *if and only if* $\mu_0 \leqslant 0$ *and either* $x_0 = \varphi\left(\sqrt{-\mu_0}\right)$ *or* $x_0 = \varphi\left(-\sqrt{-\mu_0}\right)$.

Proof. Since $D(0,\mu)\equiv 0$, there exists an analytic function $\widetilde{D}(x,\mu)$ such that $D(x,\mu)=x\widetilde{D}(x,\mu)$. Now, on account of

$$\widetilde{D}_\mu(0,0)=D_{x\mu}(0,0) \text{ and } \widetilde{D}_{xx}(0,0)=\frac{1}{3}D_{xxx}(0,0),$$

the result follows from applying Lemma 1 to the function $\widetilde{D}$. □

Next result studies a special type of differential equations on a strip. As we will see, this type of differential equations will appear when we study the Andronov–Hopf bifurcation, the generalized Hopf bifurcation, the Hopf bifurcation at infinity and the bifurcation of semi-stable periodic orbits.

Lemma 3. *Let $A(r,\theta;\mu)$ and $B(r,\theta;\mu)$ be analytic functions on $\mathbb{R}\times[0,\ell]\times\Lambda$ with $A(0,\theta;\mu)=0$ and $B(0,\theta;\mu)>0$. Consider the system of ordinary differential equations*

$$\begin{cases} \dfrac{dr}{dt}=A(r,\theta;\mu)\,r^m, \\ \dfrac{d\theta}{dt}=B(r,\theta;\mu)\,r^m, \end{cases} \tag{2}$$

where $m\in\mathbb{Z}$. Associated to (2), *consider*

$$\frac{dr}{d\theta}=\frac{A(r,\theta;\mu)}{B(r,\theta;\mu)}=:\sum_{i\geqslant 1}C_i(\theta;\mu)\,r^i. \tag{3}$$

For x small enough, let $r(\theta,x;\mu)$ be the solution of (3) *satisfying $r(0,x;\mu)=x$ and denote by $T(x;\mu)$ the time t that spends the solution of* (2) *starting at $(r,\theta)=(x,0)$ to arrive to $\theta=\ell$. Then $r(\theta,x;\mu)$ is an analytic function at $x=0$ verifying $r(\theta,x;\mu)=x\sum_{i\geqslant 1}r_i(\theta;\mu)x^{i-1}$ where*

$$r_1(\theta;\mu)=\exp\left(\int_0^\theta C_1(\psi;\mu)\,d\psi\right),\quad r_2(\theta;\mu)=r_1(\theta;\mu)\int_0^\theta C_2(\psi;\mu)r_1(\psi;\mu)\,d\psi$$

and

$$r_3(\theta;\mu)=r_2^2(\theta;\mu)/r_1(\theta;\mu)+r_1(\theta;\mu)\int_0^\theta C_3(\psi;\mu)r_1^2(\psi;\mu)\,d\psi.$$

In addition, $T(x;\mu)=\widehat{T}(x;\mu)/x^m$ where $\widehat{T}$ is an analytic function at $x=0$. Finally if we set $B(r,\theta;\mu)=\sum_{i\geqslant 0}B_i(\theta;\mu)\,r^i$, then it holds

$$\widehat{T}(0;\mu)=\int_0^\ell\frac{d\theta}{r_1^m(\theta;\mu)B_0(\theta;\mu)}$$

and

$$\widehat{T}_x(0;\mu) = -\int_0^{\ell}\left(\frac{r_1(\theta;\mu)B_1(\theta;\mu)}{B_0(\theta;\mu)} + m\frac{r_2(\theta;\mu)}{r_1(\theta;\mu)}\right)\frac{d\theta}{r_1^m(\theta;\mu)B_0(\theta;\mu)}.$$

Proof. That $r(\theta,x;\mu)$ is analytic at $x=0$ follows from using that $B(0,\theta;\mu)>0$ for all $\theta\in[0,\ell]$. Notice moreover that, on account of $A(0,\theta;\mu)=0$, $r(\theta,x;\mu)=x\,\widehat{r}(\theta,x;\mu)$. The concrete expression of the functions $r_i(\theta;\mu)$ in the statement follow easily by solving the recurrent ordinary differential equations obtained by replacing the expansion of $r(\theta,x;\mu)$ in (3). (These computations are not included here for the sake of brevity.) Note next that, from (2), $T(x;\mu)=x^{-m}\widehat{T}(x;\mu)$ with

$$\widehat{T}(x;\mu) := \int_0^{\ell}\frac{d\theta}{\widehat{r}(\theta,x;\mu)^m B\left(x\widehat{r}(\theta,x;\mu),\theta;\mu\right)}.$$

Since $\widehat{r}(\theta,0;\mu)=r_1(\theta;\mu)>0$ and $B(0,\theta;\mu)>0$ for all $\theta\in[0,\ell]$, it is clear that $\widehat{T}(x;\mu)$ is analytic at $x=0$. Finally, on account of $\widehat{r}(\theta,x;\mu)=\sum_{i\geqslant 1}r_i(\theta;\mu)x^{i-1}$ and $B(r,\theta;\mu)=\sum_{i\geqslant 0}B_i(\theta;\mu)\,r^i$, some easy computations show that

$$\widehat{T}(x;\mu) = \int_0^{\ell}\left(1-\left(\frac{r_1(\theta;\mu)B_1(\theta;\mu)}{B_0(\theta;\mu)} + m\frac{r_2(\theta;\mu)}{r_1(\theta;\mu)}\right)x + R(\theta,x;\mu)\right)\frac{d\theta}{r_1^m(\theta;\mu)B_0(\theta;\mu)}$$

with $\lim_{x\to 0} R(\theta,x;\mu)/x^2=0$ uniformly on θ. This proves the expression of $\widehat{T}(0;\mu)$ and $\widehat{T}'(0;\mu)$. $\square$

The next three lemmas are well-known results. They will be used in the study of the saddle loop bifurcation.

Lemma 4. *Let E be a measurable set of $\mathbb{R}$ and consider a collection of measurable functions $\{f_n\}_{n\in\mathbb{N}}$. If $\int_E\sum_{n=1}^{\infty}|f_n(x)|\,dx<\infty$ then*

$$\int_E\sum_{n=1}^{\infty}f_n(x)\,dx = \sum_{n=1}^{\infty}\int_E f_n(x)\,dx.$$

Lemma 5. *Let E be a measurable set of $\mathbb{R}$ and consider a collection of measurable positive functions $\{f_n\}_{n\in\mathbb{N}}$. Then*

$$\int_E\sum_{n=1}^{\infty}f_n(x)\,dx = \sum_{n=1}^{\infty}\int_E f_n(x)\,dx,$$

where the infinity value is also allowed.

Lemma 6 (*Cauchy's estimates*). *Let $f(z)$ be an analytic function on $D_R = \{z \in \mathbb{C} : |z| < R\}$ such that, for all $z \in D_R$, it holds $f(z) = \sum_{i=0}^{\infty} a_i z^i$ and $|f(z)| < M$. Then $|a_i| < M/R^i$ for all i.*

3. Hopf-like Bifurcations

This section is devoted to study two similar bifurcations: a generalization of the Hopf bifurcation at the origin in Section 3.1 and the Hopf bifurcation at infinity for polynomial vector fields in Section 3.2.

3.1. Generalized Andronov–Hopf bifurcation

Hopf-like bifurcations typically occur when a monodromic singular point (i.e., such that a Poincaré map can be defined in a neighbourhood of it) reverses its stability as the parameter varies.

Let us suppose that there exists an open interval Λ containing zero such that the vector field X_μ has a monodromic critical point with no characteristic directions for all $\mu \in \Lambda$. It is not restrictive to assume that the critical point is fixed at the origin and that its associated differential equation can be written as in (1), being k an odd number. Taking polar coordinates $r^2 = x^2 + y^2$ and $\theta = \arctan(y/x)$, it writes as

$$\begin{cases} \dfrac{dr}{dt} = \displaystyle\sum_{n \geqslant k} R_n(\theta;\mu) r^n, \\ \dfrac{d\theta}{dt} = \displaystyle\sum_{n \geqslant k} F_n(\theta;\mu) r^{n-1} \end{cases} \tag{4}$$

with $k \geqslant 1$, and where $R_n(\theta;\mu) = \cos\theta\, P_n(\cos\theta, \sin\theta;\mu) + \sin\theta\, Q_n(\cos\theta, \sin\theta;\mu)$ and $F_n(\theta;\mu) = \cos\theta\, Q_n(\cos\theta, \sin\theta;\mu) - \sin\theta\, P_n(\cos\theta, \sin\theta;\mu)$ are trigonometric polynomials of degree $n+1$ in θ and analytic in μ. Notice that the monodromy condition for the critical point at the origin, together with the fact that it has not characteristic directions, implies that $F_k(\theta;\mu)$ does not vanish.

Theorem 7 (*Generalized Andronov–Hopf bifurcation*). *Let $\{X_\mu\}_{\mu\in\Lambda}$ be an analytic family of planar vector fields such that its expression in polar coordinates is given by* (4) *with $k = 2p+1$ and $F_k(\theta;0) > 0$ for all θ. Let $S_n(\theta;\mu)$ be given by the relation*

$$\frac{\sum_{n\geqslant k} R_n(\theta;\mu) r^n}{\sum_{n\geqslant k} F_n(\theta;\mu) r^{n-1}} = \sum_{n\geqslant 1} S_n(\theta;\mu) r^n,$$

and define

$$V_1(\mu) = \exp\left(\int_0^{2\pi} S_1(\theta;\mu)\, d\theta\right) \quad \textit{and} \quad V_3 = \int_0^{2\pi} S_3(\theta;0) \exp\left(\int_0^{\theta} S_1(\psi;0)\, d\psi\right) d\theta.$$

Then, if $V_1(0) = 1$ *and* $V_1'(0)\,V_3 \neq 0$, *the following holds*:

(a) *Exactly one limit cycle* γ_μ *bifurcates from the critical point of* X_μ *at the origin for* $\mu \gtrsim 0$ (*respectively* $\mu \lesssim 0$) *if* $V_1'(0)\,V_3$ *is negative* (*respectively positive*). *Moreover no periodic orbits bifurcate from the origin on the opposite side of* $\mu = 0$.

(b) *The period of the periodic orbit* γ_μ *is*

$$T(\mu) = \begin{cases} T_0 + T_1\mu + O\left(|\mu|^{3/2}\right) & if \;\; p = 0, \\ T_0\,\mu^{-p}\,\left(1 + O\left(|\mu|^{1/2}\right)\right) & if \;\; p \geqslant 1, \end{cases}$$

where $T_0 > 0$ *and* T_1 *may be zero* (*see Example* 8).

Proof. For μ small enough the Poincaré return map of vector field X_μ with respect to the transversal section $\{\theta = 0\}$ is well-defined in a neighbourhood of the origin. Let $r(\theta, x; \mu)$ be the solution of the polar expression of X_μ given in (4) with $r(0, x; \mu) = x$. Then the Poincaré map can be computed as $r(2\pi, x; \mu)$ and so the displacement map is given by $D(x; \mu) := r(2\pi, x; \mu) - x$. Notice that the zeros of $D(x; \mu)$ correspond to the limit cycles of X_μ in a neighbourhood of the origin. Several derivatives of this displacement map can be computed by using Lemma 3. In particular we get that

$$D(0; 0) = D_x(0; 0) = V_1(0) - 1 = D_{xx}(0; 0) = 0 \quad \text{and}$$
$$D_{xxx}(0; 0) D_{x\mu}(0; 0) = 6V_1'(0)\,V_3 \neq 0.$$

Therefore (a) is a direct consequence of Lemma 2 applied to the displacement function.

In order to prove (b) let us assume for instance that $V_1'(0)\,V_3 < 0$. Denote the x-coordinate of the point $\gamma_\mu \cap \{\theta = 0\}$ by $x_l(\mu)$. Then Lemma 2 shows that $x_l(\mu) = \varphi(\sqrt{\mu})$ where φ is an analytic function with $\varphi(0) = 0$ and $\varphi'(0) = \sqrt{\left|V_1'(0)/V_3\right|} =: \alpha$. Now the expression of $T(\mu)$ follows from applying Lemma 3 with $m = k - 1$. Indeed, using the notation in that result, we have that $T(\mu) = T\,(x_l(\mu); \mu)$ and, on the other hand, we can assert that $T(x; \mu) = \widehat{T}(x; \mu)/x^{k-1}$ where $\widehat{T}(x; \mu)$ is an analytic function at $x = 0$. Thus, since $x_l(\mu) = \alpha\sqrt{\mu} + \beta\mu + O(\mu^{3/2})$, it turns out that

$$\begin{aligned} T(\mu) &= T(x_l(\mu); \mu) \\ &= \left(\alpha\sqrt{\mu} + \beta\mu + O(\mu^{3/2})\right)^{1-k} \left(\widehat{T}(0; 0) + \alpha\widehat{T}'(0; 0)\sqrt{\mu} + O(\mu)\right) \\ &= \alpha^{1-k}\mu^{(1-k)/2}\left(\widehat{T}(0; 0) + \left((1-k)\frac{\beta}{\alpha}\widehat{T}(0; 0) + \alpha\widehat{T}'(0; 0)\right)\sqrt{\mu} + O(\mu)\right). \end{aligned}$$

Notice that the result will follow once we show that $\widehat{T}(0; 0) > 0$ and that if $k = 1$, then $\widehat{T}'(0; 0) = 0$. By applying Lemma 3 the first inequality is straightforward because

$$\widehat{T}(0;0)=\int_0^{2\pi}\exp\left((1-k)\int_0^{\theta}\frac{R_k(\psi;0)}{F_k(\psi;0)}d\psi\right)\frac{d\theta}{F_k(\theta;0)}>0.$$

From Lemma 3 we also obtain that

$$\widehat{T}'(0;0)=(1-k)K-\int_0^{2\pi}\frac{F_{k+1}(\theta;0)}{F_k^2(\theta;0)}\exp\left((1-k)\int_0^{\theta}\frac{R_k(\psi;0)}{F_k(\psi;0)}d\psi\right)d\theta. \qquad (5)$$

We do not specify the value K because we are only interested in the case $k=1$. Let us prove that in fact the integral in (5) is zero for any k. To see this notice first that, for all $\theta\in[0,2\pi]$, $F_k(\theta+\pi;0)=F_k(\theta;0)$, $R_k(\theta+\pi;0)=R_k(\theta;0)$ and $F_{k+1}(\theta+\pi;0)=-F_{k+1}(\theta;0)$. Note moreover that the hypothesis $V_1(0)=0$ implies that $\int_0^{2\pi}\frac{R_k(\theta;0)}{F_k(\theta;0)}\,d\theta=2\int_0^{\pi}\frac{R_k(\theta;0)}{F_k(\theta;0)}\,d\theta=0$. Hence the function $\theta\longmapsto\exp\left((1-k)\int_0^{\theta}\frac{R_k(\psi;0)}{F_k(\psi;0)}\,d\psi\right)$ is π-periodic. Consequently, if we denote the integrand appearing in (5) by $I(\theta)$, we have shown that $I(\theta+\pi)=-I(\theta)$. Therefore $\widehat{T}'(0;0)=(1-k)K$ as desired. $\square$

Next example shows that the constant T_1 that appears in Theorem 7 may be zero. It also shows that the period of the limit cycle of an Andronov–Hopf bifurcation tends to a constant value with many different possible speeds.

Example 8. Consider a polynomial system of the form (1) such that in polar coordinates writes as

$$\begin{cases}\dfrac{dr}{dt}=\mu r-r^3,\\[4pt] \dfrac{d\theta}{dt}=1-\delta r^{2m}\end{cases}$$

with $\delta\in\{0,1\}$. One can easily check that the hypotheses in Theorem 7 are fulfilled. It has limit cycles only for $\mu>0$, and in this case the limit cycle is unique and given by $\gamma_\mu=\{r=\sqrt{\mu}\}$. Furthermore its period is

$$T(\mu)=\frac{2\pi}{1-\delta\mu^m}=\begin{cases}2\pi(1+\mu^m+O(\mu^{2m})) & when\ \ \delta=1,\\ 2\pi & when\ \ \delta=0.\end{cases}$$

Remark 9. Notice that the hypotheses in Theorem 7 for $k=1$ (modulus the regularity of X_μ) are the same as the ones in the classical Andronov–Hopf bifurcation. In particular, $V_1(0)=1$ indicates that the origin is a weak focus for X_0, V_3 is the first Lyapunov constant of X_0, and the condition $V_1'(0)\neq 0$ implies that the eigenvalues of the linear part of X_μ at the origin cross the imaginary axis transversally when μ moves from negative to positive values.

3.2. Hopf bifurcation at infinity

Given a family of planar polynomial vector fields $\{X_\mu\}$, it is said that a Hopf bifurcation at infinity occurs for μ crossing 0 if "the infinity changes its stability" giving rise to a periodic orbit (see [15] or Theorem 10 for a rigorous definition). To study the period of the periodic orbit appearing in this bifurcation it is more convenient to compactify the polynomial vector field defined on the plane to an analytic vector field on the sphere. Instead of this well-known procedure, called the Poincaré compactification, we will consider a simpler coordinate transformation that consists in changing the radius r of the polar coordinates to $\rho = 1/r$. Suppose that the polynomial family $\{X_\mu\}$ writes in polar coordinates as

$$\begin{cases} \dot{r} = \sum_{n=k}^{m} R_n(\theta;\mu) r^n, \\ \dot{\theta} = \sum_{n=k}^{m} F_n(\theta;\mu) r^{n-1}, \end{cases} \tag{6}$$

where $m \in \mathbb{N}$ is the maximum degree of the components of X_μ and R_n and F_n are defined as in (4). With this notation we prove the following:

Theorem 10 (*Hopf bifurcation at infinity*). *Let $\{X_\mu\}$ be an analytic family of planar polynomial vector fields such that its expression in polar coordinates is given by* (6). *Assume that $m = 2p+1$ and that $F_m(\theta;0) > 0$ for all $\theta \in [0, 2\pi]$. Let $S_n(\theta;\mu)$ be given by the relation*

$$-\frac{\sum_{n=1}^{m+1-k} R_{m+1-n}(\theta;\mu)\rho^n}{\sum_{n=1}^{m+1-k} F_{m+1-n}(\theta;\mu)\rho^{n-1}} = \sum_{n \geqslant 1} S_n(\theta;\mu)\rho^n,$$

and define

$$W_1(\mu) = \exp\left(\int_0^{2\pi} S_1(\theta;\mu)\,d\theta\right) \quad \text{and} \quad W_3 = \int_0^{2\pi} S_3(\theta;0)\exp\left(\int_0^{\theta} S_1(\psi;0)\,d\psi\right) d\theta.$$

Then, if $W_1(0) = 1$ and $W_1'(0)\,W_3 \neq 0$, the following holds:

(a) *Exactly one limit cycle γ_μ bifurcates from the infinity for $\mu \gtrsim 0$* (*respectively $\mu \lesssim 0$*) *in case that $W_1'(0)\,W_3$ is negative* (*respectively positive*). *Moreover no periodic orbits bifurcate from infinity on the opposite side of $\mu = 0$.*

(b) *The period of the periodic orbit γ_μ is $T(\mu) = T_0\,\mu^p\left(1 + O(\sqrt{|\mu|})\right)$ with $T_0 > 0$.*

Proof. The change $\rho = 1/r$ in (6) yields to

$$\begin{cases} \dot{\rho} = -\left(\sum_{n=1}^{m+1-k} R_{m+1-n}(\theta;\mu)\rho^n\right)\rho^{1-m}, \\ \dot{\theta} = \left(\sum_{n=1}^{m+1-k} F_{m+1-n}(\theta;\mu)\rho^{n-1}\right)\rho^{1-m}. \end{cases}$$

Notice that the "infinity" of the original system is now $\rho = 0$. By applying Lemma 3 it can be seen that if $\rho_l(\mu)$ denotes the inverse of the x-coordinate of the point $\gamma_\mu \cap \{\theta = 0\}$, then there exists an analytic function φ such that $\rho_l(\mu) = \varphi(\sqrt{\mu})$ (respectively $\rho_l(\mu) = \varphi(\sqrt{-\mu})$) satisfying that $\varphi(0) = 0$ and $\varphi'(0) = \sqrt{\left|W_1'(0)/W_3\right|}$. To end the proof we can follow the same steps that in the proof of Theorem 7. □

4. Bifurcation from a semi-stable periodic orbit

Let $\{X_\mu\}_{\mu\in\Lambda}$ be a one-parameter family of planar analytic vector fields. The bifurcation from a *semi-stable periodic orbit* is characterized by the sudden emergence of a double periodic orbit Γ, for let us fix $\mu = 0$, which afterwards gives rise to two hyperbolic periodic orbits with different stability.

By means of the arc-length and the normal coordinates, the study of the above bifurcation is settled into an analogous framework than the one when using the polar coordinates in the Hopf bifurcation. Following [16, Chapter 2], see this reference for further details, we introduce local coordinates with respect to the emerging singular limit cycle Γ of X_0. We assume, without loss of generality, that this limit cycle turns in clockwise sense. Fix an arbitrary point $p \in \Gamma$ and consider the arc-length parametrization of Γ from p, say $s \longmapsto \big(\varphi(s), \psi(s)\big)$ for $s \in [0, \ell]$, being ℓ the length of Γ and taking also the clockwise sense. Let n denote the length of the normal to Γ, whose outward direction is taken to be positive. Then any point (x, y) in a sufficiently small neighbourhood of Γ can be parameterized by the curvilinear coordinates (n, s). If $X_\mu(x, y) = P(x, y; \mu)\partial_x + Q(x, y; \mu)\partial_y$ as usual, then the relation between both coordinate systems is given by

$$x = \varphi(s) - n\psi'(s), \quad y = \psi(s) + n\varphi'(s), \tag{7}$$

where

$$\begin{aligned} \big(\varphi'(s), \psi'(s)\big) = {} & \frac{1}{\sqrt{P^2\left(\varphi(s), \psi(s); 0\right) + Q^2\left(\varphi(s), \psi(s); 0\right)}} \\ & \times\Big(P\left(\varphi(s), \psi(s); 0\right),\ Q\left(\varphi(s), \psi(s); 0\right)\Big). \end{aligned} \tag{8}$$

Notice therefore that Γ is located at $\{n = 0\}$. Define $\widetilde{P}(n, s; \mu) := P\left(\varphi(s) - n\psi'(s),\right.$ $\left.\psi(s) + n\varphi'(s); \mu\right)$ and $\widetilde{Q}(n, s; \mu) := Q\left(\varphi(s) - n\psi'(s), \psi(s) + n\varphi'(s); \mu\right)$. Then one can verify that the coordinate transformation (7) brings system (1) to

$$\begin{cases} \dfrac{dn}{dt} = \dfrac{\widetilde{Q}(n, s; \mu)\varphi'(s) - \widetilde{P}(n, s; \mu)\psi'(s) - n\left(\widetilde{P}(n, s; \mu)\varphi''(s) + \widetilde{Q}(n, s; \mu)\psi''(s)\right)}{1 + n\left(\psi'(s)\varphi''(s) - \varphi'(s)\psi''(s)\right)}, \\ \dfrac{ds}{dt} = \dfrac{\widetilde{P}(n, s; \mu)\varphi'(s) + \widetilde{Q}(n, s; \mu)\psi'(s)}{1 + n\left(\psi'(s)\varphi''(s) - \varphi'(s)\psi''(s)\right)}. \end{cases} \tag{9}$$

Associated to the above system we consider the differential equation

$$\frac{dn}{ds} = F(n, s; \mu), \tag{10}$$

where

$$F(n, s; \mu) := \frac{\widetilde{Q}(n, s; \mu)\varphi'(s) - \widetilde{P}(n, s; \mu)\psi'(s) - n\left(\widetilde{P}(n, s; \mu)\varphi''(s) + \widetilde{Q}(n, s; \mu)\psi''(s)\right)}{\widetilde{P}(n, s; \mu)\varphi'(s) + \widetilde{Q}(n, s; \mu)\psi'(s)}.$$

It is easy to check that $F(n, s; \mu)$ is analytic at $n = 0$ and ℓ-periodic with respect to s. We can now state the main result of this section.

Theorem 11 (*Bifurcation from a double-periodic orbit*). *Let $\{X_\mu\}_{\mu\in\Lambda}$ be a family of planar analytic vector fields such that X_0 has a periodic orbit Γ of length ℓ. Consider the curvilinear coordinates (n, s) associated to Γ given by* (7) *and let $n_0(s; \mu)$ be the solution of* (10) *with $n_0(0; \mu) = 0$. Define*

$$W_1 = \exp\left(\int_0^\ell \frac{\partial F(0, s; 0)}{\partial n}\, ds\right), \quad W_2 = \frac{1}{2}\int_0^\ell \frac{\partial^2 F(0, s; 0)}{\partial n^2} \exp\left(\int_0^s \frac{\partial F(0, \zeta; 0)}{\partial n} d\zeta\right) ds$$

and $R(\mu) = n_0(\ell; \mu)$. Then, if $W_1 = 1$ and $R'(0)\, W_2 \neq 0$, the following holds:

(a) *Exactly two limit cycles, γ_μ^+ and γ_μ^-, bifurcate from Γ for $\mu \gtrsim 0$* (*respectively $\mu \lesssim 0$*) *when $R'(0)\, W_2$ is negative* (*respectively positive*). *Moreover no periodic orbits bifurcate from Γ on the opposite side of $\mu = 0$.*

(b) *The period of the emerging limit cycles is given by $T^\pm(\mu) = T_0 \pm T_1\sqrt{|\mu|} + O(\mu)$, where $T_0 > 0$ is the period of Γ and T_1 may be zero* (*see Example* 12).

Proof. Consider the transversal section to Γ given by $\Sigma := \{s = 0, n \in (-\delta, \delta)\}$ for some $\delta > 0$ small enough. Notice that the return map of X_μ with respect to Σ is well defined for $\mu \approx 0$. This return map leads to the displacement map

$$D(x;\mu) := n(\ell, x;\mu) - x,$$

where $n(s,x;\mu)$ is the solution of (10) satisfying $n(0,x;\mu) = x$. It is clear that, for $x \approx 0$, the zeroes of $x \longmapsto D(x;\mu)$ correspond to the limit cycles of X_μ near Γ. Note also that, by definition, $n_0(s;\mu) = n(s,0;\mu)$ and $n_0(s;0) \equiv 0$.

We claim that $D_x(0;0) = W_1 - 1$, $D_{xx}(0;0) = 2W_2$ and $D_\mu(0;0) = R'(0)$ hold. Note that once we show this then, on account of the hypothesis, we will have that $D(0;0) = 0$, $D_x(0;0) = 0$ and $D_{xx}(0;0)D_\mu(0;0) \neq 0$. Thus the assertions in (a) will follow by applying Lemma 1. In order to prove the claim we perform the change of variables $w = n - n_0(s;\mu)$ to the differential equation (10), which yields to

$$\frac{dw}{ds} = F(w + n_0(s;\mu), s;\mu) - \frac{\partial n_0(s;\mu)}{\partial s} = \frac{\partial F(n_0(s;\mu), s;\mu)}{\partial n} w + \frac{1}{2}\frac{\partial^2 F(n_0(s;\mu), s;\mu)}{\partial n^2} w^2 + O(w^3).$$

(More precisely, the remainder term above is a function $f(w,s;\mu)$ such that $\lim_{w\to 0} f(w,s;\mu)/w^3 = 0$ uniformly on s and μ.) Then, by Lemma 3, we can assert that $n(s,x;\mu) = n_0(s;\mu) + w(s,x;\mu)$ with

$$w(s,x;\mu) = x\, e^{\int_0^s \frac{\partial F(n_0(\tau;\mu),\tau;\mu)}{\partial n} d\tau} \left(1 + \frac{x}{2}\int_0^s \frac{\partial^2 F(n_0(\tau;\mu),\tau;\mu)}{\partial n^2}\, e^{\int_0^\tau \frac{\partial F(n_0(\zeta;\mu),\zeta;\mu)}{\partial n} d\zeta} d\tau\right) + O(x^3).$$

Consequently, taking also $n_0(s;0) \equiv 0$ into account, the above expansion shows that $D(0;\mu) = R(\mu)$ and $D(x;0) = (W_1 - 1)\,x + W_2\,x^2 + O(x^3)$. So the claim is proved and the assertions in (a) follow from Lemma 1. This result also shows that if $x_l^\pm(\mu)$ denotes the n-coordinate of the point $\gamma_\mu^\pm \cap \{s = 0\}$, then there exists an analytic function ϕ defined in a neighbourhood of $\mu = 0$, with $\phi(0) = 0$ and $\phi'(0) = \sqrt{|R'(0)/W_2|} =: \alpha$, such that $x_l^\pm(\mu) = \phi(\pm\sqrt{|\mu|})$.

Next, to prove (b) let us denote by $T(x;\mu)$ the time that spends the solution of (9) starting at a point in Σ with $(n,s) = (x,0)$ to return to Σ. It is clear then that $T^\pm(\mu) = T\left(x_l^\pm(\mu);\mu\right)$. By applying Lemma 3 to system (9) we have that $T(x;\mu)$ is an analytic function at $x = 0$ with

$$T(0;\mu) = \int_0^\ell \frac{1}{\widetilde{P}(0,s;\mu)\varphi'(s) + \widetilde{Q}(0,s;\mu)\psi'(s)}\, ds.$$

Accordingly, taking (8) also into account, it turns out that

$$T(0;0) = \int_0^{\ell} \frac{ds}{\sqrt{P^2\left(\varphi(s),\psi(s);0\right) + Q^2\left(\varphi(s),\psi(s);0\right)}} = \int_0^{T_0} dt = T_0,$$

where T_0 is the period of the periodic orbit Γ of X_0. Here we used that the relation between the initial time t and the arc-length s is given by $dt/ds = \left(P^2(\varphi(s),\psi(s);0) + Q^2(\varphi(s),\psi(s);0)\right)^{-1/2}$. On the other hand, since $x_l^{\pm}(\mu) = \pm\alpha\sqrt{|\mu|} + O(\mu)$ and $T(x;\mu) = T(0;\mu) + T'(0;\mu)x + x^2 g(x;\mu)$, we can conclude that

$$T^{\pm}(\mu) = T\left(x_l^{\pm}(\mu);\mu\right) = T(0;0) \pm \alpha T'(0;0)\sqrt{|\mu|} + O(\mu).$$

Consequently, since $T(0;0) = T_0$, this completes the proof of the result. $\square$

Next example plays a similar role to Example 8. It shows that the speed at which the period of the hyperbolic periodic orbits tend to the period of Γ can be any power of $|\mu|^{1/2}$.

Example 12. Fix a neighbourhood of $\Gamma = \{x^2 + y^2 = 1\}$ not containing the origin. Consider there the analytic family $\{X_\mu\}$ which in polar coordinates writes as

$$\begin{cases} \dfrac{dr}{dt} = r\left((r-1)^2 - \mu\right), \\ \dfrac{d\theta}{dt} = 1 - \delta(r-1)^m \end{cases}$$

with $\delta \in \{0,1\}$. Note then that it is under the hypotheses of Theorem 11. There are limit cycles only when $\mu > 0$ and, in this case, they are given by $\gamma_\mu^{\pm} = \{r = 1 \pm \sqrt{\mu}\}$. Furthermore their periods are

$$T^{\pm}(\mu) = \frac{2\pi}{1 - (\pm 1)^m \delta \mu^{m/2}} = \begin{cases} 2\pi\left(1 + (\pm 1)^m \mu^{m/2} + O(\mu^m)\right) & if\ \delta = 1, \\ 2\pi & if\ \delta = 0. \end{cases}$$

Remark 13. It is clear from the proof of Theorem 11 that the conditions $W_1 = 1$ and $W_2 \neq 0$ correspond to require that Γ is a double limit cycle. In fact it is not difficult to verify (see [1,16]) that using the original (x,y)-coordinates,

$$W_1 = \exp\left(\int_0^{T_0}\left(P_x\,(x(t),y(t);0) + Q_y\,(x(t),y(t);0)\right)dt\right),$$

where $t \longmapsto (x(t),y(t))$ is the "time" parametrization of Γ and T_0 its period. Thus W_1 is the characteristic exponent of Γ.

5. Saddle-node loop bifurcation

Consider a one-parameter family of vector fields $\{X_\mu\}_{\mu\in\Lambda}$ such that for $\mu = 0$, X_0 has a singularity p_0 which is a semi-hyperbolic saddle-node of multiplicity two. Assume also that the vector field X_0 presents a homoclinic orbit Γ connecting the non-hyperbolic separatrix of p_0 with its nodal sector, not through the boundary of this sector. If the dependence of $\{X_\mu\}$ with respect to μ is such that the saddle-node presents the local saddle-node bifurcation then, for those μ such that the saddle node disappears, a hyperbolic limit cycle γ_μ emerges from Γ. This bifurcation is known as the *saddle-node loop bifurcation*. This section is devoted to study the behaviour of the period of γ_μ as $\mu \longrightarrow 0$. The main result of this section is, essentially, a reformulation of results in [6, pp. 1011–1013, 8]. It reads as follows:

Theorem 14 (*Saddle-node loop bifurcation*). *Let $\{X_\mu\}_{\mu\in\Lambda}$ be a one-parameter $\mathcal{C}^\infty$ family of planar vector fields such that*:

(a) *For $\mu = 0$, X_0 has a semi-hyperbolic saddle-node point p_0 of multiplicity two.*
(b) *The vector field X_0 has a homoclinic connection, say Γ, at p_0. This orbit Γ connects the non-hyperbolic separatrix of the hyperbolic sector of p_0 with its nodal sector but not through the boundary of this sector.*
(c) *The family $\{X_\mu\}_{\mu\in\Lambda}$ provides a generic unfolding of the saddle-node* (*see Remark* 15 *for a precise formulation of this condition*).

Then there exists a neighbourhood U of Γ and a neighbourhood V of $\mu = 0$ such that, for all $\mu \in V$ lying on one side of $\mu = 0$, X_μ has a unique periodic orbit γ_μ in U, which tends to Γ as $\mu \longrightarrow 0$. Furthermore, denoting its period by $T(\mu)$, then

$$T(\mu) \sim T_0/\sqrt{|\mu|}$$

for some $T_0 > 0$. For all $\mu \in V$ on the opposite side of $\mu = 0$, X_μ has no periodic orbits in U.

Proof. We take first a convenient normal form for $\{X_\mu\}$ near the singularity p_0. Thus, on account of the assumption in (a), one can show (see [6,9] for instance) that for each $k \in \mathbb{N}$ there exist a $\mathcal{C}^k$ diffeomorphism Φ_k such that, in some neighbourhood of $(p_0, 0) \in \mathbb{R}^2 \times \Lambda$,

$$X_\mu = (\Phi_k)_* \left(f(x;\mu) \left(g(x;\mu) \left(x^2 + \alpha(\mu) \right) \partial_x + y\partial_y \right) \right), \tag{11}$$

where $f(x;\mu)$, $g(x;\mu)$ and $\alpha(\mu)$ are $\mathcal{C}^k$ functions with $f(0;0)\,g(0;0) \neq 0$ and $\alpha(0) = 0$. Clearly we can assume that $f(0;0)\,g(0;0) > 0$ (otherwise we reverse time). The generic condition in (c) corresponds to require that $\alpha'(0) \neq 0$. Let us fix for instance that $\alpha'(0) > 0$ (otherwise we perform the change in the parameter given by $\mu \longmapsto -\mu$). In this case it is well known (see [8] for instance) that for $\mu \gtrsim 0$, a unique (hyperbolic and stable) limit cycle γ_μ bifurcates from Γ. In the study of the bifurcations in the two preceding sections and, as we will see, also in the next one, the dominant term of the

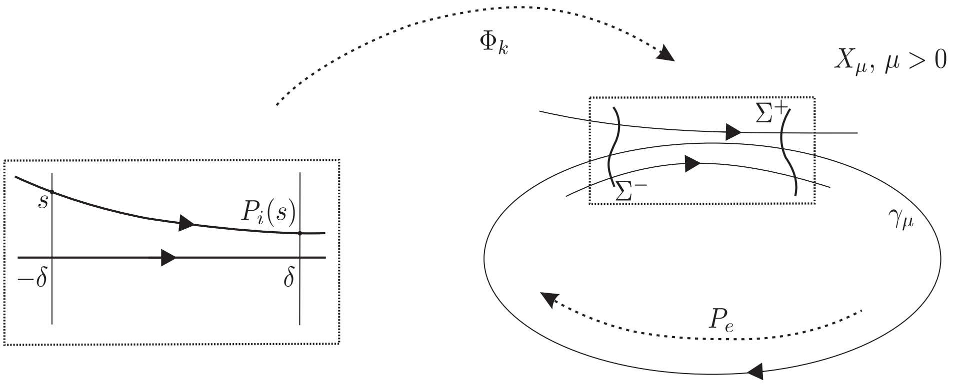

Fig. 1. Poincaré map in the saddle-node loop bifurcation.

period $T(\mu)$ of the limit cycle γ_μ strongly depends on its distance to Γ when μ varies. Fortunately, in this case, the leading term of the asymptotic behaviour of $T(\mu)$ can be computed without locating γ_μ. This fact makes the study of this case easier than the other ones.

Take any $k \geqslant 1$ and consider the C^k diffeomorphism $\Phi_k =: \Phi$ given in (11). Define

$$\Sigma^- := \left\{\Phi(-\delta, s) : s \in (-\varepsilon, \varepsilon)\right\} \quad \text{and} \quad \Sigma^+ := \left\{\Phi(\delta, s) : s \in (-\varepsilon, \varepsilon)\right\}.$$

For $\varepsilon > 0$ and $\delta > 0$ small enough, it is clear that Σ^- and Σ^+ are transversal sections for X_0 to the homoclinic connection Γ. Thus the same happens for X_μ in a neighbourhood of Γ and $\mu \approx 0$. Note in addition that a Poincaré return map for X_μ with $\mu \gtrsim 0$ is well defined in Σ^-. Let us denote this return map by $P(s;\mu)$ and its associated time function by $T(s;\mu)$. In order to study them we first consider the Poincaré and time mappings of X_μ from Σ^- to Σ^+ (see Fig. 1), which we denote by $P_i(s;\mu)$ and $T_i(s;\mu)$, respectively. More precisely, they are defined implicitly by means of

$$\varphi\left(T_i(s;\mu), \Phi(-\delta, s); \mu\right) = \Phi\left(\delta, P_i(s;\mu)\right),$$

where $\varphi(t, q; \mu)$ is the solution of X_μ passing through $q \in \mathbb{R}^2$ at $t = 0$. Similarly, let $P_e(s;\mu)$ and $T_e(s;\mu)$ be respectively the Poincaré and time mappings of X_μ from Σ^+ to Σ^-, which verify

$$\varphi\left(T_e(s;\mu), \Phi(\delta, s); \mu\right) = \Phi\left(-\delta, P_e(s;\mu)\right).$$

Recall that, for $\mu \gtrsim 0$, there exists a periodic orbit γ_μ which tends to Γ as μ tends to zero. Note moreover that γ_μ has a unique intersection point with Σ^-, which we fix to be $\Phi\left(-\delta, s_\ell(\mu)\right)$ for some $s_\ell(\mu) \in (-\varepsilon, \varepsilon)$. It is clear that $s_\ell(\mu) \longrightarrow 0$ as $\mu \longrightarrow 0^+$. Notice in addition that the period of γ_μ is

$$T(\mu) = T_i\left(s_\ell(\mu); \mu\right) + T_e\left(P_i(s_\ell(\mu); \mu); \mu\right). \tag{12}$$

Due to the continuous dependence with respect to initial conditions and parameters, the second term in the above equality tends to a constant value when μ tends to zero, i.e.,

$$\lim_{\mu \longrightarrow 0^+} T_e\left(P_i(s_\ell(\mu); \mu); \mu\right) = T_e^\Gamma > 0, \tag{13}$$

where T_e^Γ is the time that spends the homoclinic solution Γ of X_0 for going from Σ^+ to Σ^-.

Let us turn now to study the first term in (12), which tends to infinity as $\mu \longrightarrow 0^+$. Taking (11) into account we can assert that

$$T_i\left(s_\ell(\mu); \mu\right) = \int_{-\delta}^{\delta} \frac{R(x; \mu)}{x^2 + \alpha(\mu)}\, dx,$$

where $R(x; \mu) := 1/\left(f(x; \mu)\, g(x; \mu)\right)$. Thus $R(0; 0) > 0$. To study this integral notice first that

$$\int_{-\delta}^{\delta} \frac{R(0; \mu)}{x^2 + \alpha(\mu)}\, dx = \frac{2R(0; \mu)}{\sqrt{\alpha(\mu)}} \arctan\left(\frac{\delta}{\sqrt{\alpha(\mu)}}\right) \sim \frac{T_0}{\sqrt{\mu}} \quad \text{with } T_0 := \frac{\pi R(0; 0)}{\sqrt{\alpha'(0)}}.$$

On the other hand, by applying the mean value theorem,

$$\begin{aligned} \frac{T_i\left(s_\ell(\mu); \mu\right)\sqrt{\mu}}{T_0} &= \int_{-\delta}^{\delta} \frac{\sqrt{\mu}\, R(x; \mu)}{T_0\left(x^2 + \alpha(\mu)\right)}\, dx \\ &= \int_{-\delta}^{\delta} \frac{\sqrt{\mu}\, R(0; \mu)}{T_0\left(x^2 + \alpha(\mu)\right)}\, dx + \int_{-\delta}^{\delta} \frac{\sqrt{\mu}\, R_x\left(\xi(x; \mu); \mu\right)\, x}{T_0\left(x^2 + \alpha(\mu)\right)}\, dx, \end{aligned}$$

where $\xi(x; \mu)$ is between 0 and x, in particular inside $[-\delta, \delta]$. Notice at this point that if we define $K := \sup\left\{R_x(x; \mu)\, ;\, x \in [-\delta, \delta], \mu \approx 0\right\}$, then it turns out that

$$\left|\int_{-\delta}^{\delta} \frac{\sqrt{\mu}\, R_x(\xi(x; \mu); \mu)\, x}{T_0\left(x^2 + \alpha(\mu)\right)}\, dx\right| \leqslant K\sqrt{\mu} \int_0^{\delta} \frac{2x\, dx}{x^2 + \alpha(\mu)} = K\sqrt{\mu}\, \ln\left|\frac{\delta^2 + \alpha(\mu)}{\alpha(\mu)}\right|,$$

which one can easily verify that tends to zero as $\mu \longrightarrow 0$. Accordingly

$$\lim_{\mu\longrightarrow 0^+} \frac{T_i(s_\ell(\mu);\mu)\sqrt{\mu}}{T_0} = 1 + \lim_{\mu\longrightarrow 0^+} \int_{-\delta}^{\delta} \frac{\sqrt{\mu}\, R_x\left(\xi(x;\mu);\mu\right)\, x}{T_0\left(x^2+\alpha(\mu)\right)}\, dx = 1.$$

This, together with (12) and (13), proves that $T(\mu) \sim T_0/\sqrt{\mu}$ as desired. □

Remark 15. The hypothesis in (c) for the family $\{X_\mu\}$ in Theorem 14 corresponds to require that the function $\alpha(\mu)$ in (11) verifies $\alpha'(0) \neq 0$.

6. Saddle loop bifurcation

Let $\{X_\mu\}_{\mu\in\Lambda}$ be a one-parameter $\mathcal{C}^\infty$ family of planar vector fields. Suppose that for $\mu = 0$, X_0 presents a saddle loop Γ, being the saddle point p_0 hyperbolic and strong (i.e., $\operatorname{div} X_0(0) \neq 0$). This section is devoted to study, for $\mu \approx 0$, the dominant term of the asymptotic development of the period of the periodic orbit that bifurcates from Γ when the connection is broken. Note that the hyperbolicity of the saddle point p_0 of X_0 forces that, for $\mu \approx 0$, each vector field X_μ has also a hyperbolic saddle point p_μ. We denote by $\lambda_2(\mu) < 0 < \lambda_1(\mu)$ its eigenvalues and by $r(\mu) = -\lambda_2(\mu)/\lambda_1(\mu)$ its *ratio of hyperbolicity.*

Theorem 16 (*Saddle loop bifurcation*). *Let $\{X_\mu\}_{\mu\in\Lambda}$ be an one-parameter $\mathcal{C}^\infty$ family of planar vector fields. Assume that for $\mu = 0$, X_0 has a hyperbolic saddle point p_0 with hyperbolicity radio $r(0) > 1$ (respectively, $r(0) < 1$). Suppose also that X_0 has a saddle connection, say Γ, at p_0. Under a generic assumption (to be specified in* Remark 20), *there exists a neighbourhood U of Γ and a neighbourhood V of $\mu = 0$ such that for all $\mu \in V$ lying on one side of $\mu = 0$, X_μ has a unique periodic orbit γ_μ in U, which tends to Γ as $\mu \longrightarrow 0$. Furthermore, denoting its period by $T(\mu)$, then*

$$T(\mu) = c\, \ln|\mu| + O(1),$$

where $c = -1/\lambda_1(0)$ (respectively, $c = 1/\lambda_2(0)$). For all $\mu \in V$ on the opposite side of $\mu = 0$, X_μ has no periodic orbits in U.

Let us point out that the assertions concerning the existence and location of γ_μ are common knowledge (see [4,8]). For related results concerning the period of γ_μ see [3,11]. Our first goal will be to prove Lemma 18, that will provide us a convenient normal form to study the time and Dulac functions associated to the passage near a saddle point. This is an easy application of the following result of Bonckaert [2]:

Lemma 17. *For each $k \in \mathbb{N}$ there exists $K(k) \in \mathbb{N}$ such that if $\{Y_\mu\}$ is any $\mathcal{C}^\infty$ family of vector fields verifying that*

$$j^{K(k)}\left(Y_\mu - X_\mu\right)(p_\mu) = 0,$$

then the two families $\{X_\mu\}$ and $\{Y_\mu\}$ are $\mathcal{C}^k$ conjugate. (This means that there exists a $\mathcal{C}^k$ family of diffeomorphisms Φ_μ such that $(\Phi_\mu)_*\left(Y_\mu\right) = X_\mu$.)

Lemma 18. *Let $\{X_\mu\}_{\mu\in\Lambda}$ be the family defined above. Fix some parameter $\mu_0 \in \Lambda$ and any $k \in \mathbb{N}$.*

(a) *If $r(\mu_0) = p/q$ with $(p,q) = 1$, then there exists a $\mathcal{C}^k$ family of diffeomorphisms Φ_μ such that, in some neighbourhood of $(p_{\mu_0}, \mu_0) \in \mathbb{R}^2 \times \Lambda$,*

$$X_\mu = (\Phi_\mu)_*\left(\frac{1}{f(u;\mu)}\left(x\partial_x + yg(u;\mu)\partial_y\right)\right),$$

where $f(u;\mu)$ and $g(u;\mu)$ are polynomials in $u := x^p y^q$ with coefficients $\mathcal{C}^\infty$ functions in μ. In particular it holds $f(0;\mu) = 1/\lambda_1(\mu)$ and $g(0;\mu) = -r(\mu)$.

(b) *If $r(\mu_0) \notin \mathbb{Q}$ then there exists a $\mathcal{C}^k$ family of diffeomorphisms Φ_μ such that it holds*

$$X_\mu = (\Phi_\mu)_*\left(\lambda_1(\mu)\, x\, \partial_x + \lambda_2(\mu)\, y\, \partial_y\right)$$

in some neighbourhood of $(p_{\mu_0}, \mu_0) \in \mathbb{R}^2 \times \Lambda$.

Proof. Clearly we can assume that $p_\mu = (0,0)$ and $j^1 X_\mu(0) = \lambda_1(\mu)\, x\, \partial_x + \lambda_2(\mu)\, y\, \partial_y$ for all μ. Fix some parameter μ_0 and let $k \in \mathbb{N}$ be given. Consider in addition the natural number $K(k)$ that provides Lemma 17.

Let us study first the case $r(\mu_0) \in \mathbb{Q}$ and assume that $r(\mu_0) = p/q$ with $(p,q) = 1$. Recall (see [4] for instance) that the resonant monomials of order i for the first and second components of X_μ are given respectively by

$$\lambda_1(\mu) = n\lambda_1(\mu) + m\lambda_2(\mu) \text{ and } \lambda_2(\mu) = n\lambda_1(\mu) + m\lambda_2(\mu),$$

where $n + m = i \geqslant 2$. Consequently all the resonant monomials for X_{μ_0} are generated by the unique relation $p\lambda_1(\mu_0) + q\lambda_2(\mu_0) = 0$. Thus, on account of the continuity of $r(\mu)$, there exists a neighbourhood U_0 of μ_0 such that if $\mu \in U_0$ then the resonances of X_μ with order $\leqslant K(k)$ are also given by $p\lambda_1(\mu) + q\lambda_2(\mu) = 0$. Then, by using standard techniques (see again [4]), we can construct a conjugation ($\mathcal{C}^\infty$ on μ and analytic on x and y) between $\{X_\mu\}_{\mu\in U_0}$ and

$$X_\mu^1 := \left(xP(u;\mu) + \mathrm{o}\left(\|x,y\|^{K(k)}\right)\right)\partial_x + \left(yQ(u;\mu) + \mathrm{o}\left(\|x,y\|^{K(k)}\right)\right)\partial_y,$$

where P and Q are polynomial in $u := x^p y^q$ with $P(0;\mu) = \lambda_1(\mu)$ and $Q(0;\mu) = \lambda_2(\mu)$. Next, by applying Lemma 17, we can assert the existence of a C^k conjugation between X_μ^1 and

$$X_\mu^2 := xP(u;\mu)\partial_x + yQ(u;\mu)\partial_y.$$

Consider now any $\kappa \in \mathbb{N}$ verifying that $(p+q)\kappa + 1 > K(k)$. We define $f(u;\mu)$ and $g(u;\mu)$ as the Taylor polynomial of degree κ at $u = 0$ of

$$u \longmapsto \frac{1}{P(u;\mu)} \quad \text{and} \quad u \longmapsto \frac{Q(u;\mu)}{P(u;\mu)},$$

respectively. Therefore, since by construction we have that

$$\frac{1}{f(u;\mu)} = P(u;\mu) + \mathrm{o}(u^\kappa) \quad \text{and} \quad \frac{g(u;\mu)}{f(u;\mu)} = Q(u;\mu) + \mathrm{o}(u^\kappa),$$

taking $(p+q)\kappa + 1 > K(k)$ into account, Lemma 17 shows that X_μ^2 is $\mathcal{C}^k$ conjugate to

$$X_\mu^3 := \frac{1}{f(u;\mu)} \left(x\partial_x + yg(u;\mu)\partial_y \right).$$

This completes the proof in the rational case.

Consider finally the case $r(\mu_0) \notin \mathbb{Q}$ and note that then X_{μ_0} has no resonant monomials. Hence, due to the continuity of $r(\mu)$, there exists a neighbourhood U_0 of μ_0 such that if $\mu \in U_0$ then X_μ has no resonant monomials of order $\leqslant K(k)$. In this situation, exactly the same way as before, we can construct a conjugation between $\{X_\mu\}_{\mu \in U_0}$ and

$$X_\mu^1 := \left(\lambda_1(\mu)\, x + \mathrm{o}\left(\|x, y\|^{K(k)} \right) \right) \partial_x + \left(\lambda_2(\mu)\, y + \mathrm{o}\left(\|x, y\|^{K(k)} \right) \right) \partial_y.$$

Then, by Lemma 17, there exists a $\mathcal{C}^k$ conjugation between X_μ^1 and $X_\mu^2 := \lambda_1(\mu)\, x \;\; \partial_x + \lambda_2(\mu)\, y\, \partial_y$. This shows the result in the irrational case and completes the proof. $\square$

Taking $\mu_0 = 0$ and any $k \geqslant 1$, we consider the C^k diffeomorphism Φ in Lemma 18. Define

$$\Sigma_1 = \{\Phi(s, 1) : s \in (-\varepsilon, \varepsilon)\} \quad \text{and} \quad \Sigma_2 = \{\Phi(1, s) : s \in (-\varepsilon, \varepsilon)\}.$$

For $\varepsilon > 0$ small enough, it is clear that Σ_1 (respectively Σ_2) is a transversal section for X_μ in the stable (respectively unstable) manifold of p_μ.

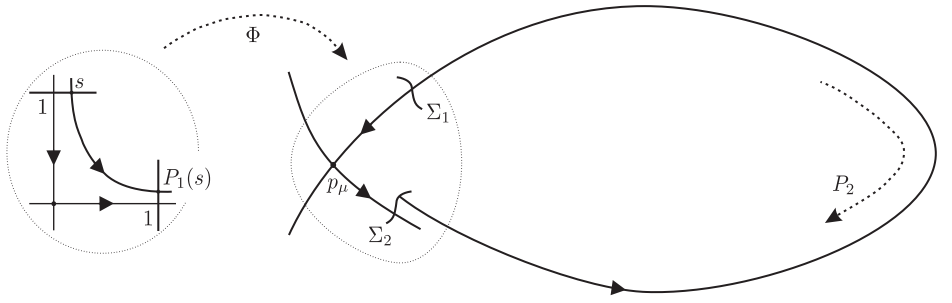

Fig. 2. Transversal sections in Definition 19.

Definition 19. We denote the Dulac and time mappings associated to the passage from Σ_1 to Σ_2 for X_μ by P_1 and T_1 respectively (see Fig. 2). To be more precise, for each $s \in (0, \varepsilon)$ we define $P_1(s; \mu)$ and $T_1(s; \mu)$ by means of the relation

$$\varphi\left(T_1(s;\mu), \Phi(s,1);\mu\right) = \Phi\left(1, P_1(s;\mu)\right),$$

where $\varphi(t, q; \mu)$ is the solution of X_μ passing through $q \in \mathbb{R}^2$ at $t = 0$. Similarly, let P_2 and T_2 be respectively the Poincaré and time mappings from Σ_1 to Σ_2 for $-X_\mu$. More precisely, for each $s \in (0, \varepsilon)$, we define $P_2(s; \mu)$ and $T_2(s; \mu)$ by means of $\varphi\left(-T_2(s;\mu), \Phi(s,1);\mu\right) = \Phi\left(1, P_2(s;\mu)\right)$.

Let us point out that T_1 and T_2 are *positive* functions. It is well known that P_2 and T_2, which are only well defined for $\mu \approx 0$, are $\mathcal{C}^k$ functions at $s = 0$. Note in particular that

$$P_2(s;\mu) = a_0(\mu) + a_1(\mu)s + \mathrm{o}(s) \;\text{ with } a_0(0) = 0 \text{ and } a_1(0) \neq 0.$$

Remark 20. The generic assumption in the statement of Theorem 16 is $a_0'(0) \neq 0$. It is important to note that this condition does not depend on the particular transversal sections Σ_1 and Σ_2 used to define P_2. We construct them using the normal form only for convenience.

Definition 21. Let $g(s; \mu)$ be a $\mathcal{C}^1$ function in $(0, \varepsilon) \times \Lambda$ for some $\varepsilon > 0$. We shall say that g belongs to $\mathcal{B}$ if setting $g(0; \mu) := 0$ then g is a $\mathcal{C}^1$ function at $(s; \mu) = (0; \mu_0)$ for $\mu_0 \approx 0$ and $g_s(0; \mu_0) = 0$. In other words, $g \in \mathcal{B}$ if there exists a $\mathcal{C}^1$ function $\widetilde{g}$ in $(-\varepsilon, \varepsilon) \times \Lambda$ with $\widetilde{g}(0; \mu) = \widetilde{g}_s(0; \mu) = 0$ such that $g(s; \mu) = \widetilde{g}(s; \mu)$ for $s > 0$. (Note that if $g \in \mathcal{B}$, then it also holds $g_\mu(0; \mu_0) = 0$ for $\mu_0 \approx 0$.)

Definition 22. The function defined for $s > 0$ and $\alpha \in \mathbb{R}$ by means of

$$\omega(s;\alpha) = \begin{cases} \frac{s^{-\alpha}-1}{\alpha} & \text{if } \alpha \neq 0, \\ -\ln s & \text{if } \alpha = 0, \end{cases}$$

is called the Roussarie–Ecalle compensator.

It is well known that in general the functions P_1 and T_1, involved in the passage near the saddle point, are not smooth at $s = 0$. Concerning these functions we shall prove the following:

Proposition 23. *With the definitions introduced above,*
(a) *If* $r(0) > 1$ *then* $P_1(s;\mu) = s^{r(\mu)}\left(1 + \psi_1(s;\mu)\right)$ *and* $T_1(s;\mu) = \frac{-1}{\lambda_1(\mu)} \ln s + \psi_2(s;\mu)$ *with* $\psi_i \in \mathcal{B}$.
(b) *If* $r(0) = 1$ *then, setting* $\alpha_1(\mu) = 1 - r(\mu)$,

$$P_1(s;\mu) = s^{r(\mu)}\bigg(1 + \alpha_2(\mu) s\,\omega\,(s;\alpha_1(\mu)) + \psi_1(s;\mu)\bigg),$$

and

$$T_1(s;\mu) = \frac{-1}{\lambda_1(\mu)} \ln s + \beta_1(\mu) s\,\omega\,(s;\alpha_1(\mu)) + \psi_2(s;\mu),$$

where $\psi_i \in \mathcal{B}$ *and* α_2 *and* β_1 *are* $\mathcal{C}^\infty$.

In order to prove Theorem 16, about which we recall that it deals with the case $r(0) \neq 1$, it is enough to consider the case $r(0) > 1$. As we will see, the assertion concerning the case $r(0) < 1$ is straightforward once it is proved the one for $r(0) > 1$. This is the reason why Proposition 23 does not contemplate the case $r(0) < 1$. On the other hand, since little effort has to be made to study also the case $r(0) = 1$, we include it for the sake of completeness. Let us also point out that to prove Theorem 16 it suffices that the function ψ_2 in Proposition 23 is bounded for $s \approx 0$. We show that $\psi_2 \in \mathcal{B}$ because we think that it is an interesting result by itself. Finally it is worth noting that β_1 and α_2 are related to the polynomials $f(u;\mu)$ and $g(u;\mu)$ of the normal form that we use in the resonant case (see (a) in Lemma 18). More concretely, $\beta_1(\mu) = f_u(0;\mu)$ and $\alpha_2(\mu) = q g_u(0;\mu)$. So we prefer to keep the notation of the proof although they are unspecified in the statement.

In the proof of Proposition 23 we shall use the following result:

Lemma 24. *Let* $\alpha(\mu)$ *and* $\beta(\mu)$ *be* $\mathcal{C}^\infty$ *functions in a neighbourhood of* $\mu = 0$ *with* $\alpha(0) = 0$ *and* $\beta(0) > 1$. *Then the function* $G(s;\mu) = s^{\beta(\mu)}\omega\,(s;\alpha(\mu))^n$ *belongs to* $\mathcal{B}$ *for any* $n \in \mathbb{N}$.

Proof. Define $G(0;\mu) := 0$. To show the result it is convenient to write the Ecalle–Roussarie compensator as

$$\omega(s;\alpha) = F(\alpha \ln s) \ln s \ \text{ with } \ F(u) := \frac{e^{-u}-1}{u}.$$

It is easy to verify that $|F(u)| \leqslant e^{|u|}$ and $|F'(u)| \leqslant e^{|u|}$. Then, using the first inequality, it turns out that $|G(s;\mu)| = |s^{\beta(\mu)} F\,(\alpha(\mu) \ln s)^n \,(\ln s)^n| \leqslant s^{\beta(\mu) - n|\alpha(\mu)|} (\ln s)^n$. Hence, since $\beta(0) - n|\alpha(0)| > 1$,

$$\frac{\partial G(0;\mu_0)}{\partial s} = \lim_{s \longrightarrow 0} \frac{G(s;\mu_0)}{s} = 0 \ \text{ for } \mu_0 \approx 0.$$

It is clear in addition that $G_\mu(0;\mu_0) = 0$. On the other hand, by using also the bound for F', some computations yield to

$$\left| \frac{\partial G(s;\mu)}{\partial \mu} \right| \leqslant s^{\beta(\mu) - n|\alpha(\mu)|} (\ln s)^{n+1} \left(\beta'(\mu) + n\alpha'(\mu) \right)$$

and

$$\left| \frac{\partial G(s;\mu)}{\partial s} \right| \leqslant \Big(\big(\beta(\mu) + n\alpha(\mu)\big) \ln s + n \Big) s^{\beta(\mu) - n|\alpha(\mu)| - 1} (\ln s)^{n-1}.$$

Note that both upper bounds tend to zero as $(s,\mu) \longrightarrow (0,\mu_0)$ with $\mu_0 \approx 0$ because $\beta(0) > 1$ and $\alpha(0) = 0$. Thus $G_s(s;\mu) \longrightarrow 0$ and $G_\mu(s;\mu) \longrightarrow 0$ as $(s,\mu) \longrightarrow (0,\mu_0)$ and so the result follows. $\square$

Proof of Proposition 23. Recall that the diffeomorphism Φ, which we use to define Σ_1 and Σ_2, verifies $X_\mu = \Phi_* \left(X_\mu^N \right)$, where X_μ^N denotes the normal form of X_μ. This normal form depends on $r(0) \notin \mathbb{Q}$ and $r(0) \in \mathbb{Q}$. In the first case $X_\mu^N = \lambda_1(\mu) x \partial_x + \lambda_2(\mu) y \partial_y$ and one can easily show, without using that $r(0) \geqslant 1$, that $P_1(s;\mu) = s^{r(\mu)}$ and $T_1(s;\mu) = \frac{-1}{\lambda_1(\mu)} \ln s$.

So consider the case $r(0) \in \mathbb{Q}$ and assume that $r(0) = p/q$ with $(p,q) = 1$. Let us fix that the functions that appear in X_μ^N are

$$f(u;\mu) = \frac{1}{\lambda_1(\mu)} + \sum_{i=1}^{n} \beta_i(\mu)\, u^i \ \text{ and } \ g(u;\mu) = -r(\mu) + \frac{1}{q} \sum_{i=1}^{n} \alpha_{i+1}(\mu)\, u^i,$$

where recall that $u = x^p y^q$. It will be clear later the reason why we fix the coefficients of $g(u;\mu)$ in this way. For the same reason it is convenient to introduce

$$\alpha_1(\mu) := p - qr(\mu).$$

Note that the coefficients α_i and β_i are $\mathcal{C}^\infty$ functions defined for $\mu \approx 0$ and that $\alpha_1(0) = 0$.

Let us show first the result concerning the Dulac map. It follows from the tools developed by Roussarie in [12] to prove the so called Mourtada's form for the Dulac map (see also [5,10]). Indeed, according to Propositions 10 and 11 in [12] there exist $\kappa \in \mathbb{N}$ and $\psi \in \mathcal{B}$ such that

$$P_1(s;\mu) = s^{r(\mu)} \left(1 + \sum_{i=1}^{\kappa} s^{ip} Q_i(s;\mu) + \psi(s;\mu)\right)^{1/q}, \tag{14}$$

where each $Q_i(s;\mu)$ is a polynomial of degree $\leqslant i$ in $\omega(s;\alpha_1(\mu))$ with its coefficients polynomial in $\alpha_1(\mu), \dots, \alpha_{i+1}(\mu)$. In particular one can easily verify that $Q_1(s;\mu) = \alpha_2(\mu)\omega(s;\alpha_1(\mu))$. For each i we consider the function

$$(s;\mu) \longmapsto s^{ip} Q_i(s;\mu). \tag{15}$$

Assume first that $r(0) > 1$, and note that in consequence $p \geqslant 2$. Thus $ip \geqslant 2$ and hence, by applying Lemma 24, the function in (15) belongs to $\mathcal{B}$ for any i. On account of (14) this easily shows that the assertion concerning $P_1(s;\mu)$ in (a) is true. In order to prove the one in (b) note that $r(0) = 1$ implies that $p = q = 1$. According to (14), since $Q_1(s;\mu) = \alpha_2(\mu)\omega(s;\alpha_1(\mu))$, it suffices to verify that the function in (15) belongs to $\mathcal{B}$ for $i \geqslant 2$. However, by applying Lemma 24, this is also clear because then $ip \geqslant 2$.

Let us study next the time function T_1 associated to the passage through the saddle. Notice that $T_1(s;\mu)$ is precisely the time that spends the solution of X_μ^N passing through $(s,1)$ to reach $\{x = 1\}$. Consider the family of vector fields $Y_\mu := x\partial_x + yg(u;\mu)\partial_y$, which it is clear that provides the same foliation as X_μ^N. To study the solutions of Y_μ we follow the same approach as Roussarie [12] for the Dulac map. We thus perform the singular change of variables $\{x = x, u = x^p y^q\}$, which one can easily show that brings Y_μ to

$$\begin{cases} \dot{x} = x, \\ \dot{u} = \sum_{i=1}^{n+1} \alpha_i(\mu)\, u^i. \end{cases}$$

Note that this differential system has separated variables. The solution of the first equation is $x(t,x_0) = x_0 e^t$. Let us denote by $u(t,u_0;\mu)$ the solution of the second equation passing through u_0 at $t=0$. This solution can be expanded as

$$u(t,u_0;\mu)=\sum_{i=1}^{\infty} g_i(t;\mu)u_0^i. \tag{16}$$

In particular one can easily verify that $g_1(t;\mu)=e^{\alpha_1(\mu)\,t}$. Moreover Lemma 19 in [12] shows that there exist positive constants C and C_0 such that

$$|g_i(t;\mu)|\leqslant C_0\left(Ce^{t/2}\right)^i \quad \text{for } t\geqslant 0 \text{ and } \mu\approx 0. \tag{17}$$

This implies that (16) is convergent for $|u_0|<\frac{1}{Ce^{t/2}}$ and, since $\sum_{i=1}^{\infty} r^i<1$ for $0<r<1/2$, that

$$|u(t,u_0;\mu)|<C_0 \ \text{ if } \ |u_0|<\frac{1}{2Ce^{t/2}}. \tag{18}$$

Note at this point that, on account of $x(-\ln s,s)=1$ and $Y_\mu=f(u;\mu)\,X_\mu^N$, we have that

$$T_1(s;\mu)=\int_0^{-\ln s} f\left(u(t,s^p;\mu);\mu\right)\,dt.$$

In order to study this function let us first note that

$$f\,(u(t,u_0;\mu);\mu)=\frac{1}{\lambda_1(\mu)}+\sum_{i=1}^{\infty} a_i(t;\mu)\,u_0^i, \tag{19}$$

with

$$\begin{aligned} a_i(t) := \ &\beta_1\, g_i(t)+\beta_2\sum_{m_1+m_2=i} g_{m_1}(t)\,g_{m_2}(t) \\ &+\cdots+\beta_n\sum_{m_1+\ldots+m_n=i} g_{m_1}(t)\cdots g_{m_n}(t). \end{aligned} \tag{20}$$

In the above equality (and in the sequel when there is no risk of ambiguity) we omit the parameter dependence for the sake of shortness. Note in particular that $a_1(t)=\beta_1 e^{\alpha_1 t}$.

The above expansion yields to

$$T_1(s;\mu) = -\frac{1}{\lambda_1(\mu)} \ln s + \int_0^{-\ln s} \sum_{i=1}^{\infty} a_i(t;\mu)\, s^{pi}\, dt.$$

Our next goal is to commute the sum and integral in the above expression of $T_1(s;\mu)$. To this end note that, since $u \longmapsto f(u;\mu)$ is polynomial, the series in (19) has the same radius of convergence than the one in (16). Consequently, on account of (18), if we define $C_1 := \sup\{|f(u;\mu)| : |u| \leqslant C_0, \mu \approx 0\}$ then by applying Lemma 6 with $R = \frac{1}{2Ce^{t/2}}$ it follows that

$$|a_i(t;\mu)| \leqslant C_1 (2Ce^{t/2})^i \quad \text{for } t \geqslant 0 \text{ and } \mu \approx 0. \tag{21}$$

This easily shows that the condition in Lemma 4 is verified and hence that

$$T_1(s;\mu) = -\frac{1}{\lambda_1(\mu)} \ln s + \sum_{i=1}^{\infty} s^{pi} \int_0^{-\ln s} a_i(t;\mu)\, dt \tag{22}$$

for $s > 0$ small enough. In order to develop the above expression we take advantage of Proposition 10 in [12], which shows that $g_i(t) = e^{\alpha_1 t} Q_i(t)$ where Q_i is a polynomial of degree $\leqslant i-1$ in

$$\Omega(\alpha_1, t) := \begin{cases} \frac{e^{\alpha_1 t}-1}{\alpha_1} & \text{if } \alpha_1 \neq 0, \\ t & \text{if } \alpha_1 = 0 \end{cases}$$

with its coefficients polynomial in $\alpha_1, \ldots, \alpha_i$. Consequently from (20) it follows that

$$a_i(t) = \beta_1 e^{\alpha_1 t} P_i^1(\Omega) + \beta_2 e^{2\alpha_1 t} P_i^2(\Omega) + \cdots + \beta_n e^{n\alpha_1 t} P_i^n(\Omega),$$

where P_i^j is a polynomial of degree $i-j$ in Ω with its coefficients polynomial in $\alpha_1, \alpha_2, \ldots, \alpha_i$ for $j \leqslant i$ and $P_i^j \equiv 0$ for $j > i$ (here we use that when $j > i$ there is not any combination of j natural numbers verifying $m_1 + \ldots + m_j = i$). Note moreover that the change $\xi = \Omega(\alpha_1, t)$ yields to

$$\int_0^{-\ln s} e^{j\alpha_1 t} P_i^j\,(\Omega(\alpha_1, t))\, dt = \int_0^{\omega(s;\alpha_1)} (\alpha_1 \xi + 1)^{j-1} P_i^j(\xi)\, d\xi$$

and accordingly this proves that

$$\int_0^{-\ln s} a_i(t;\mu)\, dt = R_i\,(\omega(s;\alpha_1))\,, \tag{23}$$

where R_i is a polynomial of degree i with its coefficients polynomial in $\alpha_1, \ldots, \alpha_i$ and $\beta_1, \ldots, \beta_i$. Thus, if we define

$$\psi(s;\mu) := \sum_{i=3}^{\infty} s^{pi} \int_0^{-\ln s} a_i(t;\mu)\,dt, \tag{24}$$

then, on account of the expression of $T_1(s;\mu)$ in (22) and the relation in (23), we get

$$T_1(s;\mu) = \frac{-1}{\lambda_1(\mu)} \ln s + s^p R_1(\omega(s;\alpha_1)) + s^{2p} R_2(\omega(s;\alpha_1)) + \psi(s;\mu). \tag{25}$$

Next we shall see that $\psi \in \mathcal{B}$, and to this end we need the following:

Claim 1. *There exists a positive constant C_3 such that*

$$\left|\frac{\partial a_i(t;\mu)}{\partial \mu}\right| < C_3 \left(8Ce^{t/2}\right)^i \quad \textit{for } t \geqslant 0 \textit{ and } \mu \approx 0.$$

Since the proof of this claim is rather technical, for the sake of clarity in the exposition we defer it until we show the assertions concerning the time function. Some computations, using the above claim and (21), show that if $\mu \approx 0$ then

$$\left|s^{pi} \int_0^{-\ln s} a_i(t;\mu)\,dt\right| \leqslant C_1 (2Cs^p)^i \int_0^{-\ln s} e^{it/2} dt \leqslant 4C_1(2Cs^{p-1/2})^i, \tag{26}$$

$$\left|\frac{d}{d\mu}\left(s^{pi} \int_0^{-\ln s} a_i(t;\mu)\,dt\right)\right| \leqslant s^{pi} \int_0^{-\ln s} \left|\frac{\partial a_i(t;\mu)}{\partial \mu}\right| dt \leqslant 4C_3(8Cs^{p-1/2})^i \tag{27}$$

and

$$\begin{aligned}\left|\frac{d}{ds}\left(s^{pi} \int_0^{-\ln s} a_i(t;\mu)\,dt\right)\right| &= s^{pi-1}\left|pi \int_0^{-\ln s} a_i(t;\mu)\,dt - a_i(-\ln s;\mu)\right| \\ &\leqslant C_1 \frac{4pi+1}{s}\left(2Cs^{p-1/2}\right)^i.\end{aligned} \tag{28}$$

(To obtain these inequalities we assume that $0 < s < 1$.) Define $\psi(0;\mu) := 0$ for all μ. Note that from (26) we get that, for $\mu_0 \approx 0$,

$$\lim_{s \to 0} \left|\frac{\psi(s;\mu_0)}{s}\right| \leqslant 4C_1 \lim_{s \to 0} \frac{1}{s} \sum_{i=3}^{\infty} \left(2Cs^{p-1/2}\right)^i = 4C_1 \lim_{s \to 0} \frac{1}{s} \frac{\left(2Cs^{p-1/2}\right)^3}{1 - 2Cs^{p-1/2}} = 0$$

because $p \geqslant 1$. Therefore $\frac{\partial \psi(0;\mu_0)}{\partial s} = 0$. The inequality in (28) shows on the other hand that we can compute $\frac{\partial \psi}{\partial s}$ derivating (24) term by term and that $\frac{\partial \psi(s;\mu)}{\partial s} \longrightarrow 0$ as $(s,\mu) \longrightarrow (0,\mu_0)$. Accordingly $\frac{\partial \psi}{\partial s}$ is continuous at $(0,\mu_0)$. Exactly the same way but using (27) one can easily verify that $\frac{\partial \psi}{\partial \mu}$ is also continuous. This proves that $\psi \in \mathcal{B}$ as desired.

We are now in position to conclude the proof of the assertions concerning the time function. Suppose first that $r(0) > 1$ (i.e., $p/q > 1$). Then $p \geqslant 2$ and Lemma 24 shows that $s^p R_1(\omega(s;\alpha_1))$ and $s^{2p} R_2(\omega(s;\alpha_1))$ belong to $\mathcal{B}$. On account of (25) this proves (a). Assume finally that $r(0) = 1$. Then $p = 1$ and, again by Lemma 24, $s^2 R_2(\omega(s;\alpha_1)) \in \mathcal{B}$. This proves (b) because, due to $g_1(t;\mu) = e^{\alpha_1(\mu)t}$, it is easy to check that $sR_1(\omega(s;\alpha_1)) = \beta_1 s\omega(s;\alpha_1)$.

Finally we must show Claim 1, and to this end we use an intermediate step:

Claim 2. *For any $t \geqslant 0$ and $\mu \approx 0$, the function $u_0 \longmapsto \frac{\partial u(t,u_0;\mu)}{\partial \mu}$ can be written as a power series in u_0 with radius of convergence greater than $\frac{1}{4Ce^{t/2}}$ and there exists a positive constant C_2 such that*

$$\left| \frac{\partial u(t,u_0;\mu)}{\partial \mu} \right| < C_2 \quad \textit{if} \quad |u_0| < \frac{1}{8Ce^{t/2}}. \tag{29}$$

To see this note that, setting $p(u;\mu) := \sum_{i=1}^{n+1} i\alpha_i(\mu)u^{i-1}$ and $q(u;\mu) := \sum_{i=1}^{n+1} \alpha_i'(\mu)u^i$, then the function $t \longmapsto \frac{\partial u(t,u_0;\mu)}{\partial \mu}$ is the solution of the linear differential equation

$$x'(t) - p\,(u(t,u_0;\mu);\mu)\,x(t) = q\,(u(t,u_0;\mu);\mu)$$

with initial condition $x(0) = 0$. (Here we apply the theorem on differentiability of solutions with respect to parameters.) Consequently one can verify that

$$\begin{aligned} \frac{\partial u(t,u_0;\mu)}{\partial \mu} &= \exp\left(\int_0^t p\,(u(s,u_0))\,ds \right) \\ &\quad \times \left\{ \int_0^t q\,(u(s,u_0)) \exp\left(-\int_0^s p\left(u(\xi,u_0)\right)\,d\xi \right) ds \right\}. \end{aligned} \tag{30}$$

Notice that, since $u \longmapsto p(u;\mu)$ is polynomial, the series $p\left(u(\xi,u_0)\right) = \sum_{i=1}^{\infty} p_i(\xi)u_0^i$ is convergent for $|u_0| < \frac{1}{Ce^{\xi/2}}$. In addition, if we define $C_2' := \sup\{|p(u;\mu)| : |u| \leqslant C_0,\ \mu \approx 0\}$, then from (18) we have that $|p\left(u(\xi,u_0)\right)| < C_2'$ for $|u_0| < \frac{1}{2Ce^{\xi/2}}$. Thus, by applying Lemma 6 with $R = \frac{1}{2Ce^{\xi/2}}$, we can assert that $|p_i(\xi)| < C_2'(2Ce^{\xi/2})^i$.

Therefore

$$\left|\int_0^s p\left(u(\xi,u_0)\right)\,d\xi\right| \leqslant \sum_{i=1}^{\infty}\int_0^s |p_i(\xi)u_0^i|\,d\xi \leqslant C_2'\sum_{i=1}^{\infty}\int_0^s \left(2Ce^{\xi/2}|u_0|\right)^i\,d\xi$$
$$= 2C_2'\sum_{i=1}^{\infty}\frac{e^{is/2}-1}{i}\,(2C|u_0|)^i$$
$$\leqslant 4C_2'\sum_{i=1}^{\infty}\left(2Ce^{s/2}|u_0|\right)^i. \tag{31}$$

(Here we use Lemma 5 in the first inequality.) In particular, by applying Lemma 4, this shows that the series

$$\int_0^s p\left(u(\xi,u_0)\right)\,d\xi = \sum_{i=1}^{\infty}\left(\int_0^s p_i(\xi)\,d\xi\right)u_0^i$$

is convergent for $|u_0| < \frac{1}{2Ce^{s/2}}$. Consequently, since $x \longmapsto e^{-x}$ is an entire function, the series

$$\exp\left(-\int_0^s p\left(u(\xi,u_0)\right)\,d\xi\right) = \sum_{i=1}^{\infty}\bar{p}_i(s)u_0^i$$

is also convergent for $|u_0| < \frac{1}{2Ce^{s/2}}$. On the other hand, since $\sum_{i=1}^{\infty} r^i < 1$ in case that $0 < r < 1/2$, from (31) it is also clear that the above function is bounded by $e^{4C_2'}$ for $|u_0| < \frac{1}{4Ce^{s/2}}$. Notice moreover that $q(u(s,u_0)) = \sum_{i=1}^{\infty} q_i(s)u_0^i$ has radius of convergence greater than $\frac{1}{Ce^{s/2}}$ because $u \longmapsto q(u;\mu)$ is polynomial. We can conclude therefore that

$$q(u(s,u_0))\exp\left(-\int_0^s p\big(u(\xi,u_0)\big)\,d\xi\right) = \sum_{i=1}^{\infty}\bar{q}_i(s)u_0^i$$

is convergent for $|u_0| < \frac{1}{2Ce^{s/2}}$. Note also that if we define $C_2'' := \sup\{|q(u;\mu)| : |u| \leqslant C_0, \mu \approx 0\}$ then, due to (18), this function is bounded by $C_2''e^{4C_2'}$ for $|u_0| < \frac{1}{4Ce^{s/2}}$. Thus by applying Lemma 6 again, now with $R = \frac{1}{4Ce^{s/2}}$, it turns out that $|\bar{q}_i(s)| < C_2''e^{4C_2'}\left(4Ce^{s/2}\right)^i$. In addition,

$$\left|\int_0^t \sum_{i=1}^{\infty}\bar{q}_i(s)u_0^i\,ds\right| \leqslant \sum_{i=1}^{\infty}\int_0^t |\bar{q}_i(s)u_0^i|\,ds \leqslant C_2''e^{4C_2'}\sum_{i=1}^{\infty}\int_0^t \left(4Ce^{s/2}|u_0|\right)^i\,ds$$

$$= 2C_2''e^{4C_2'}\sum_{i=1}^{\infty}\frac{e^{it/2}-1}{i}\,(4C|u_0|)^i$$
$$\leqslant 4C_2''e^{4C_2'}\sum_{i=1}^{\infty}\left(4Ce^{t/2}|u_0|\right)^i. \tag{32}$$

(Here we use Lemma 5 in the first inequality.) According to Lemma 4 this shows that the series

$$\int_0^t q(u(s,u_0))\exp\left(-\int_0^s p\big(u(\xi,u_0)\big)\,d\xi\right)ds = \sum_{i=1}^{\infty}\left(\int_0^t \bar{q}_i(s)\,ds\right)u_0^i$$

is convergent for $|u_0| < \frac{1}{4Ce^{t/2}}$. It is clear then that the function between brackets in (30) can be written as a convergent series in $u_0 = 0$ for $|u_0| < \frac{1}{4Ce^{t/2}}$. Note in addition that, on account of (32), it is bounded by $4C_2''e^{4C_2'}$ for $|u_0| < \frac{1}{8Ce^{t/2}}$. On the other hand, from (31) taking $s = t$, it follows that

$$\exp\left(\int_0^t p(u(s,u_0))\,ds\right) < e^{4C_2'} \text{ for } |u_0| < \tfrac{1}{4Ce^{t/2}}$$

and, since $x \longmapsto e^x$ is entire, that this function can be written as a series in $u_0 = 0$ with radius of convergence greater than $\frac{1}{2Ce^{t/2}}$. In brief, we have shown that $\frac{\partial}{\partial\mu}u(t,u_0;\mu)$ is the product of two series with radius of convergence greater than $\frac{1}{4Ce^{t/2}}$ and that $|\frac{\partial}{\partial\mu}u(t,u_0;\mu)| < C_2$ for $|u_0| < \frac{1}{8Ce^{t/2}}$ with $C_2 := e^{4C_2'}\left(4C_2''e^{4C_2'}\right)$. This shows the validity of Claim 2.

We are now in position to prove Claim 1. To do so note first that if we define $f_1(u;\mu) := \frac{\partial f(u;\mu)}{\partial u}$ and $f_2(u;\mu) := \frac{\partial f(u;\mu)}{\partial \mu}$, then from (19) we obtain that

$$\sum_{i=1}^{\infty}\frac{\partial a_i(t;\mu)}{\partial\mu}u_0^i = f_1\left(u(t,u_0;\mu)\right)\frac{\partial u(t,u_0;\mu)}{\partial\mu} + f_2\left(u(t,u_0;\mu)\right) + \frac{\lambda_1'(\mu)}{\lambda_1(\mu)^2}. \tag{33}$$

Recall in addition that $u(t,u_0;\mu)$ and $\frac{\partial u(t,u_0;\mu)}{\partial\mu}$ can be written as a series in $u_0 = 0$ with radius of convergence greater than $\frac{1}{4Ce^{t/2}}$. (This follows from (17) and Claim 2 respectively.) Consequently the series in (33) has also radius of convergence greater than $\frac{1}{4Ce^{t/2}}$ because each f_i is polynomial in u. Now if we define

$$C_3 := \sup\left\{|f_1(u;\mu)|\,C_2 + \left|f_2(u;\mu) + \frac{\lambda_1'(\mu)}{\lambda_1(\mu)^2}\right| : |u| < C_0,\, \mu \approx 0\right\}$$

then, taking the upper bounds in (18) and (29) into account and applying Lemma 6 with $R = \frac{1}{8Ce^{t/2}}$, the claim follows. $\square$

Proof of Theorem 16. Since the transversal sections Σ_1 and Σ_2 are $\mathcal{C}^k$, with $k \geqslant 1$, it is well known that P_2 and T_2 are $\mathcal{C}^k$ functions at $s = 0$ (see Fig. 2). It is also clear that if

$$P_2(s;\mu) = a_0(\mu) + a_1(\mu)s + \mathrm{o}(s) \text{ and } T_2(s;\mu) = b_0(\mu) + b_1(\mu)s + \mathrm{o}(s)$$

are the respective Taylor's developments at $s = 0$, then it holds $a_0(0) = 0$, $a_1(0) > 0$ and $b_0(0) > 0$. The (generic) assumption that we make is that $a_0'(0) \neq 0$.

Let us consider first the case $r(0) > 1$ and define

$$\mathcal{F}(s;\mu) := P_1(s;\mu) - P_2(s;\mu) \text{ and } \mathcal{T}(s;\mu) := T_1(s;\mu) + T_2(s;\mu).$$

Thus, for $\mu_0 \approx 0$, the periodic orbits of X_{μ_0} near the saddle connection Γ are precisely the positive roots of $\mathcal{F}(s;\mu_0) = 0$ near $s = 0$. In addition, if $\mathcal{F}(s_0;\mu_0) = 0$ then the period of the corresponding periodic orbit is given by $\mathcal{T}(s_0;\mu_0)$. The idea will be to track down the periodic orbits by applying the Implicit Function Theorem to $\mathcal{F}$. To this end note that, by (a) in Proposition 23, P_1 is the restriction to $s > 0$ of a function, say $\widetilde{P}_1$, which is $\mathcal{C}^1$ on neighbourhood of $(s;\mu) = (0;0)$ and verifies

$$\widetilde{P}_1(0;0) = \frac{\partial \widetilde{P}_1(0;0)}{\partial s} = \frac{\partial \widetilde{P}_1(0;0)}{\partial \mu} = 0.$$

(Here we took Definition 21 into account.) To be precise, we shall apply the Implicit Function Theorem using this "extended" function instead of the original one. However, to avoid introducing new notation, let us maintain the name of $\mathcal{F}$. Thus, since

$$\mathcal{F}(0;0) = 0, \ \mathcal{F}_s(0;0) = -a_1(0) \neq 0 \text{ and } \mathcal{F}_\mu(0;0) = a_0'(0) \neq 0,$$

by the Implicit Function Theorem, there exists a $\mathcal{C}^1$ function $x_l(\mu)$, defined for $\mu \in (-\varepsilon, \varepsilon)$, verifying that $\mathcal{F}(x_l(\mu);\mu) \equiv 0$. We can assert in addition that

$$x_l(\mu) = c\mu + \mathrm{o}(\mu) \text{ with } c := -\frac{a_0'(0)}{a_1(0)}.$$

Consequently, if $c > 0$ (respectively $c < 0$) then the solution of X_μ passing through $\Phi(x_l(\mu), 1)$ is a limit cycle for $\mu \in (0, \varepsilon)$ (respectively $\mu \in (-\varepsilon, 0)$). It is also clear that, in each case, the period of this limit cycle is given by $T(\mu) := \mathcal{T}(x_l(\mu);\mu)$.

Finally, by applying (a) in Proposition 23, some computations show that

$$T(\mu) = \frac{-1}{\lambda_1(0)} \ln|\mu| + O(1).$$

In order to prove the assertion when $r(0) < 1$ we take the family of vector fields $\widetilde{X}_\mu := -X_\mu$. Following the obvious notation, it is clear that $\widetilde{\lambda}_1(\mu) = -\lambda_2(\mu)$ and $\widetilde{\lambda}_2(\mu) = -\lambda_1(\mu)$. Consequently $\widetilde{r}(\mu) = 1/r(\mu)$, and so the assertion follows by applying the result in case of ratio of hyperbolicity greater than one. $\square$

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