RATIONAL PERIODIC SEQUENCES FOR THE LYNESS RECURRENCE

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Abstract. Consider the celebrated Lyness recurrence $x_{n+2} = (a + x_{n+1})/x_n$ with $a \in \mathbb{Q}$. First we prove that there exist initial conditions and values of a for which it generates periodic sequences of rational numbers with prime periods $1, 2, 3, 5, 6, 7, 8, 9, 10$ or 12 and that these are the only periods that rational sequences $\{x_n\}_n$ can have. It is known that if we restrict our attention to positive rational values of a and positive rational initial conditions the only possible periods are $1, 5$ and 9. Moreover 1-periodic and 5-periodic sequences are easily obtained. We prove that for infinitely many positive values of a, positive 9-period rational sequences occur. This last result is our main contribution and answers an open question left in previous works of Bastien & Rogalski and Zeeman. We also prove that the level sets of the invariant associated to the Lyness map is a two-parameter family of elliptic curves that is a universal family of the elliptic curves with a point of order $n, n \geq 5$, including n infinity. This fact implies that the Lyness map is a universal normal form for most birational maps on elliptic curves.

1. **Introduction and main results.** The dynamics of the Lyness recurrence

$$x_{n+2} = \frac{a + x_{n+1}}{x_n}, \tag{1}$$

specially when $a > 0$, has focused the attention of many researchers in the last years and it is now completely understood in its main features after the independent research of Bastien & Rogalski [3] and Zeeman [21], and the later work of Beukers & Cushman [5]. See also [2, 11]. In particular all possible periods of the recurrences

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generated by (1) are known and for any $a \notin \{0,1\}$ infinitely many different prime periods appear.

However there are still some open problems concerning the dynamics of rational points. With the computer experiments in mind, and following [3, 21], it is interesting to know the existence of rational periodic sequences. The Lyness map

$$F_a(x,y) = (y,(a+y)/x), \tag{2}$$

associated to (1), leaves invariant the elliptic curves[a]

$$C_{a,h} := \{(x+1)(y+1)(x+y+a) - hxy = 0\}$$

and the map action can be described in terms of the group law action of them. In consequence several tools for studying the rational periodic orbits on them are available. In particular, from Mazur's Torsion Theorem (see for example [19]), we know that, under the above hypotheses, the rational periodic points can only have (prime) periods $1,2,\ldots,9,10$ and 12. Our first result proves that almost all these periods appear for the Lyness recurrence for suitable $a \in \mathbb{Q}^+$ and rational initial conditions.

Theorem 1. *For any $n \in \{1,2,3,5,6,7,8,9,10,12\}$ there are $a \in \mathbb{Q}^+ \cup \{0\}$ and rational initial conditions x_0, x_1 such that the sequence generated by (1) is n-periodic. Moreover these values of n are the only possible prime periods for rational initial conditions and $a \in \mathbb{Q}$.*

Notice that the value $n = 4$ is the only one in Mazur's list that is not in the list given in the theorem. Following [21] it is possible to interpret that this period corresponds to the case $a = \infty$, see Remark 7 in Section 3.

Concerning the rational periodic points in $\mathbb{Q}^+ \times \mathbb{Q}^+$ for the Lyness map with $a > 0$, it is proved in [3] that it only can have periods 1, 5 and 9.

Taking $a = n^2 - n$ and $x_0 = x_1 = n \in \mathbb{N}$ we obtain trivially 1-periodic integer sequences. The existence of positive rational periodic points of period 5 is well known and simple: they only exist when $a = 1$ and in this case all rational initial conditions give rise to them because the recurrence (1) is globally 5-periodic. On the other hand, as far as we know, the case of period 9 has resisted all previous analysis. In particular, the Conjecture[b] 2 of Zeeman, [21] says that there are no such points and their existence is left in Problem[c] 1 bis of [3] as an open question.

We prove that there are some values of a for which the Lyness recurrences (1) have positive rational periodic sequences of period 9 and, even more, that this happens for infinitely many values of a, see more details in Theorem 2.

It is known, see again [3, 21], that the periodic points of period 9 of F_a are on the elliptic curve

$$a(x+1)(y+1)(x+y+a) - (a-1)(a^2-a+1)xy = 0, \tag{3}$$

and they have positive coordinates only when $a > a_* \simeq 5.41147413$ where a_* is the biggest root of $a^3 - 6\,a^2 + 3\,a + 1 = 0$, see also Subsection 2.1.

By using MAGMA ([6])[d], and after several trials, we have found some positive rational points on the above curve proving that the Zeeman's Conjecture 2 has a

[a] For some concrete values of h and a the curve is not elliptic. We study these values separately.

[b] Conjecture 1 of Zeeman was about the monotonicity of certain rotation number function associated to the invariant ovals of the Lyness map and was proved in [5].

[c] Problem 1 of [3] is about rational 5-periodic points and it is recalled and solved in Section 5.

[d] It is also possible to use SAGE ([16]).

negative answer. The simplest one that we have obtained is $(a; x, y) = (7; 3/2, 5/7)$. Notice that the sequence (1), taking $a = 7$ and the initial condition $x_0 = 3/2$, $x_1 = 5/7$, gives

$$\frac{3}{2}, \frac{5}{7}, \frac{36}{7}, 17, \frac{14}{3}, \frac{35}{51}, \frac{28}{17}, \frac{63}{5}, \frac{119}{10}, \frac{3}{2}, \frac{5}{7}, \ldots$$

Other positive rational points that we have found are

$$\left(11; \frac{29}{82}, \frac{19}{22}\right), \quad \left(13; \frac{1584676}{61133}, \frac{335937}{856427}\right), \quad \left(19; \frac{4259697}{16150}, \frac{5178617}{168283}\right)$$

and many others with much bigger entries. Our main result proves that there are infinitely many positive rational values of the parameter a giving rise to 9–periodic positive rational orbits.

Theorem 2. *There are infinitely many values $a \in \mathbb{Q}^+$ for which there exist initial conditions $x_0(a), x_1(a) \in \mathbb{Q}^+$ such that the sequence given by the Lyness recurrence (1) is 9–periodic. Furthermore, the closure of these values of a contains the real interval $[a_1, +\infty)$, where $a_1 \simeq 5.41147624$ is the biggest root of $a^3 - \frac{2019}{529}a^2 - \frac{777}{92}a - 1$.*

Notice that there is a small gap between the values a_* and a_1 where we do not know if there are or not rational values of a for which the Lyness recurrence has positive rational periodic orbits of period 9. As we will see in the proof of the theorem the gap is provoked by our approach and it seems to us that it is not intrinsic to the problem, see the comments in Section 4, after the proof of the theorem.

From the classical results of Mordell, see for example [18, Ch.VIII], it is well known that the set of rational points on an elliptic curve E over $\mathbb{Q}$, together with the point at infinity, form an additive group $E(\mathbb{Q})$ and

$$E(\mathbb{Q}) \cong \mathbb{Z}^r \times \Phi,$$

where $r \in \mathbb{N}$ is called the rank of $E(\mathbb{Q})$ and Φ is the torsion of the group. Notice that r is a measure of the amount of rational points that the curve contains. The torsion part Φ is well understood from the results of Mazur already quoted, and it contains at most 16 points. For the elliptic curve (3) it is easy to check that $\Phi = \mathbb{Z}/9\mathbb{Z}$. Fixed any Φ_0, among all the allowed possibilities, it is not known if the rank of the elliptic curves having $E(\mathbb{Q}) \cong \mathbb{Z}^r \times \Phi_0$, for some $r \in \mathbb{N}$, has an upper bound, but it is believed that it has not (see the related Conjecture VIII.10.1 in [18]). As far as we know, nowadays when $\Phi_0 = \{0\}$ the highest known rank is greater or equal than 28 while when $\Phi_0 = \mathbb{Z}/9\mathbb{Z}$ the highest known rank is 4, see [9]. We have found values of a for which the algebraic curve (3) has ranks $0, 1, 2, 3$ or 4. For instance 0 appears for $a = 6$ and the value 4 happens for $a = 408/23$. A key point to obtain these results is the following theorem, which extends the results of [4], given for the cases of torsion points with order 5 or 6, and proves the universality of the level curves invariant for the Lyness map.

Theorem 3. *(Lyness normal form) The family of elliptic curves $C_{a,h}$,*

$$(x+1)(y+1)(x+y+a) - hxy = 0,$$

together with the points $\mathcal{O} = [1 : -1 : 0]$ and $Q = [1 : 0 : 0]$, is the universal family of elliptic curves with a point of order n, $n \geq 5$ (including $n = \infty$). This means that for any elliptic curve E over any field $\mathbb{K}$ (not of characteristic 2 or 3) with a point $R \in E(\mathbb{K})$ of order n, there exist unique values $a_{(E,R)}, h_{(E,R)} \in \mathbb{K}$ and a *unique isomorphism between E and $C_{a_{(E,R)},h_{(E,R)}}$ sending the zero point of E to $\mathcal{O}$ and the point R to Q. Moreover, given a finite n, the relation between a and h can be easily obtained from the computations made in Subsection 2.1.*

From a dynamical viewpoint the above result implies that the Lyness Map is an affine model for most birational maps on elliptic curves. This result is similar to the one described by Jogia, Roberts and Vivaldi in [13, Theorem 3], where they use the Weierstrass normal form. Moreover, notice that in our result the sum operation takes the extremely easy form $\tilde{F}([x:y:z]) = [xy : az^2 + yz : xz]$ (see Section 2.1).

Once for some value $a \in \mathbb{Q}^+$ a periodic orbit in $\mathbb{Q}^+ \times \mathbb{Q}^+$ of period 9 for F_a is obtained, it is not difficult to obtain infinitely many different 9-periodic orbits by using the group law on the curve. For instance, for a given a, we know that if a point $P = (x, y)$ is on the oval of (3) then $(2k+1)P, k \in \mathbb{Z}$ is also on it. In particular for $a = 7$ if we write $3P = (z, w)$ we get

$$z = \frac{2(6727x+913y+913)(90272x-415y-2905)}{(5583410x^2+858451819xy-28403347y^2-187465799x-227226776y-198823429)}$$

and w can be obtained for instance by plugging $x = z$ in (3) and choosing a suitable solution y. Taking $P = (3/2, 5/7)$ we find

$$3P = \left(\frac{260143588}{23256135}, \frac{337001111}{246029869}\right),$$

which also gives rise to a 9-periodic orbit in $\mathbb{Q}^+ \times \mathbb{Q}^+$. Taking now $3P$ as starting point for the procedure we obtain a new initial condition for a 9-periodic orbit:

$$9P = \left(\frac{3147471926986755321149021}{2260910710326066258309 25}, \frac{8915221428528882132657 18}{8517462828850687723197 5}\right),$$

and so on. In general we have the following result, see Subsection 2.1.

Proposition 4. *If $a \in \mathbb{Q}^+$ is a value for which there exist an initial condition $(x_0, x_1) \in \mathbb{Q}^+ \times \mathbb{Q}^+$ such that the sequence (1) is 9–periodic, then there exist infinite many different rational initial conditions giving rise to 9–periodic sequences. Moreover the points corresponding to these initial conditions fill densely the elliptic curve (3).*

Observe that such a point $(x_0, x_1) \in \mathbb{Q}^+ \times \mathbb{Q}^+$ always gives rise to a point on the elliptic curve given by (3) which is not a torsion point, because all the 9-torsion points are not in $\mathbb{Q}^+ \times \mathbb{Q}^+$, see again Subsection 2.1.

The case of 5-periodic points is studied in Section 5, proving that for $a = 1$ there are some elliptic curves $C_{1,h}, h \in \mathbb{Q}$, with rank 0 and, as a consequence, ovals without points with rational coordinates.

Finally, last section is devoted to give rational values of a for which the Lyness map F_a has as many periods as possible, of course with rational initial conditions.

The structure of this paper is the following. In Subsection 2.1 we recall some known results which describe the action of the Lyness map in terms of a linear translation over elliptic curves. Subsection 2.2 is devoted to prove Theorem 3 about the Lyness normal form. In Section 3 we prove Theorem 1, while the proof our main result, Theorem 2, and all our outcomes on rational 9-periodic points of (2) are presented in Section 4. Section 5 studies the number of rational points on the elliptic curves invariant for F_1 and the last section deals with the problem of finding a concrete F_a with as many periods as possible.

2. **Preliminary results.**

2.1. **Lyness recurrence from group law's action viewpoint.**
The results of this subsection are well known, we refer the reader to references [3, 13, 21] for instance to get more details.

As mentioned above, the phase space of the discrete dynamical system defined by the map (2) is foliated by the family curves

$$C_{a,h} := \{(x+1)(y+1)(x+y+a) - hxy = 0\}.$$

This family is formed by elliptic curves except for a few values of h. When $h = 0$ it is a product of straight lines; when $h = a-1$, $a \neq 1$, it is the formed by a straight line and an hyperbola; and when

$$h = h_c^{\pm} := \frac{2\,a^2 + 10\,a - 1 \pm (4\,a+1)\sqrt{4\,a+1}}{2\,a}, \tag{4}$$

with $h_c^{\pm} \neq a-1, 0$, it is a rational cubic, having an isolated real singularity. In fact the values $h_c^{\pm}$ correspond to the level sets containing the fixed points of F_a, $((1 \pm \sqrt{4a+1})/2, (1 \pm \sqrt{4a+1})/2)$.

The first quadrant $Q_1 = \{(x,y), x > 0, y > 0\}$ is invariant under the action of F_a and it is foliated by ovals with energy $h > h_c^{+}$.

In summary, on most energy levels, F_a is a birational map on an elliptic curve and therefore it can be expressed as a linear action in terms of the group law of the curve [13, Theorem 3]. Indeed, taking homogeneous coordinates on the projective plane $P\mathbb{R}^2$ the curves $C_{a,h}$ have the form

$$\tilde{C}_{a,h} := \{(x+z)(y+z)(x+y+az) - hxyz = 0\},$$

and F can be seen as the map $\tilde{F}([x:y:z]) = [xy : az^2 + yz : xz]$, except for the points $[x:0:0]$, $[0:y:0]$ and $[0:-a:1]$. Taking the point $\mathcal{O} = [1:-1:0]$ as the neutral element, each elliptic curve $\tilde{C}_{a,h}$ is an abelian group with respect the sum defined by the usual *secant–tangent chord process* (i.e. if a line intersects the curve in three points P,Q, and R then $P+Q+R = \mathcal{O}$). According to these operation the Lyness map can be seen as the linear action

$$\tilde{F} : P \longrightarrow P + Q, \tag{5}$$

where $Q = [1:0:0]$.

When $a(a-1) \neq 0$, using the group operation on each elliptic curve, it is not difficult to get that $2Q = [-1:0:1]$, $3Q = [0:-a:1]$,

$$4Q = \left[-a : \frac{ah-a+1}{a-1} : 1\right],$$
$$5Q = \left[\frac{ah-a+1}{a-1} : \frac{-a^2-ah+2a-1}{a(a-1)} : 1\right],$$
$$6Q = \left[\frac{-a^2-ah+2a-1}{a(a-1)} : \frac{a^3-2a^2-ah+2a-1}{a(ah-a+1)} : 1\right],$$
$$7Q = \left[\frac{-a^2-ah+2a-1}{a(a-1)} : \frac{-a^4h+a^3h+a^3+a^2h-3a^2-ah+3a-1}{a^3h-a^3+a^2h^2-3a^2h+3a^2+2ah-3a+1} : 1\right],$$

$-Q=[0:1:0]$, $-2Q=[0:-1:1]$, $-3Q=[-a:0:1]$,

$$-4Q=\left[\frac{ah-a+1}{a-1}:-a:1\right],$$
$$-5Q=\left[\frac{-a^2-ah+2a-1}{a(a-1)}:\frac{ah-a+1}{a-1}:1\right],$$

see also [3].

In terms of the recurrence, the linear action (5) can be seen as follows: taking the initial conditions $P:=[x_0:x_1:1]$ then $[x_{n+1}:x_{n+2}:1]=P+(n+1)Q$, where $+$ is the group operation. Notice also that, from this point of view, the condition of existence of rational periodic orbits is equivalent to the condition that Q is in the torsion of the group given by the rational points of $\tilde{C}_{a,h}$, which, as we have already commented, is described by Mazur's Theorem.

Hence, from the above expressions of kQ we can obtain the values of h corresponding to a given period. For instance, for period 9 we impose that $4Q=-5Q$, or equivalently $9Q=\mathcal{O}$, which gives $-a=(-a^2-ah+2a-1)/(a(a-1))$. From this equality we have that

$$h=\frac{(a-1)(a^2-a+1)}{a},$$

which corresponds to the elliptic curve (3). For these values of h the points corresponding to the torsion subgroup are:

$$\begin{aligned}Q=&[1:0:0],\,[-1:0:1],\,[0:-a:1],\,[-a:a(a-1):1],\\ &[a(a-1):-a:1],\,[-a:0:1],\,[0:-1:1],\,[0:1:0],\,\mathcal{O}=[1:-1:0].\end{aligned}$$

It is also important from a dynamical point of view the following well know property of the secant–tangent chord process defined on any real non-singular elliptic curve E. Let P be a point of E such that kP, for $k\in\mathbb{Z}$, is never the neutral element $\mathcal{O}$. Recall that when the elliptic curve is defined over $\mathbb{Q}$ there are at most sixteen points P with rational entries of finite order, due the Mazur classification of the torsion subgroup of $E(\mathbb{Q})$. Then the adherence of the set $\{kP\}_{k\in\mathbb{Z}}$ is:

- either all the curve E, when P belongs to the connected component of E which does not contains the neutral element $\mathcal{O}$; or
- the connected component containing $\mathcal{O}$ when P belongs to it.

This result is due to the fact that there is a continuous isomorphism between E with this operation and the group $\{e^{it}\,:\,t\in[0,2\pi)\}\times\{1,-1\}$, with the operation $(u,v)\cdot(z,w)=(uz,vw)$, see for instance Corollary 2.3.1 of [17, Ch. V.2]. Clearly, by using this construction, Proposition 4 follows.

Notice that when an elliptic curve is given by $C_{a,h}$ the bounded component never contains $\mathcal{O}$.

2.2. **A new normal form for elliptic curves.**

Proof of Theorem 3. It is known that any elliptic curve having a point R that is not a 2 or a 3 torsion point can be written in the so called Tate normal form

$$Y^2+(1-c)XY-bY=X^3-bX^2,$$

where R is sent to $(0,0)$, see [12, Ch. 4, Sec. 4]. Our proof will follow by showing that the curves $C_{a,h}$ can be transformed into the ones of the Tate normal form. In projective coordinates these curves write as

$$Y^2Z + (1-c)XYZ - bYZ^2 = X^3 - bX^2Z$$

and the curves $C_{a,h}$ as

$$(x+z)(y+z)(x+y+az) - hxyz = 0.$$

With the change of variables

$$X = bz, \quad Y = bc(y+z), \quad Z = c(x+y) + (c+1)z$$

and the relations

$$h = -\frac{b}{c^2}, \quad a = \frac{c^2+c-b}{c^2},$$

both families of curves are equivalent and the theorem follows. Observe that the case $c = 0$ corresponds to a curve with a 4–torsion point. As we will see in the proof of Theorem 2 one advantage of this normal form is that it is symmetric with respect to x and y. ∎

From Theorem 3 we have that all the known results on elliptic curves with a point of order greater than 4 can be applied to the corresponding Lyness curves. In particular we find inside $C_{a,h}$, the curves with high rank and prescribed torsion given in [9] or we can use the list of "Elliptic Curve Data" for curves in Cremona form ([8]), taking advantage of the MAGMA or SAGE softwares that allow to identify a given elliptic curve in it.

3. **Possible periods for rational points.**

3.1. **The non-elliptic curves case.**

As was explained in Subsection 2.1 the curves $C_{a,h}$ are elliptic for all values of a and h except for $h \in H := \{0, a-1, h_c^{\pm}\}$, with $h_c^{\pm}$ given in (4). On the curves corresponding to these values there could be, for the rational periodic orbits, some periods that are not in the list given by Mazur's theorem. In this section we prove that no new period appears.

Lemma 5. *The periods of the rational periodic orbits of F_a lying on the curves $C_{a,h}$ for $h \in H$ are $1, 2, 3, 6, 8$ and 12.*

Proof. It is well known that the case $a = 0$ is globally 6-periodic. So from now on we consider that $a \neq 0$. We start the study of the rational cubic curves $C_{a,h_c^{\pm}}$, where $h_c^{\pm}$ are given in (4) and moreover $h_c^{\pm} \notin \{0, a-1\}$. By setting $b = \pm\sqrt{4a+1}$, we have

$$\frac{(b+3)^3}{4(b+1)} = \begin{cases} h_c^+ & \text{when} \quad b \geq 0, \quad b \neq 1, \\ h_c^- & \text{when} \quad b \leq 0, \quad b \notin \{-1,-2,-3\}. \end{cases}$$

Note that $a = 0$ implies that $b = \pm 1$; $h_c^{\pm} = a - 1 \neq 0$ implies that $b = -2$; and $h_c^{\pm} = 0$ implies that $b = -3$. Observe also that since a and $h_c^{\pm}$ are in $\mathbb{Q}$ then $b \in \mathbb{Q}$.

By using again a computer algebra software we obtain the following joint parametrization of both curves $C_{a,h_c^{\pm}}$:

$$t \to (x(t), y(t)) = \left(\tfrac{(3t+tb-2)(2tb+4t-b-1)}{2(b+1)(t-1)}, -\tfrac{(3t+tb-b-1)(2tb+4t-3-b)}{2t(b+1)} \right).$$

Moreover

$$t = \frac{x(t) - (b+1)/2}{x(t) + y(t) - (b+1)}. \tag{6}$$

On each curve $C_{a,h_c^\pm}$ the Lyness map F_a can be seen as

$$F_a|_{C_{a,h_c^\pm}} : \big(x(t),y(t)\big) \longrightarrow \left(y(t), \frac{(b^2-1)/4+y(t)}{x(t)}\right) = \big(x(f(t)),y(f(t))\big),$$

where $f(t)$ has to be determined. By using (6) we obtain that f is the linear fractional transformation

$$f(t) = \frac{t-(b+1)/(2b+4)}{t},$$

where recall that $b \in \mathbb{Q}$. Therefore the dynamics of F_a on each of the curves $C_{a,h_c^\pm}$ is completely determined by the dynamics of the maps $f(t)$.

It is a well-known fact that if a linear fractional map, $g(t) = (At+B)/(t+D)$, has a periodic orbit of prime period p, $p>1$, then it is globally p-periodic. Moreover this happens if and only if either:

- $\Delta := (D-A)^2+4B > 0$ and $A=-D$ and in this case g is 2-periodic; or
- $\Delta < 0$ and $\xi := (A+D-\sqrt{|\Delta|}\,i)/(A+D+\sqrt{|\Delta|}\,i)$ is a primitive p-root of the unity and in this case g is p-periodic.

Hence, apart of the fixed points, the maps $f(t)$ can have p-periodic solutions, $p>1$, if and only if

$$\Delta = -\frac{b}{b+2} < 0 \text{ and } \xi = \frac{1}{b+1} - \frac{b+2}{b+1}\sqrt{\frac{b}{b+2}}\,i \text{ is a primitive } p\text{-root of the unity.}$$

The condition that ξ is a primitive p-root of the unity, implies that

$$\cos\left(2\pi\frac{q}{p}\right) = \frac{1}{b+1} \in \mathbb{Q}, \text{ for some } q \in \mathbb{Z}.$$

It is also a well-known fact that the only rational values of $\cos(x)$, where x is a rational multiple of π, are 0,$\pm 1/2$,± 1 (see [14, Theorem 6.16] or [20]). This fact implies that $b \in \{-3,-2,0,1\}$. The only allowed valued is $b=0$, which implies that $\Delta = 0$, and so the corresponding map f is not periodic. Hence, only the fixed points of F_a appear on this family of curves.

Concerning the case $h=0$, observe that $C_{a,0} = \{(x+1)(y+1)(a+x+y)=0\}$. When $a=1$ there are no periodic orbits on this level set. When $a \neq 1$, the three straight lines forming this set are mapped one into the other in cyclical order by F_a, and so they are invariant under F_a^3. The cyclical order determined by F_a is:

$$\{x+1=0\} \to \{a+x+y=0\} \to \{y+1=0\} \to \{x+1=0\}.$$

The restriction of F_a^3 on $\{x+1=0\}$ is given by

$$F_a^3(-1,y) = \left(-1, \frac{1-a}{y+a}\right).$$

Hence the dynamics of F_a^3 is determined by the dynamics of the linear fractional map

$$f(y) = \frac{1-a}{y+a}.$$

Observe that the fixed point $y=-1$ corresponds to the continua of three periodic points of F_a given in Table 1. By using again the characterization of the periodicity of the linear fractional maps we obtain that f is periodic only when $a=0$.

It remains to study the case $h = a-1, a \neq 1$. In this situation $C_{a,a-1} = \{(x+y+1)(a+x+y+xy)=0\}$ and F_a sends the straight line to the hyperbola and vice versa. So both level sets are invariant under F_a^2. The restriction of F_a^2 on $\{x+y+1=0\}$ is given by

$$F_a^2(x,-1-x)=\left(\frac{-x+a-1}{x},\frac{1-a}{x}\right).$$

Hence the dynamics of F_a^2 is determined by the linear fractional map

$$f(x)=\frac{-x+a-1}{x},$$

which has fixed points only when $a\geq 3/4$. They give rise to 2-periodic points of F_a when $a>3/4$ and to a fixed point when $a=3/4$. Arguing as in the case $h=h_c^{\pm}$, the map f is p-periodic, $p\geq 2$, only when

$$\Delta=4a-3<0 \text{ and } \xi=\frac{1-2a-i\sqrt{3-4a}}{2a-2} \text{ is a primitive } p-\text{root of the unity.}$$

This happens if and only if $a<3/4$ and $(1-2a)/(2a-2)\in\{0,\pm 1/2,\pm 1\}$, or equivalently when $a\in\{1/2,2/3\}$ (recall that the case $a=0$ is already considered). The case $a=1/2$ gives a 4-periodic map f and $a=2/3$ a 6-periodic map. These cases correspond, respectively, to the existence of continua of 8 and 12 periodic points for F_a on $C_{a,a-1}$. ∎

3.2. **Proof of Theorem 1.**

Proof of Theorem 1. From Lemma 5 we know that when $h\in H$ the possible periods on $C_{a,h}$ are in the list given in the statement. For those points on the elliptic curves $C_{a,h}$ for all values of a and $h\notin H$ we can apply Mazur's theorem and we obtain that the only possible periods are the ones of the statement together with the period 4. The points of (prime) period 4 can be discarded by observing that in Subsection 2.1 we prove that $4Q\neq\mathcal{O}$. It is also possible to perform a direct study with resultants of the system $F_a^4(x,y)=(x,y)$. From this study we get that that its only solutions are the ones corresponding to fix or 2-periodic points.

These results together with the ones presented on Table 1 prove the theorem. ∎

Period	a	x_0	x_1	Comments
1	u^2-u	u	u	$u\in\mathbb{Q}\setminus\{0\}$
2	u^2+u+1	u	$-u-1$	$u\in\mathbb{Q}\setminus\{-1,0\}$
3	a	-1	-1	$a\in\mathbb{Q}\setminus\{1\}$
5	1	x_0	y_0	Almost for all x_0, y_0 in $\mathbb{Q}$
6	0	x_0	y_0	Almost for all x_0, y_0 in $\mathbb{Q}$
7	$\frac{u^2-1}{2u-1}$	$\frac{u^2-1}{u^2-u+1}$	$-x_0$	Almost for all u in $\mathbb{Q}$
8	$\frac{u^2-1}{u^2+2u-1}$	$\frac{u^2-1}{u^2+1}$	$-x_0$	Almost for all u in $\mathbb{Q}$
9	7	3/2	5/7	See also Theorem 2
10	3/2	-2	3/5	There are infinitely many
12	12/13	-4/9	-10/13	There are infinitely many

Table 1. Examples of rational periodic sequences for recurrence (1). See also Remark 6.

Next remark collects some comments on the results presented in Table 1.

Remark 6. *(i) The continua of sequences of periods 1 and 2 are not the most general ones, we have chosen simple one-parameter families.*

(ii) It is already known that there exist one-parameter families of elliptic curves having points of 7 (or 8) torsion and rank 1. By using Theorem 3 we know that they should appear also in the Lyness normal form. Nevertheless, we have got them by a direct study, only searching points satisfying $x+y=0$.

(iii) The values corresponding to periods 10 and 12 are obtained with MAGMA. By using similar arguments to the ones used to study the case of period 9 (see the proof of Theorem 2) we can prove the existence of rational periodic points, with these periods, for infinitely many rational values of a. *In fact there exist infinitely many points satisfying that* $x+y=0$, *parameterized, in each case, by an elliptic curve of rank 1 with Cremona's number 592a1 and 148a1, respectively. It is not known the existence of a continua of them.*

(iv) Note that we have obtained infinite values of a *for each period 7,8, 10 and 12 only asking for solutions satisfying that* $x+y=0$. *It seems that in all these cases the family of solutions that we have obtained is either known (some can be found in [15]) or they are parameterized by an elliptic curve with the same rank than the ones already known. Anyway we consider interesting that we have found them in such a uniform way. Note also that in the case of period 9 we can also get infinitely many cases under this condition, with base curve having again rank 1 (Cremona's number 92b1). As we will see in the proof of Theorem 2 (see also Remark 9), in this case it is possible to find a base curve with rank 3 if we impose* $x+y=23/4$.

Remark 7. *By introducing* $y_n := x_n/\sqrt{a}, n\in\mathbb{N}$, *the Lyness recurrence* (1) *writes as* $y_{n+2} = \left(1+y_{n+1}/\sqrt{a}\right)/y_n$. *When* a *tends to infinity we obtain the recurrence* $y_{n+2}=1/y_n$, *which is globally 4-periodic.*

4. **Proof of Theorem 2.**

In the proof of Theorem 2 we will use the following Lemma, see for example [1].

Lemma 8. *The curve* $K^2 = A^4 + w_2A^2 + w_1A + w_0$ *is isomorphic to the elliptic curve*

$$Y^2 = X^3 - \left(\frac{w_2^2}{48} + \frac{w_0}{4}\right)X + \frac{w_1^2}{64} + \frac{w_2^3}{864} - \frac{w_0w_2}{24},$$

where the change of variables is given by

$$X = \frac{1}{2}\left(A^2 + K + \frac{w_2}{6}\right), \quad Y = \frac{A}{2}\left(A^2 + K + \frac{w_2}{2}\right) + \frac{w_1}{8}.$$

Note that if in the above lemma w_0, w_1 and w_2 are rational numbers, then a point (X,Y) on the elliptic curve, with $X \neq -w_2/6$, is in $\mathbb{Q}^2$ if and only if the corresponding point on the quartic, (A,K) is also in $\mathbb{Q}^2$.

Proof of Theorem 2. As we have already explained in the introduction, following [3, 21] we already know that all the real *positive* initial conditions that give rise to 9-periodic recurrences (1) correspond to the points of the positive oval of the elliptic curves (3),

$$S_a := \{(a;x,y) \,:\, a(x+1)(y+1)(x+y+a) \\ -(a-1)(a^2-a+1)xy = 0,\ x>0, y>0, a>a_*\}.$$

We want to find infinitely many points $(x(a), y(a), a) \in (\mathbb{Q}^+ \times \mathbb{Q}^+ \times \mathbb{Q}^+) \cap S_a$. As we will see bellow, we will find first one point $(a; x, y)$, satisfying $x + y = 23/4$, and from it we will construct infinitely many via a multiplication process.

We start by applying the transformation $S = x + y$, $P = xy$ to each surface S_a. We obtain,

$$a(1 + S + P)(a + S) - (a - 1)\left(a^2 - a + 1\right)P = 0,$$

or equivalently,

$$P = \frac{a(1 + S)(a + S)}{a^3 - 3a^2 + (2 - S)a - 1}. \tag{7}$$

It is easy to see that if $(S, P) \in \mathbb{Q}^+ \times \mathbb{Q}^+$ and $\Delta := S^2 - 4P$ is a perfect square then the corresponding $(x, y) \in \mathbb{Q}^+ \times \mathbb{Q}^+$. So, we want to find a value of S such that Δ has a suitable expression that facilitates to find values of a for which Δ is a perfect square.

By using (7), $\Delta = S^2 - 4P$ writes as

$$\Delta = \frac{S^3 a - \left(a^3 - 3\,a^2 - 2\,a - 1\right)S^2 + \left(4\,a^2 + 4\,a\right)S + 4\,a^2}{a^3 - 3a^2 + (2 - S)a - 1}.$$

We will fix a value of S such that the value of the discriminant with respect a of the denominator of the last expression vanishes. The discriminant is $(4S - 23)(S + 1)^2$, so we fix $S = 23/4$, obtaining

$$\Delta_2 := \Delta|_{S=23/4} = \frac{1}{16}\,\frac{2116\,a^3 - 8076\,a^2 - 17871\,a - 2116}{(a - 4)(1 + 2\,a)^2}.$$

In order to avoid the terms which are already perfect squares and to have an algebraic expression, we consider

$$\Delta_3 := \left(\frac{4(1 + 2a)(a - 4)}{46}\right)^2 \Delta_2 = (a - 4)\left(a^3 - \frac{2019}{529}a^2 - \frac{777}{92}a - 1\right),$$

and we try to find rational values of a on the quartic $k^2 = \Delta_3$, that is

$$k^2 = (a - 4)\left(a^3 - \frac{2019}{529}a^2 - \frac{777}{92}a - 1\right).$$

In order to apply Lemma 8 to the above equation we perform the following translation $A = a - 4135/2116$, obtaining the new elliptic quartic

$$K^2 = A^4 - \frac{36024561}{2238728}A^2 - \frac{38272338}{148035889}A + \frac{1009624858257249}{20047612231936}, \tag{8}$$

which, by Lemma 8, is isomorphic to the elliptic cubic

$$Y^2 = X^3 - \frac{1288423179}{71639296}X + \frac{8775405707427}{303177500672}. \tag{9}$$

Consider the additive group of rational points on the elliptic curve (9):

$$E(\mathbb{Q}) = \{(X, Y) \in \mathbb{Q} \times \mathbb{Q} \text{ satisfying (9)}\} \cup \mathcal{O}$$

where $\mathcal{O}$ is the usual point at infinity acting as a neutral element, by using MAGMA we have found the following rational point $R = (\frac{18243}{8464}, \frac{81}{184})$, which lies on the compact oval of (9), see Figure 1.

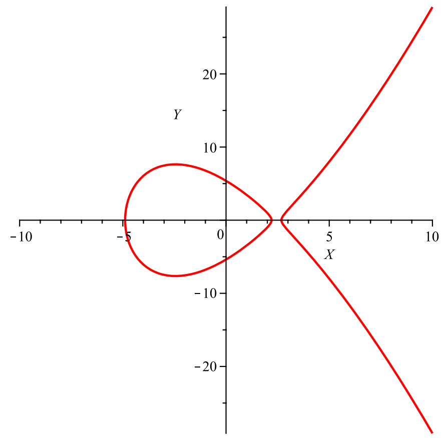

Figure 1: The elliptic curve $Y^2 = X^3 - \frac{1288423179}{71639296}X + \frac{8775405707427}{303177500672}$.

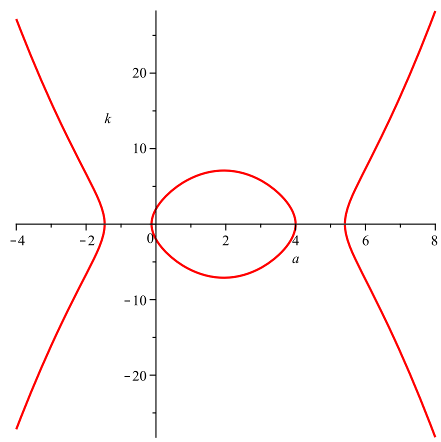

Figure 2: The elliptic quartic $k^2 = \Delta_3 = (a-4)\left(a^3 - \frac{2019}{529}a^2 - \frac{777}{92}a - 1\right)$.

It is easy to check that R is not in the torsion of $E(\mathbb{Q})$. In fact, $E(\mathbb{Q})$ has trivial torsion. Then, as has been explained in Section 2.1, we know that the set of rational points $\{kR\}_{k\in\mathbb{Z}}$, obtained by the secant-tangent chord process, fill densely the full elliptic curve (9). So, their images, trough the transformations given in our proof form a dense set of rational points on the quartic (8). The corresponding projections give a set of rational values of a in $(-\infty, a_4] \cup [a_3, a_2] \cup [a_1, +\infty)$ such that the corresponding map F_a has 9-periodic rational points[e]. Here $a_4 < a_3 < a_2 = 4 < a_1$ are the four roots of Δ_3, see Figure 2. Of course, these periodic points can not have

[e]Notice that not all the rational points on (9) are good seeds for obtaining the density of values of a in $[a_1, +\infty)$ by the secant-tangent chord process. For instance, although the rational point $(X, Y) = (23947/8464, 1781/2116)$ gives

$$(a; x(a), y(a)) = \left(\frac{50025}{6344}; \frac{4231448}{8351929}, \frac{175168575}{33407716}\right),$$

which is a counterexample to Zeeman's Conjecture 2, it is not useful for our purposes because it lies on the unbounded connected component of (9).

both coordinates positive when a belongs to the first two intervals. For instance one of two points corresponding to R is

$$(a; x(a), y(a)) = \left(\frac{391}{370}; \frac{28543}{4224}, -\frac{4255}{4224}\right).$$

Finally we prove that the rational points found on the quartic (8) corresponding to values of $a \in [a_1, +\infty)$ give rise to positive rational 9-periodic orbits. Observe that for $S = 23/4$, the value of P given by equation (7) is

$$P = \frac{27}{4}\,\frac{a(4a+23)}{(a-4)(1+2a)^2}.$$

Hence for all values of $a \geq a_*$, the value P is positive and therefore the corresponding values of x and y are also positive. Hence the theorem follows. ∎

Notice that our proof of Theorem 2 searches positive rational points $(a; x, y)$ in S_a, such that $x + y = 23/4$. Hence a first necessary condition for their existence is that both curves intersect in the first quadrant of the (x, y)-plane. It is easy to prove that when $a \geq a_*$ this only happens when $a \geq a_1 > a_*$.

Some values of a in (a_*, a_1) with not large numerators and denominators[f] are:

$$\frac{14243}{2632}, \frac{14335}{2649}, \frac{18675}{3451}, \frac{197021}{36408}, \frac{216459}{40000}, \frac{333060}{61547},$$
$$\frac{11034}{2039}, \frac{21976}{4061}, \frac{60641}{11206}, \frac{96005}{17741}, \frac{96860}{17899}.$$

Notice that $a_1 - a_* \simeq 0.21 \times 10^{-5}$. For the above values we have not been able to find rational points on the corresponding elliptic curve (3). Nevertheless it is not difficult, by using again MAGMA or SAGE, to compute the root numbers associated to these elliptic curves, obtaining -1 for the values of the first row and $+1$ for the ones of the second row. Hence, if one believes that the Birch and Swinnerton-Dyer Conjecture is true (see for example [7], Conjecture 8.1.7.), as most people do, or more concretely, on the trueness of the so-called Parity Conjecture (see for example [7], section 8.5), we obtain that for the first six values the corresponding curve would have rational points (not necessarily with positive coordinates).

Remark 9. *The elliptic curve given by the equation (9) has minimal model*

$$Y^2 + XY + Y = X^3 - X^2 - 994154X + 376423337.$$

A computation with either SAGE or MAGMA reveals that this curve has rank 3*. As far as we know, this is the highest known rank for the base elliptic curve of a family of elliptic curves with a* 9 *torsion point parameterized by such curve, see* [9] *or* [15, App. B.5]*.*

Next, we show the non-existence of rational 9-periodic points for some $F_a, a > a_*$.

Proposition 10. *(i) For $a \in \{6, 8\}$ there are no initial conditions in $\mathbb{Q} \times \mathbb{Q}$ such that the sequence* (1) *has period 9.*

(ii) For $a = 9$ there are no initial conditions in $\mathbb{Q}^+ \times \mathbb{Q}^+$ such that the sequence (1) *has period 9, but there are infinitely many in $\mathbb{Q} \times \mathbb{Q}$.*

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[f] We have obtained them by computing several convergents of the expansion in continuous fractions of some points in the interval.

Proof. Consider the elliptic curves (3) for $a = 6, 8$ and 9. By using MAGMA we can identify them in the list of Cremona curves ([8]) as numbers 17670bb1, 122094bl3 and 118698i1, respectively. From the list we know that for the first two cases the rank is 0. Hence they have no rational points apart of the 9-torsion points. In the case $a = 9$ we have that the rank of the elliptic curve is 1, so there are infinitely many rational points on it. But a generator of the free part is $(-3/70, -1273/105)$, which has not positive coordinates. This implies that no rational point in the elliptic curve has positive coordinates as well. ∎

The result (i) for $a = 6$ of the above proposition can also be proved in different ways. It is possible to use the 2-descent method, or, even better, since our curves have 9-torsion points and, hence, 3-torsion points, one can instead use the descent via 3-isogeny method. It is also possible to use an indirect method: showing that the value of its L-function at $s = 1$ is not zero. Then, by the known true cases of the Birch and Swinnerton-Dyer Conjecture (see [7], Theorem 8.1.8), we can deduce its rank is 0. By using MAGMA or SAGE, one shows that $L(E, 1) \simeq 6.10077623$, and hence E has rank 0. The same argument applies for $a = 8$.

By using Theorem 3, together with the known example of an elliptic curve of rank 4 with a torsion point of order 9, see [9], we get an example of rank 4 for the curves C_a. This curve gives also a counterexample of Zeeman's Conjecture 2 with many (positive) rational periodic orbits.

Corollary 11. *For $a = 408/23$, the curve C_a has rank 4 over $\mathbb{Q}$, that is $C_a(\mathbb{Q}) \cong \mathbb{Z}^4 \times \mathbb{Z}/9\mathbb{Z}$. Moreover the non-torsion points are generated by*

$$\left(\frac{15708}{38617}, \frac{1275}{4346}\right), \left(\frac{117348775936}{1130069373}, \frac{17875982344}{22803541107}\right),$$
$$\left(\frac{-5313}{5186}, \frac{199644}{17}\right), \left(\frac{96539240}{980237}, \frac{892914}{1232041}\right).$$

5. **Period 5 points.** When $a = 1$, the map $F_1(x, y) = (y, (1 + y)/x)$ is the celebrated, globally 5-periodic, Lyness map. We define $E_h := C_{1,h}$ as the family of elliptic curves

$$E_h := \{(x + 1)(y + 1)(x + y + 1) - h\,xy = 0\}.$$

It is clear that all them are filled of 5-periodic orbits. Following [3] we know the torsion of the group of rational points of each of the curves, $E_h(\mathbb{Q})$ is either $\mathbb{Z}/5\mathbb{Z}$ or $\mathbb{Z}/10\mathbb{Z}$. In that paper it is proved that $(1, 4) \in E_{15}$ or that $(1, 3) \in E_{40/3}$ and that they are not in the corresponding torsion subgroups. Hence the corresponding elliptic curves have infinitely many positive rational points and moreover the ranks of both groups are greater or equal than 1. In [3, Prob. 1] the authors ask the following question: Is the rank of every $E_h(\mathbb{Q})$ for $h \in \mathbb{Q}$, $h > h_c^+$ always positive? Here $h_c^+ = (11 + 5\sqrt{5})/2 \simeq 11.09$ is the value where E_h starts to have points in the first quadrant. Next result proves that the answer to the above question is negative.

Proposition 12. *There exist rational values of h, $h > h_c^+$, such that $E_h(\mathbb{Q})$ has rank 0.*

Proof. We use the same method that in the proof of Proposition 10. Consider E_h with $h = 13$. Its corresponding number in Cremona's list ([8]) is 325e1, obtaining rank 0 and torsion $\mathbb{Z}/5\mathbb{Z}$. Hence the result follows. ∎

Remark 13. *(i) Arguing as in the previous proof, for the values $h = 12, 14$ and 15 we obtain ranks 0,0 and 1, and torsions $\mathbb{Z}/10\mathbb{Z}$, $\mathbb{Z}/5\mathbb{Z}$ and $\mathbb{Z}/5\mathbb{Z}$, respectively. In fact, in the 100 first integers values of $h > 11$, there are 48 values with rank 0 (and 43 values with rank 1, and 9 values of rank 2).*

(ii) For the case $h = 12$, one can also use the existence of a point with 2-torsion to give an easy proof that the rank of $E_{12}(\mathbb{Q})$ is 0, by using the descent via 2-isogeny method.

(iii) Recall that the Lyness map can be written as $P \to P+Q$, where $Q = [1 : 0 : 0]$ and this point has 5-torsion. Thus when the initial point P_0 has 2-torsion we obtain that

$$P_0 + (2m + 1)Q = (2m + 1)(P_0 + Q), \quad m \geq 0$$

and so the odd multiples of P_0+Q by the secant–tangent chord process coincide with the points generated by the Lyness map.

(iv) The case $h = 12$ is also interesting because, although the rang of E_{12} is zero, there is a unique rational (integer) orbit corresponding to half of the torsion points. This orbit is given by the initial conditions $x_0 = 1$, $x_1 = 1$ and is

$$1, 1, 2, 3, 2, 1, 1, \ldots$$

As it is explained in item (iii), this situation happens because the point $(1, 1)$ has 2-torsion.

(iv) By using once more the universality of the level sets of the Lyness curves $C_{a,h}$ and the results given in [9], where four examples of elliptic curves with torsion $\mathbb{Z}/5\mathbb{Z}$ and rank 8 appear, we have obtained the following values of h:

$$\frac{2308482}{11325881}, \quad \frac{283551345}{1294328864}, \quad \frac{2489872}{6474845}, \quad \frac{9645545}{74914011}$$

(and also the ones corresponding to $-1/h$) for which the rank of E_h is 8. Since all these values are in $(0, 1)$ none of them satisfies $h > h_c^+$.

(v) By using the same method than in the previous item we know that $E_{\tilde{h}}$ and $E_{-1/\tilde{h}}$ for $\tilde{h} = 12519024/498355 \simeq 25.12 > h_c^+$ have rank 7.

6. **Single Lyness recurrences with many periods.** Looking at Theorem 1 it is natural to wonder if there exists some rational fixed value of a for which there are rational initial conditions such that the corresponding sequences generated by (1) has all the allowed periods $\{1, 2, 3, 5, 6, 7, 8, 9, 10, 12\}$ given the theorem. It is easy to see that the answer is no, because period 5 (resp. 6) only happens when $a = 1$ (resp. $a = 0$). Moreover we will prove in next proposition that rational sequences with prime periods 1 and 12 or 2 and 12 can not coexist for a given value of a. On the other hand, as it is shown in Table 2, for each $a \neq 1$, the point $(-1, -1)$ always gives rise to a 3-periodic orbit for F_a.

Hence the interesting problem is to find values of $a \in \mathbb{Q}$ such that F_a has one of the following sets of prime periods realized for rational periodic points:

$$\mathcal{P}_1 = \{1, 2, 3, 7, 8, 9, 10\} \quad \text{or} \quad \mathcal{P}_2 = \{3, 7, 8, 9, 10, 12\}.$$

Table 2 shows that for $a = 21/37$ the set of periods of F_a is $\mathcal{P}_2$. The search of rational points with periods 9 and 10 is quite involved. We give some details of how we have obtained them in the end of this section.

This table also shows that for the integer value $a = 20$ the set of achieved periods is $\mathcal{P}_1 \setminus \{2\}$ and it is easy to see that period 2 does not appear. To find explicit rational values a, candidates to have as set of periods with rational entries the full set $\mathcal{P}_1$, is not very difficult. First we give a rational parametrization of the values of a for which rational fixed and period 2-points appear. After, among these values, we search for "small" values for which the root numbers of the elliptic curves corresponding to the points of period $7, 8, 9$ and 10 is -1. We have obtained the list

$$\frac{88401}{18496}, \quad \frac{136353}{23104}, \quad \frac{139971}{36100}, \quad \frac{9633}{12544}, \quad \frac{17301}{19600}, \quad \frac{157521}{92416}, \quad \frac{31003}{39204}.$$

For all these values of a, if one believes once more in the trueness of the Birch and Swinnerton-Dyer Conjecture or of the Parity Conjecture, the set of periods of the corresponding F_a should be $\mathcal{P}_1$. Unfortunately for none of these values of a we have been able to find explicitly periodic points of all the periods.

Per	(x_0, x_1) for $a = 20$	(x_0, x_1) for $a = 21/37$
1	$(5, 5)$	-
2	-	-
3	$(-1, -1)$	$(-1, -1)$
7	$(-\frac{11}{3}, -\frac{35}{32})$	$(\frac{455}{1679}, -\frac{9394}{6693})$
8	$(-\frac{95}{2}, -\frac{31}{12})$	$(\frac{221}{14}, -\frac{645}{658})$
9	$(\frac{5}{166}, -\frac{95}{12})$	$(-\frac{2719003411664}{4342282089993}, \frac{25886110233337}{102273997737527})$
10	$(-\frac{60905}{253889}, -\frac{5756625}{291104})$	$(\frac{1657822032572550308388507}{4355431052669166166335275}, -\frac{1803238432370002727833401}{2680435796120980996248701})$
12	-	$(-\frac{51}{35}, -\frac{32}{7})$

Table 2. Values of a and rational initial conditions for the recurrences (1) with periodic sequences of several periods.

Proposition 14. *Given $a \in \mathbb{Q}$, the Lyness map F_a has not simultaneously rational periodic points with prime periods 1 and 12, or 2 and 12.*

Proof. From the results of Subsection 3.1 we know that these couple of periods do not coexist when the corresponding curves $C_{a,h}$ are not elliptic curves. So, from now one we can assume that the 12 periodic points lie on an elliptic curve $C_{a,h}$. First of all, given $a \in \mathbb{Q}$, it is easy to obtain that there is a rational periodic point with period 1 if and only if $4a+1$ is a square in $\mathbb{Q}$ and that there is one with period 2 if and only if $4a-3$ is a square in $\mathbb{Q}$. By using the expressions of kQ given in Section 2.1 (or by using the known results for the Tate curve [1]), we get that the explicit parameterization of the values of a and h for which Q can have order 12 is

$$a = \frac{2t(1+t)}{3t^2+1} \quad \text{and,} \quad h = -\frac{(t-1)^2(t^2+1)}{t(1+t)(3t^2+1)},$$

for some $t \neq 0, \pm 1$. Hence, to have a rational periodic points with periods 1 and 12, we need rational numbers t such that $\frac{8t(1+t)}{3t^2+1} + 1$ is a square in $\mathbb{Q}$. We will show that this only happens for the values $t = 0$ and 1, which do not give prime period 12. Multiplying by $(3t^2+1)^2$, we need to search for rational solutions of the equation $z^2 = (3t^2+1)(11t^2+8t+1)$. A standard change of variables, similar to the one used used in Section 4, shows that this genus one curve is isomorphic to the elliptic curve with corresponding number in the Cremona's list equal to 15a8, which has only four rational points. These points correspond to the points with $t = 0$ and $t = -1$.

Similarly, to have rational periodic points with periods 2 and 12, we need rational numbers t such that $\frac{8t(1+t)}{3t^2+1} - 3$ is a square in $\mathbb{Q}$. Arguing as in the previous case we arrive to the equation $z^2 = (3t^2+1)(-t^2+8t-3)$ which number in Cremona's list is 39a4 and has only the two rational points corresponding to $t=1$. So the result follows. ∎

6.1. **Searching 9 and 10 rational periodic points for $F_{21/37}$.** In order to find 9 or 10 rational periodic points for $F_{21/37}$, one cannot just naively search for points, since their coordinates are too big. So we use the following strategy. First, with the same formulas that in the proof of Theorem 3, we transform the equations $C_{21/37,h}$, for their corresponding $h = -\frac{16528}{28749}$ and $h = -\frac{296}{609}$, to a Weierstrass equation. We need to find non-torsion points on these elliptic curves. Since the Weierstrass equations have too big coefficients, we apply 2-descent procedure with MAGMA in order to get an equivalent quartic equation with smaller coefficients for the corresponding elliptic curves. We get that they are equivalent respectively to

$$y^2 = -57376476x^4 + 66683940x^3 + 800552377x^2 - 118125576x + 209901456$$

and

$$y^2 = -7734191x^4 + 116312038x^3 + 178646017x^2 - 246594696x - 138820464.$$

These are still not sufficient simple to be able to find (non-torsion) rational points, so we do another transformation. In the first case, we apply 4-descent in order to obtain another form, this case as intersection of two quadrics (in the projective space). We get the equations

$$15X^2+104XY-16Y^2+12XZ-42YZ+18Z^2+62XT-14YT-8ZT+22T^2=0,$$

$$123X^2-460XY-233Y^2-24XZ-398YZ-153Z^2+122XT-320YT+688ZT+321T^2=0.$$

Finally, an easy search finds the point given in projective coordinates by $[-\frac{10}{21} : \frac{9}{14} : \frac{19}{6} : 1]$. Using the transformation rules given by MAGMA one gets the corresponding point in $C_{21/37,h}$, shown in Table 2. For the second equation, corresponding to the 10 rational periods, we directly transform the quartic equation to an intersection of two quadrics, and then we apply an algorism due to Elkies ([10]) to search for rational points in this type of curves (as implemented in MAGMA).

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