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RATIONAL PERIODIC SEQUENCES FOR THE LYNESS RECURRENCE

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Abstract. Consider the celebrated Lyness recurrence $x_{n+2} = (a + x_{n+1})/x_n$ with $a \in \mathbb{Q}$. First we prove that there exist initial conditions and values of a for which it generates periodic sequences of rational numbers with prime periods $1, 2, 3, 5, 6, 7, 8, 9, 10$ or 12 and that these are the only periods that rational sequences $\{x_n\}_n$ can have. It is known that if we restrict our attention to positive rational values of a and positive rational initial conditions the only possible periods are $1, 5$ and 9. Moreover 1-periodic and 5-periodic sequences are easily obtained. We prove that for infinitely many positive values of a, positive 9-period rational sequences occur. This last result is our main contribution and answers an open question left in previous works of Bastien & Rogalski and Zeeman. We also prove that the level sets of the invariant associated to the Lyness map is a two-parameter family of elliptic curves that is a universal family of the elliptic curves with a point of order $n, n \geq 5$, including n infinity. This fact implies that the Lyness map is a universal normal form for most birational maps on elliptic curves.

1. Introduction and main results.

The dynamics of the Lyness recurrence

$$x_{n+2} = \frac{a + x_{n+1}}{x_n}, \tag{1}$$

specially when $a > 0$, has focused the attention of many researchers in the last years and it is now completely understood in its main features after the independent research of Bastien & Rogalski [3] and Zeeman [21], and the later work of Beukers & Cushman [5]. See also [2, 11]. In particular all possible periods of the recurrences

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