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Quadratic and Cubic Systems with Degenerate Infinity

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We obtain an affine classification of quadratic systems with degenerate infinity in eight types and prove that they present 13 phase portraits on the Poincaré sphere. For cubic systems with degenerate infinity the situation is much more complicated. We show that there are examples of such systems with at least three concentric limit cycles and we give a criterion to ensure that, in some cases, this number is less than or equal to two.

1. INTRODUCTION

We consider vector fields X associated with two-dimensional autonomous systems of differential equations of the form

$$\begin{aligned} \dot{x} &= \frac{dx}{dt} = P_0(x,y) + P_1(x,y) + P_2(x,y) + P_3(x,y), \\ \dot{y} &= \frac{dy}{dt} = Q_0(x,y) + Q_1(x,y) + Q_2(x,y) + Q_3(x,y), \end{aligned} \tag{1}$$

where P_k and Q_k are real homogeneous polynomials of degree k, satisfying

$$xQ_3(x,y) - yP_3(x,y) \equiv 0 \qquad (\text{resp. } xQ_2(x,y) - yP_2(x,y) \equiv 0) \tag{2}$$

whenever $P_3^2 + Q_3^2 \not\equiv 0$ (resp. $P_3^2 + Q_3^2 \equiv 0$ and $P_2^2 + Q_2^2 \not\equiv 0$). Such systems will be called cubic systems (resp. quadratic systems) with degenerate infinity, CSDI (resp. QSDI) for short. This terminology is due to the fact that, in the Poincaré compactification of (1), the equator of S^2, i.e., the circle at infinity, consists entirely of critical points; see [9].

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We are interested in determining all possible QSDI (modulus affine transformations and changes of the scale of the independent variable t) and in obtaining all their phase portraits on the Poincaré sphere. We get exactly eight affine equivalent classes (each of them with zero or one parameter) for QSDI. Furthermore, we show that QSDI can only have 13 different phase portraits on the Poincaré sphere (reversing the time variable t if necessary). See Fig. 1 and Theorem Q in next section for more details.

Directly related with our work is the paper [2]. In that paper, entitled "Affine Classification for the Quadratic Vector Fields without the Critical Points at Infinity," the authors study a subfamily of QSDI. At this point we want to comment that it is well known that any quadratic system has critical points at infinity when we make its Poincaré compactification. So, what is the meaning of the title of the above paper? The answer is that, in their paper, the authors only consider the subclass of QSDI such that a reparametrization of its Poincaré compactification does not have critical points at infinity.

On the other hand, we also consider CSDI. Such systems appear, for instance, in a mathematical model of catalytic networks formed by two members; see [7] for more details. Here we show that CSDI are much more complicated than QSDI. In fact we give an example exhibiting at least three limit cycles surrounding the same critical point and prove a criterion that asserts that a certain subfamily of CSDI has at most two limit cycles. See Theorem C in the last section.

The organization of this paper is as follows. Section 2 contains all the material concerning QSDI, and in Section 3 we consider CSDI.

2. ON QUADRATIC SYSTEMS WITH DEGENERATE INFINITY

THEOREM Q. *Consider the vector field X associated with a quadratic system with degenerate infinity. Denote by $p(X)$ its Poincaré compactification on S^2. Then*:

(i) *there is an affine change of coordinates of X such that in some local chart of S^2, centered at infinity, a parametrization of $p(X)$ is a linear vector field*;

(ii) *there is an affine change of coordinates plus a rescaling of the time variable t such that the quadratic system with degenerate infinity has one of the* *following forms*:

$$\text{(A)} \quad \begin{cases} \dot{x} = xy, \\ \dot{y} = 1 + dy + y^2, 0 \le d < 2, \end{cases}$$

$$\text{(B)} \quad \begin{cases} \dot{x} = xy, \\ \dot{y} = x + 1 + dy + y^2, 0 \le d < 2, \end{cases}$$

$$\text{(C)} \quad \begin{cases} \dot{x} = xy, \\ \dot{y} = (y - r)(y - 1), r \le 1, \end{cases}$$

$$\text{(D)} \quad \begin{cases} \dot{x} = xy, \\ \dot{y} = x + (y - r)(y - 1), 0 \le r < 1, \end{cases}$$

$$\text{(E)} \quad \begin{cases} \dot{x} = xy, \\ \dot{y} = y^2, \end{cases} \qquad \text{(F)} \quad \begin{cases} \dot{x} = xy, \\ \dot{y} = x + y^2, \end{cases}$$

$$\text{(G)} \quad \begin{cases} \dot{x} = x^2, \\ \dot{y} = 1 + xy, \end{cases} \qquad \text{(H)} \quad \begin{cases} \dot{x} = x + x^2, \\ \dot{y} = 1 + xy; \end{cases}$$

(iii) *phase portraits on the Poincaré sphere, of the systems given in* (ii), *are plotted in Fig.* 1 *according to the values of their parameters.*

COROLLARY. *Consider a quadratic system with degenerate infinity. Then the following holds*:

(i) *it has, at least, one invariant straight line,*

(ii) *its orbits* (*without the parametrization with respect to time*) *can be expressed as the level curves of a function* $\mathcal{F}(x, y)$. *This function is obtained by integrating a linear differential equation associated with the quadratic system.*

Proof of the Corollary. (i) Follows from (ii) of Theorem Q, because $x = 0$ is always an invariant straight line. Part (ii) is an easy consequence of (i) of Theorem Q. ∎

Remarks. (i) For a lot of cases of QSDI it is also possible to determine their trajectories (that is, the orbits with their parametrization) using (ii) of the above theorem. For instance in cases (A), (C), (E), (G), and (H).

(ii) Observe that Figs. 1, 2, 3, 4, and 5 of [2] correspond with F; B, $d = 0$; B, $0 < d < 2$; D, $r = 0$, and D, $0 < r < 1$, of Fig. 1, respectively.

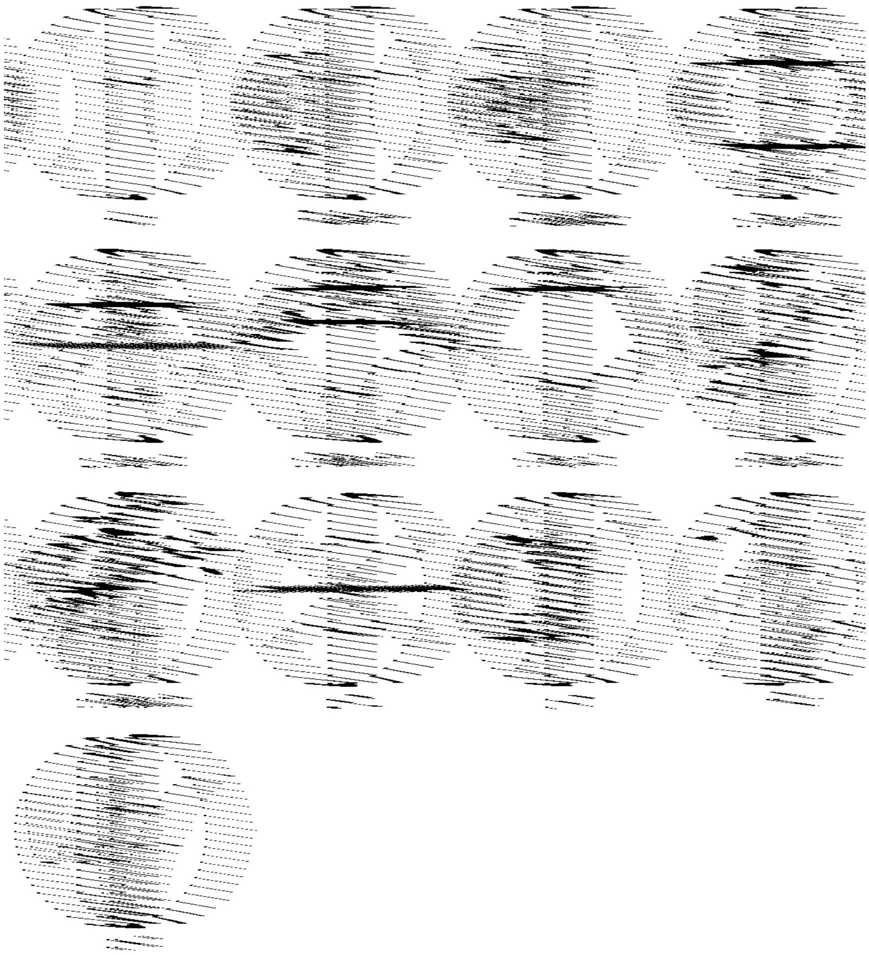

FIG. 1. QSDI except perhaps the orientation, according to (ii) of Theorem Q.

Proof of Theorem Q. (i) Any QSDI can be written as

$$\dot{x} = \alpha + ax + by + x(lx + my),$$
$$\dot{y} = \beta + cx + dy + y(lx + my), \qquad \text{where } l^2 + m^2 \neq 0.$$

Changing, if necessary, x with y and vice versa, we can assume that $l \neq 0$. Furthermore, taking the new coordinates $x_1 = lx + my$, $y_1 = y$, the QSDI can be written as

$$\begin{aligned} \dot{x} &= \alpha + ax + by + x^2, \\ \dot{y} &= \beta + cx + dy + xy, \end{aligned} \tag{3}$$

where the constants are, perhaps, different from the previous ones and where we have omitted the subscripts.

It is easy to check that system (3) has the straight line $y = px + bp^2 + (a - d)p - c$, invariant if p is a real root of

$$-b^2p^3 + b(2d - a)p^2 + (da + bc - d^2 - \alpha)p + \beta - dc = 0.$$

Therefore if $b \neq 0$; or $b = 0$ and $\alpha \neq d(a - d)$; or $b = 0$, $\alpha = d(a - d)$, and $\beta - dc = 0$, such a real root always exists. Let us call it $\bar{p}$. Hence, in the new coordinates $x_1 = d - \bar{p}b + x$, $y_1 = y - \bar{p}x - b\bar{p}^2 + (d - a)\bar{p} + c$, system (3) can be written as

$$\begin{aligned} \dot{x} &= \alpha' + a'x + b'y + x^2, \\ \dot{y} &= xy, \end{aligned}$$

where we have omitted the subscripts. Finally, calling $x_1 = y$, $y_1 = x$, renaming the constants, and again omitting the subscripts, we get

$$\begin{aligned} \dot{x} &= xy, \\ \dot{y} &= \beta + cx + dy + y^2. \end{aligned} \tag{4}$$

In the case in which we cannot assure the existence of a real root of the cubic equation, we have in (3) that $b = 0$, $\alpha = d(a - d)$, and $\beta - dc \neq 0$; that is,

$$\begin{aligned} \dot{x} &= d(a - d) + ax + x^2, \\ \dot{y} &= \beta + cx + dy + xy, \qquad \text{where } \beta - dc \neq 0. \end{aligned}$$

taking $z = x + d$, $\omega = (c + y)/(\beta - dc)$, we obtain

$$\begin{aligned} \dot{z} &= (z + a - 2d)z, \\ \dot{\omega} &= 1 + z\omega. \end{aligned}$$

Calling $x = z$, $y = \omega$ and renaming $a - 2d$ as a, we get

$$\begin{aligned} \dot{x} &= ax + x^2, \\ \dot{y} &= 1 + xy, \end{aligned} \tag{5}$$

Therefore, any QSDI can be written as (4) or (5). The expressions of these two systems in the local chart $\mathscr{U}_1$ of the Poincaré compactification are

$$\begin{aligned} \dot{z}_1 &= \frac{z_2}{\Delta(z)}(c + dz_1 + \beta z_2), \\ \dot{z}_2 &= \frac{z_2}{\Delta(z)}(-z_1), \end{aligned} \tag{4'}$$

$$\begin{aligned} \dot{z}_1 &= \frac{z_2}{\Delta(z)}(-az_1 + z_2), \\ \dot{z}_2 &= \frac{z_2}{\Delta(z)}(-1 - az_2), \end{aligned} \tag{5'}$$

respectively, where $\Delta(z) = (z_1^2 + z_2^2 + 1)^{1/2}$. Clearly, the above expressions prove (i) taking a new time variable s, such that $ds/dt = z_2/\Delta(z)$.

(ii) Consider system (4). Taking the new variables $x_1 = qx$, $y_1 = py$, and $t_1 = p^{-1}t$, we have

$$\begin{aligned} \frac{dx_1}{dt_1} &= x_1 y_1, \\ \frac{dy_1}{dt_1} &= \frac{p^2 c}{q} x_1 + p^2 \left\{ \beta + \frac{d}{p} y_1 + \frac{1}{p^2} y_1^2 \right\}. \end{aligned}$$

From the above expression it is easy to get systems from (A) to (F), with $c = 0$ or $c \neq 0$ and with the existence or not of real roots of $\beta + dy + y^2 = 0$. In systems (C) and (D) we can assume that $r \leq 1$ because, otherwise, the change $x_1 = r^{-2}x$, $y_1 = r^{-1}y$, $t_1 = rt$ reduces the system to the previous case. Furthermore, in system (D) we can suppose $0 \leq r < 1$, by using the change $x_1 = r/(r-1)^2 + (1/(r-1)^2)x - (r/(1-r^2))y$, $y_1 = r/(r-1) - (1/(r-1))y$, $t_1 = (1-r)t$ (resp. $x_1 = x - y + 1$, $y_1 = 1 - y$, $t_1 = -t$) if $r < 0$ (resp. $r = 1$). Finally, systems (G) and (H) came from (5) with $a = 0$ or $a \neq 0$, respectively, taking coordinates $x_1 = a^{-1}x$, $y_1 = ay$, $t_1 = at$, if necessary.

(iii) To obtain phase portraits, on the Poincaré sphere, of systems from (A) to (H), as in Fig. 1, we find and draw their orbits. To do this it is enough to integrate the expressions of these systems in the local chart $\mathscr{U}_1$ given by (4′) or (5′), to use the change $z_1 = y/x$, $z_2 = 1/x$ and to take into account that $x = 0$ is an invariant straight line for all of them. The integration of systems (4′) and (5′) can be done by reducing equation $dx_2/dz_1 = (a_1 z_1 + b_1 z_2 + c_1)/(a_2 z_1 + b_2 z_2 + c_2)$, to a homogeneous differential equation whenever we have $a_1 b_2 - a_2 b_1 \neq 0$, or to the equation

$dz_2/dz_1 = \bar{f}(a_1 z_1 + b_1 z_2)$, otherwise. Both equations are elementarily integrable by separating variables. ∎

3. ON CUBIC SYSTEMS WITH DEGENERATE INFINITY

In this section we will study some properties of CSDI. From its definition it is clear that they can be written as

$$\dot{x} = P(x, y) + xH(x, y),$$
$$\dot{y} = Q(x, y) + yH(x, y),$$

where P and Q are arbitrarily polynomials of degree 2 and H is a homogeneous polynomial of degree 2. We are interested in the problem of periodic orbits; so it is not restrictive to assume that the above system has at least a critical point and that it is the origin; that is, $P(0,0) = Q(0,0) = 0$. Then, its expression in polar coordinates, $x = r\cos\theta$, $y = r\sin\theta$, is

$$\begin{aligned} \dot{r} &= a(\theta)r + f(\theta)r^2 + h(\theta)r^3, \\ \dot{\theta} &= b(\theta) + g(\theta)r, \end{aligned} \tag{6}$$

where a, b, f, g, and h are homogeneous trigonometric polynomials of degrees 2, 2, 3, 3, and 4, respectively. Observe that the only difference between this expression and the general expression of a cubic system with a critical point at the origin is that in this latter case, the second differential equation has a term of the form $i(\theta)r^3$, with i trigonometrical polynomial of degree 4.

We next consider the function

$$A(\theta) = (a(\theta)g(\theta) - b(\theta)f(\theta))g(\theta) + b^2(\theta)h(\theta), \tag{7}$$

which is a homogeneous trigonometrical polynomial of degree 8 and generalizes a function frequently used to study planar systems with homogeneous nonlinearities; see, for instance, [1].

Our main result is the following.

THEOREM C. (i) *There are CSDI with at least three limit cycles surrounding the same critical point.*

(ii) *Consider a CSDI, written as in* (6) *and assume that the function A, defined in* (7), *does not change sign. Then, this system has at most two limit cycles and, when they exist, they surround the origin.*

Proof of Theorem C. (i) Taking $z = x + iy = \mathrm{Re}(z) + i\,\mathrm{Im}(z)$, it is not difficult to check that the equation

$$\dot{z} = (i + \lambda)z + 2\overline{B}z^2 + Bz\bar{z} + \frac{3}{4}\left(\overline{B} + \frac{\xi}{4\pi\, d\overline{B}}\right)\bar{z}^2 + d(z^3 + z\bar{z}^2) + ez^2\bar{z},$$

with $d, e, \lambda, \xi \in \mathbb{R}$, $B \in \mathbb{C}$, $d\overline{B} \neq 0$, corresponds to a CSDI.

Using [5], we obtain that the first Liapunov constants of this equation are

$$\begin{aligned} v_1 &= \exp(2\pi\lambda) - 1, \\ v_3 &= 2\pi e, \qquad \text{when } \lambda = 0, \\ v_5 &= \xi, \qquad \text{when } \lambda = e = 0, \\ v_7 &= \frac{-525}{256}\pi B\overline{B}\,\mathrm{Im}(\overline{B}^4) + dH(\mathrm{Re}(B), \mathrm{Im}(B), d), \\ &\qquad \text{when } \lambda = e = \xi = 0, \end{aligned}$$

where $H(x, y, z)$ is a polynomial. Hence, taking B such that $\mathrm{Im}(\overline{B}^4) \neq 0$ and d small enough, we get a system with $v_7 \neq 0$. Standard arguments imply that if we take $|\lambda| \ll |\varepsilon| \ll |\xi|$ and v_7, ξ, ε, and λ alternating in signs, we get a CSDI with three small-amplitude limit cycles.

(ii) Following similar arguments to those of [1 or 4], it can be proved that any periodic orbit of system (6) surrounds the origin and does not cut the set $\{(r, \theta) \in \mathbb{R}^2 | b(\theta) + g(\theta)r = 0\}$. Hence, if we assume that system (6) has three limit cycles, they are positive solutions of the equation

$$dr/d\theta = S(r, \theta) = \frac{a(\theta)r + f(\theta)r^2 + h(\theta)r^3}{b(\theta) + g(\theta)r},$$

satisfying $r(\theta) = r(\theta + 2\pi)$, and they can be written as $r_1(\theta) > r_2(\theta) > r_3(\theta)$. Following [8, p. 103], we consider

$$\frac{\dot{r}_1 - \dot{r}_2}{r_1 - r_2} - \frac{\dot{r}_1 - \dot{r}_3}{r_1 - r_3} - \frac{\dot{r}_2}{r_2} + \frac{\dot{r}_3}{r_3}.$$

Straightforward calculations show that this last expression coincides with

$$\frac{A(\theta)r_1(r_2 - r_3)}{(b(\theta) + g(\theta)r_1)(b(\theta) + g(\theta)r_2)(b(\theta) + g(\theta)r_3)},$$

and so,

$$0 = \log\left\{ \frac{(r_1(\theta) - r_2(\theta))r_3(\theta)}{(r_1(\theta) - r_3(\theta))r_2(\theta)} \right\}\Bigg|_0^{2\pi}$$

$$= \int_0^{2\pi} \frac{A(\theta) r_1 (r_2 - r_3)\, d\theta}{(b(\theta) + g(\theta) r_1)(b(\theta) + g(\theta) r_2)(b(\theta) + g(\theta) r_3)},$$

because r_i, $i = 1, 2, 3$, are 2π-periodic functions. Note that this last equality leads us to a contradiction because the integrating function does not change sign and it is continuous. Hence, system (6) has at most two limit cycles. ∎

FINAL REMARKS

(a) Take a quadratic system $\dot{x} = \overline{P}(x, y)$, $\dot{y} = \overline{Q}(x, y)$, with four hyperbolic limit cycles in the configuration (3, 1). It is clear that if we consider $(\dot{x}, \dot{y}) = (\overline{P}(x, y), \overline{Q}(x, y)) + \varepsilon H(x, y) \cdot (x, y)$, with H a homogeneous polynomial of degree 2 and ε small enough, we have constructed a CSDI with configuration (3, 1) and proved, again, Theorem C (i). At this point we would like to present a problem for CSDI that seems interesting. Which configuration of limit cycles (different from configurations that quadratic systems have) can they present?

(b) Following again [4], it is also possible to show, adding to the hypotheses of Theorem C (ii) the conditions $b(\theta)$ does not vanish and that $\{(r, \theta) \in \mathbb{R}^2 | b(\theta) + g(\theta) r = 0\}$ is not a closed curve, that the sum of the multiplicities of the limit cycles that system (6) can have is two. The key point in the proof is the use of the formula for the derivatives of the Poincaré return map, proved in [6], and the fact that

$$\frac{\partial^3 S}{\partial r^3}(r, \theta) = \frac{6A(\theta) b(\theta)}{(b(\theta) + g(\theta) r)^4}.$$

(c) It is easy to check that $(\dot{x}, \dot{y}) = (\lambda x - y, x + \lambda y) + a(x^2 + y^2) \cdot (x, y)$ is a CSDI that has $A(\theta) \equiv a$. Therefore, it satisfies the hypotheses of Theorem C (ii) and, so, it has at most two limit cycles. In fact, taking polar coordinates it is clear that the limit cycle is $x^2 + y^2 = -\lambda/a$ and that it is unique. It is not difficult to construct other examples of CSDI under the hypotheses of Theorem C (ii) with one limit cycle, but unfortunately we have not found any of them with two limit cycles.

(d) Another way of approaching a CSDI, written as (6), is to transform it by using the change of variables $\rho = r/(b(\theta) + g(\theta)r)$, to the Abel equation

$$\frac{d\rho}{d\theta} = \frac{A(\theta)}{b(\theta)}\rho^3 + \frac{b(\theta)f(\theta) - 2a(\theta)g(\theta) + b'(\theta)g(\theta) - b(\theta)g'(\theta)}{b(\theta)}\rho^2 + \frac{a(\theta) - b'(\theta)}{b(\theta)}\rho.$$

After this paper was written, Ref. [3] appeared. There the center problem for CSDI is solved. The author's proof uses results of [10] on the Liapunov constants for such systems.

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