LIMIT CYCLES FOR RIGID CUBIC SYSTEMS

A. GASULL, R. PROHENS, AND J. TORREGROSA

Abstract. We consider planar cubic systems with a unique rest point of center-focus type and constant angular velocity. For such systems we obtain an affine classification in three families, and their corresponding phase portraits on the Poincaré sphere. Our results include the study of the number of limit cycles as well as the bifurcations diagrams that they present. We prove that in two of the three families the maximum number of limit cycles is one. With respect the third family we give a representative one-parameter family, exhibiting two limit cycles.

1. Introduction and Main Results

Regarding Hilbert sixteenth problem, along the last decades, a lot of important results are achieved to better understand how to control the number of limit cycles that polynomial planar vector fields can have. Nevertheless to study the bifurcation diagram of a concrete family of planar vector exhibiting more than one limit cycle is usually a very difficult task.

This paper is devoted to get the bifurcation diagram of a family of cubic systems. Our results seem to indicate that it presents at most two limit cycles, but we have not been able to prove this assertion. In any case, we have been able to obtain a full description of their phase portraits in the subfamilies having at most one limit cycle, as well as to present the phase portraits of a one-parametric subfamily exhibiting at least two limit cycles.

Let us start giving our motivation to study this family. Planar systems whose angular speed is constant are usually called *rigid* or *uniformly isochronous.* When the origin is non degenerated they can be written as

$$\left\{\begin{array}{rcl}\dot{x} & = & -y + xF(x,y),\\ \dot{y} & = & x + yF(x,y),\end{array}\right. \tag{1}$$

where $F(x,y)$ is an arbitrary function in the variables x and y. For these systems the center-focus problem (i. e. the distinction between a focus and a center) is equivalent to the isochronicity problem. Probably, this is one of the reasons for which they haven already studied by several authors, see for instance [2, 3, 4, 14]. One of the simpler systems of the form (1) are

$$\left\{\begin{array}{rcl}\dot{x} & = & -y + x(F_0(x,y) + F_m(x,y) + F_n(x,y)),\\ \dot{y} & = & x + y(F_0(x,y) + F_m(x,y) + F_n(x,y)),\end{array}\right. \tag{2}$$

1991 *Mathematics Subject Classification.* 34C-05, 34C-25, 58F-14, 58F-21.
Key words and phrases. bifurcation, limit cycle, center, cubic system.
Supported by DGES No. BFM2002-04236-C02-2 and CONACIT 2001 SGR-00173.

where each $F_i(x,y)$ stands for a real homogeneous polynomial of degree i, and $0 \le m < n$ are fixed natural numbers.

Let us call $\mathcal{H}(m,n) \in \mathbb{N} \cup \{\infty\}$ the maximum number of limit cycles of system (2) in terms of m and n. It is not difficult to prove that

$$\mathcal{H}(0,n) = \begin{cases} 0, & \text{if } n \text{ is odd,} \\ 1, & \text{if } n \text{ is even.} \end{cases}$$

Furthermore, in [10] a method to decide whether the limit cycle exits for the case n even is given.

By studying the Lyapunov constants of the origin of (2), the following lower bounds for $\mathcal{H}(m,n)$, when $m \ge 1$, are obtained in [11]:

$m \setminus n$	1	2	3	4
1	–	2	2	3
2	–	–	4	4

TABLE 1. Lower bounds for $\mathcal{H}(m,n)$.

The above results motivate the main goal of this paper: the study of the case $m = 1, n = 2$, where $\mathcal{H}(1,2) \ge 2$. Notice this is the easiest case for which $\mathcal{H}(m,n)$ is not known. Before describing our results on this case we introduce some notation. Write system (2) when $m = 1, n = 2$ as

$$\left\{ \begin{array}{rcl} \dot{x} &=& -y + x(a + bx + cy + dx^2 + exy + fy^2), \\ \dot{y} &=& x + y(a + bx + cy + dx^2 + exy + fy^2), \end{array} \right. \tag{3}$$

where a, b, c, d, e, $f \in \mathbb{R}$, and $d^2 + e^2 + f^2 \neq 0$. To study its phase portrait it is useful to get some normal forms with less parameters. In Proposition 2.1.ii we will prove that it suffices to consider the following three subcases:

$$\text{(I)} \quad \left\{ \begin{array}{rcl} \dot{x} &=& -y + x(a + bx + cy + dx^2 + exy + y^2), \\ \dot{y} &=& x + y(a + bx + cy + dx^2 + exy + y^2), \quad \text{with } d > 0, \end{array} \right.$$

$$\text{(II)} \quad \left\{ \begin{array}{rcl} \dot{x} &=& -y + x(a + bx + cy + x^2), \\ \dot{y} &=& x + y(a + bx + cy + x^2), \quad \text{with } c \ge 0, \end{array} \right.$$

$$\text{(III)} \quad \left\{ \begin{array}{rcl} \dot{x} &=& -y + x(a + bx + cy + dx^2 + exy), \\ \dot{y} &=& x + y(a + bx + cy + dx^2 + exy), \quad \text{with } d \in \{0,1\}. \end{array} \right.$$

In fact, the above classification takes into account, after a re-scaling, the number of critical points at infinity of the Poincaré compactification of system (3). Families (I), (II) and (III) correspond with 0, 1 and 2 pairs of critical points at infinity, respectively. Furthermore each of these cases corresponds with negative, zero and positive values of $\Delta := e^2 - 4df$, respectively.

Our main result is the following:

Theorem 1.1. *The maximum number of limit cycles for families (I) and (II) is one and when it exists it is hyperbolic. Furthermore:*

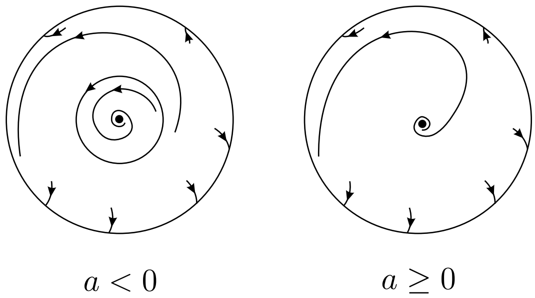


FIGURE 1. Phase portraits of families (I) and $(\mathrm{II})_{c=0}$ in accordance with parameter a.

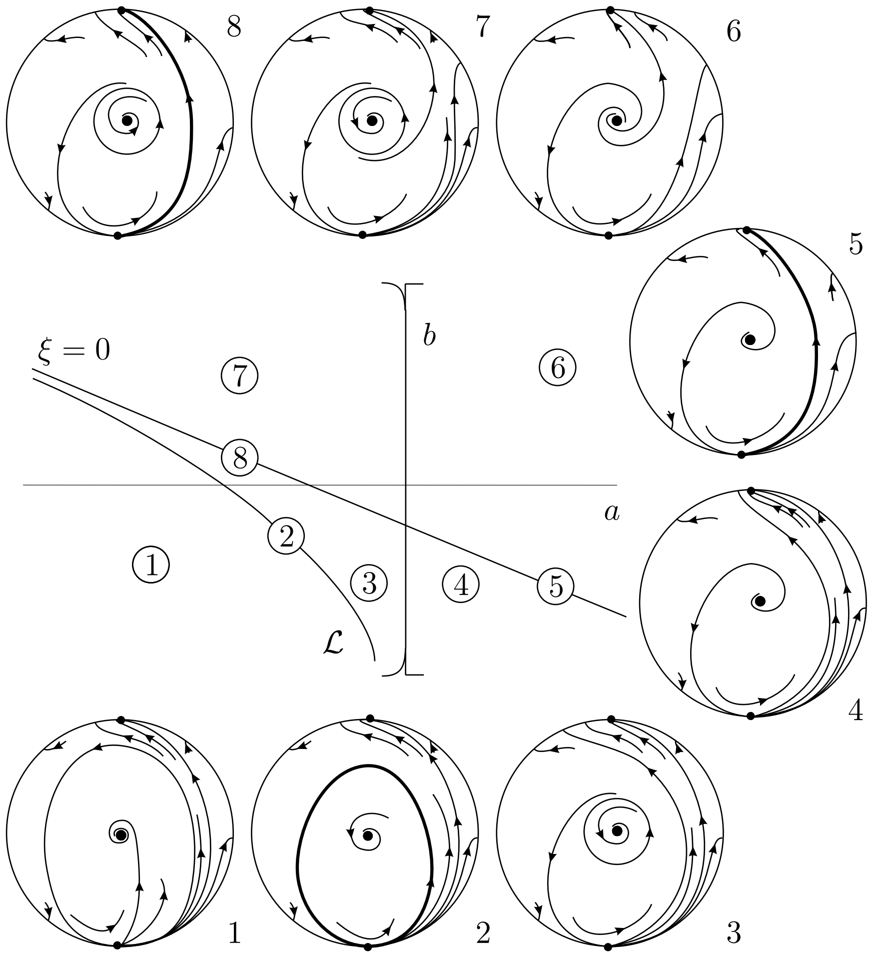


FIGURE 2. Bifurcation diagram and phase portraits of family $(\mathrm{II})_{c>0}$ for a fixed value of c.

(i) The phase portraits in the Poincaré sphere for families (I) or $(II)_{c=0}$ *are given in Figure 1.*
(ii) The bifurcation diagram and the phase portraits in the Poincaré sphere for family $(II)_{c>0}$ *are given in Figure 2. In this figure there appears a curve* $\mathcal{L}$

which is, for a fixed c, the graph of a function of b. This function gives the only value $a = a^*(b,c)$ for which the corresponding system has an infinite homoclinic saddle-node loop connection, see Figure 2.2.

A straightforward consequence of the above theorem is the following corollary.

Corollary 1.2. *(i) Families (I) or $(II)_{c=0}$ have limit cycles if and only if $a < 0$.*
(ii) Fixed b and c, there exists a negative value $a^*(b,c)$ such that family $(II)_{c>0}$ has a limit cycle if and only if $a^*(b,c) < a < 0$.

As we will see, a key point for proving the above results will be the fact that system (2), in polar coordinates, writes as an Abel differential equation.

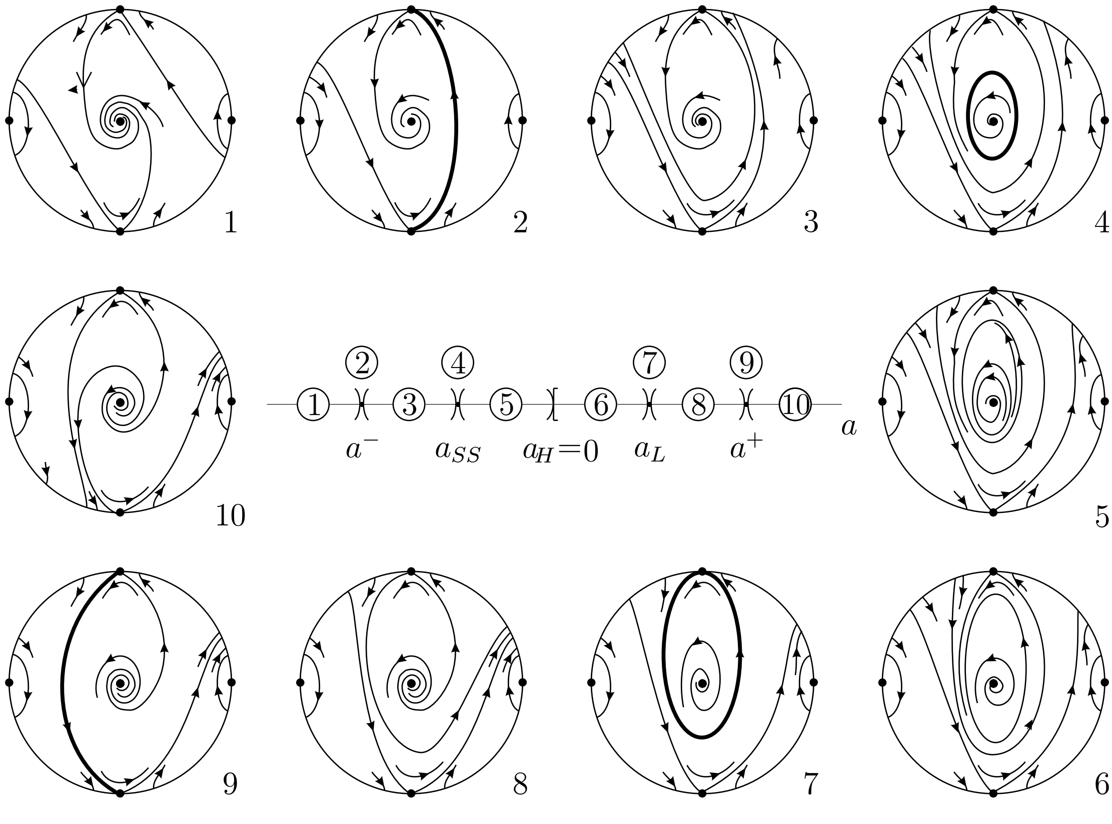


FIGURE 3. An evolution of the phase portrait of family (III) with $b = 1$, $c = 3$, $d = 1$ and $e = 5$, in terms of the parameter a exhibiting, at least, two limit cycles. Here $a^- = \frac{-2-\sqrt{29}}{25} \approx -0.29540$, $a_{SS} \approx -0.029101$, $a_L \approx 0.0135$ and $a^+ = \frac{-2+\sqrt{29}}{25} \approx 0.13540$.

Unfortunately, we have not ben able to give an upper bound for the number of limit cycles of system (3). From Table 1 we already know that there are systems of this type with at least two limit cycles. As we have already explained, this result was obtained by using Lyapunov constants. In Section 4 we choose a different approach to get as much limit cycles as it is possible for family (III). Concretely we study the Melnikov functions until order eight of the Hamiltonian system $(x^2 +$

$y^2)/2$ inside the family. We obtain again only two limit cycles, see Section 4. Finally, in Section 5 we illustrate the bifurcations that appear inside Family (III) varying only one parameter, see Figure 3.

2. Preliminary results

As a starting point we will classify system (3) in the three subfamilies (I), (II) and (III).

Let us denote by $\mathcal{P}(X)$ the Poincaré compactification on $\mathbb{S}^2$ of system (3) (see [12, 16], for instance). We note that system (3) has degenerate infinity, i.e. all the equator of $\mathbb{S}^2$ is filled up of critical points. In order to study the behaviour of the orbits when they go to infinity, we re-parametrize its Poincaré compactification obtaining, at most, two pairs of critical points at infinity. This number of pairs of infinite critical points will be the criterion to distinguish between the three families in which we will classify system (3). To establish their phase portraits we need, among others thinks, to know the configuration of the separatrices of their critical points, as well as the exact number of limit cycles that they have. To follow the evolution of the limit cycles and the separatrices when the parameters vary we will deal with semi-complete families of rotated vector fields (see [5] or [15, Sec 4.6], for instance).

Next proposition summarizes general properties and technical results on system (3).

Proposition 2.1. *Consider the vector field X associated with system (3) and set $\Delta = e^2 - 4df$. Then:*

(i) After a re-scaling of the variable t, we obtain zero, two or four infinite critical points on $\mathcal{P}(X)$, according with $\Delta < 0$, $\Delta = 0$ or $\Delta > 0$, respectively.

(ii) It is not restrictive to assume that:
- *(a) $f = 1$ and $d > 0$ when $\Delta < 0$,*
- *(b) $e = f = 0$, $d = 1$ and either $c = 0$ or $c > 0$ when $\Delta = 0$,*
- *(c) $f = 0$ and either $d = 0$ or $d = 1$ when $\Delta > 0$.*

(iii) The origin is the unique finite critical point and it is of focus or center type. Furthermore:
- *(a) Its first three Lyapunov constants are*

$$V_1 = e^{2\pi a} - 1, \quad V_3 = \pi(d + f) \quad \text{and} \quad V_5 = \pi\left(\left(c^2 - b^2\right)d - bce\right)/2.$$

- *(b) The origin is a center if and only if $V_1 = V_3 = V_5 = 0$.*
- *(c) Two limit cycles bifurcate from the origin via a degenerated Andronov-Hopf bifurcation.*

(iv) System (3) is a semi-complete family of rotated vector fields with respect the parameters a, d or f, indistinctly.

Proof. (i) The expression of system (3) in the local charts $\mathcal{U}_1$ and $\mathcal{U}_2$ of the Poincaré compactification of X, given in [12, 16], are

$$\left\{\begin{array}{rcl} \dot{z} & = & \frac{w}{D(z,w)}(w + z^2 w), \\ \dot{w} & = & \frac{-w}{D(z,w)}(d + ez + bw + fz^2 + czw + aw^2 - zw^2), \end{array}\right. \tag{4}$$

and

$$\left\{\begin{array}{lcl} \dot{z} & = & -\frac{w}{D(z,w)}(w+z^2w), \\ \dot{w} & = & -\frac{w}{D(z,w)}(f+ez+cw+dz^2+bzw+aw^2+zw^2), \end{array}\right. \tag{5}$$

respectively, where $D(z,w)=\sqrt{z^2+w^2+1}$ and in both cases $w=0$ represents the infinity of the vector field. Hence, taking a new time variable s such that $ds/dt=-w/D(z,w)$, the infinite critical points, $(z,0)$, in the local charts $\mathcal{U}_1$ and $\mathcal{U}_2$ are given by the roots of $fz^2+ez+d=0$ or $dz^2+ez+f=0$, respectively. Thus (i) follows.

(ii) Let us prove that if $\Delta\geq 0$, then there is rotation such that $f=0$ can be assumed. Consider the linear change of coordinates

$$\begin{pmatrix}\widetilde{x}\\ \widetilde{y}\end{pmatrix}=\begin{pmatrix}\cos\phi & -\sin\phi\\ \sin\phi & \cos\phi\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}.$$

Easy calculations show that if $\Delta\geq 0$ then, by taking ϕ a real solution of the equation,

$$d\tan^2\phi+e\tan\phi+f=0,$$

we get a new system of the form (3) with $f=0$ as we wanted to prove. The other assertions given in (ii) follow by using a suitable re-scaling of the form $\widetilde{x}=\alpha x$, $\widetilde{y}=\beta y$ and $\widetilde{t}=\gamma t$.

(iii) Note that since in polar coordinates $\dot{\theta}\equiv 1$, the origin is the unique finite critical point of system (3). It is also clear that it is of center or focus type. By using the usual methods, (see for instance [6] or [8]), we get the expressions of V_1, V_3 and V_5. To prove that when $V_1=V_3=V_5=0$ the origin is a center we refer to [3]. An easy example giving rise to two limit cycles is $b=e=0$, $c=d=1$, $f=\varepsilon-1$ with $a>0$ and $\varepsilon<0$ and satisfying $|a|\ll|\varepsilon|\ll 1$. Notice that these systems are inside Family (III) because $\Delta=4(1-\varepsilon)>0$.

(iv) Writing system (3) as $\dot{x}=P(x,y,\mu)$ and $\dot{y}=Q(x,y,\mu)$, where $\mu\in\{a,d,f\}$, we have

$$\begin{vmatrix} P(x,y,\mu) & Q(x,y,\mu)\\ \partial P(x,y,\mu)/\partial\mu & \partial Q(x,y,\mu)/\partial\mu\end{vmatrix}\leq 0,$$

where the equality holds only on non-invariant straight lines or nowhere, depending on the μ chosen. Thus (iv) follows. □

Remark 2.2. *(i) In* [3] *the phase portraits of the centers inside system (3) are given. Observe that from Proposition 2.1.iii, the centers only appear in family (III). Although from our results we can also obtain the results of that paper, we do not repeat that study here.*

(ii) Notice that to assume $f=0$ in system (3), when $\Delta\geq 0$, means to rotate such system in order to place one infinite critical point at $(0,0)\in\mathcal{U}_2$.

From Proposition 2.1.i, we can split the study of system (3) into families (I), (II) and (III) according with $\Delta<0$, $\Delta=0$ and $\Delta>0$, respectively.

It will be also useful to have system (3) expressed in (r,θ)-polar coordinates. This is done in the following lemma. Its proof is immediate.

Lemma 2.3. *The periodic solutions of system (3) correspond to positive solutions of the Abel differential equation*

$$\frac{dr}{d\theta} = (d\cos^2\theta + e\cos\theta\sin\theta + f\sin^2\theta)r^3 + (b\cos\theta + c\sin\theta)r^2 + ar, \qquad (6)$$

that satisfy $r(0) = r(2\pi)$.

Usually the solutions of (6) that satisfy $r(0) = r(2\pi)$ are also called periodic solutions. Furthermore, if they are isolated then are called limit cycles of (6). It is important to notice that in [13], the author proves that there is no upper bound for the number of limit cycles for Abel equations $dr/d\theta = A(\theta)r^3 + B(\theta)r^2 + C(\theta)r$, when A, B and C are arbitrary trigonometrical polynomials.

3. Proof of Theorem 1.1

The goal of this section is to obtain the bifurcation diagrams and the phase portraits on the Poincaré sphere (modulus a change of sign of the independent variable t) of system (3) with $\Delta = e^2 - 4df \le 0$. By using Proposition 2.1.i we can take $f = 1$ and $d > 0$ (resp. $e = f = 0$, $d = 1$ and $c \ge 0$) when $\Delta < 0$ (resp. $\Delta = 0$). Notice that we obtain families (I) and (II), respectively.

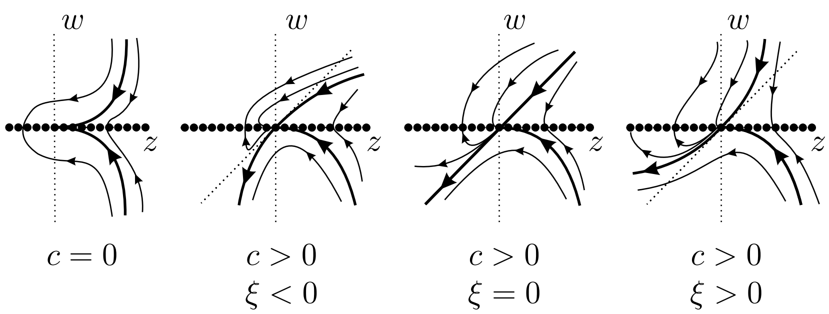


Figure 4. Phase portraits of family (II) in a neighborhood of $(0,0) \in \mathcal{U}_2$, in accordance with c and ξ. Recall that $\xi = d + bc + ac^2$.

Next lemma gives the behavior of the orbits of families (I) and (II) near infinity.

Lemma 3.1. *Consider the vector field X associated to system (3) when $\Delta \le 0$ and $d > 0$.*

(i) If $\Delta < 0$, then the equator of its Poincare compactification, $\mathcal{P}(X)$, is attractor for the flow of $\mathcal{P}(X)$.

(ii) If $\Delta = 0$, after a re-parametrization of the time, the singularity $(0,0)$ in the local chart $\mathcal{U}_2$ of $\mathcal{P}(X)$ is:
  - *(a) A cusp point with the two separatrices tangent to $w = 0$, when $c = 0$.*
  - *(b) A saddle-node point, two of whose separatrices tend to $(0,0)$ in the direction given by the straight line $w = cz$ and the other in the direction $w = 0$, when $c > 0$.*

Furthermore, in the case $\Delta = 0$ and $c = 0$ the infinity is an attractor for the flow. See Figure 4 for more details.

Proof. If $\Delta < 0$ then, from expressions (4) and (5), we have that $w'|_{\{w=0\}}$ is always positive, where the prime denotes the derivative with respect to s, being $ds/dt = -w/D(z,w)$. Hence, the equator of $\mathbb{S}^2$ is an attractor for the flow of $\mathcal{P}(X)$.

If $\Delta = 0$ then, from Proposition 2.1.ii, it is enough to consider equation (5) with $e = f = 0$ and $d = 1$. In the new time variable s introduced above, this equation becomes

$$\begin{cases} z' &= w + z^2 w, \\ w' &= cw + z^2 + bzw + aw^2 + zw^2. \end{cases}$$

If $c = 0$, we use Theorem 67 of [1] to obtain that the origin, in the local chart $\mathcal{U}_2$, is a singularity whose neighbourhood is the union of two hyperbolic sectors. After introducing again the time t, we obtain that the equator of $\mathbb{S}^2$ is attractor for the flow, see Figure 4.

If $c > 0$, after the change of variables $\widetilde{z} = cz - w$, $\widetilde{w} = w$, $\widetilde{s} = cs$ we can apply Theorem 65 of [1]. We obtain that $(0,0) \in \mathcal{U}_2$ is a saddle-node, two of whose separatrices tend to $(0,0)$ in the directions $\pi/2$ and $3\pi/2$ and the other in the direction 0. To obtain Figure 4, we undo the changes of the variables that we have made. Furthermore, by introducing the parameter $\xi = d+bc+ac^2$, we get that the line $cz - w = 0$ is invariant when $\xi = 0$ and the relative situation of this straight line and two of the separatrices depend on the sign of this parameter, see again Figure 4. □

In the next proposition we control the limit cycles of system (3) for families (I) and (II).

Proposition 3.2. *Consider system (3) when $\Delta = e^2 - 4df \leq 0$ and $d > 0$. Then it has no centers and has at most one limit cycle. Furthermore, the limit cycle can exist only when $a < 0$ and it is hyperbolic.*

Proof. If $a = 0$ then, from Proposition 2.1.iii, we have that $V_3 > 0$ for system (3). Hence, system (3) cannot have a center at the origin. Even more, by taking $a \lesssim 0$ it is possible to generate, as in the proof of Proposition 2.1.iii.c, one, and only one, small-amplitude limit cycle around it by an Andronov-Hopf bifurcation.

To prove that system (3), under our hypotheses, has at most one limit cycle, let us consider its associated Abel equation given in Lemma 2.3,

$$\frac{dr}{d\theta} = A(\theta)r^3 + B(\theta)r^2 + C(\theta)r,$$

where $A(\theta) = d\cos^2\theta + e\cos\theta\sin\theta + f\sin^2\theta$, $B(\theta) = b\cos\theta + c\sin\theta$ and $C(\theta) = a$. Observe that the hypothesis $\Delta \leq 0$ implies that $A(\theta)$ does not change sign. Therefore from [7, Th. A], we get that equation (3) has at most three limit cycles (taking into account their multiplicities) and being one of them $r = 0$. On the other hand, it can be easily checked that if $r(\theta)$ is a periodic solution of equation (3) then $-r(\pi + \theta)$ is also a periodic solution, consequently we get that our Abel equation has at most one limit cycle in the half strip $r > 0$, and that when it exists it is hyperbolic. In other words, by using Lemma 2.3, we have proved that system (3) has at most one (hyperbolic) limit cycle.

Let us see that in fact it can only exist only when $a < 0$. From the Andronov-Hopf bifurcation we have seen that the small-amplitude limit cycle is generated when $a \lesssim 0$. By using Proposition 2.1.iv, we know that our system is a complete family of rotated vector fields with respect the parameter a. Hence, the small-amplitude limit cycle born from the origin must expand, covering an annular region, when a is varied in a decreasing way, the outer boundary of it consisting of an unbounded polycycle, i.e. $(0, 0) \in \mathcal{U}_2$ (in the equator of $\mathbb{S}^2$) belongs to its outer boundary. There is an important property of one-parametric family of rotated vector fields, usually called *non intersection property.* It reads as follows: limit cycles of distinct vector fields of a semi-complete family do not intersect. Consequently, using the above facts and this property we get that, if $a \geq 0$ system (3) under our hypotheses, does not have limit cycles, because all the region of possible existence of limit cycles is covered limit cycles existing for the negative values of a. □

Remark 3.3. *In the cases either $\Delta < 0$ or $\Delta = 0$ and $c = 0$ is possible to use an easier argument to conclude that the limit cycle exists only if $a < 0$. From Proposition 3.2 the system has at most one hyperbolic limit and when it exists it is hyperbolic. On the other hand, in these cases, if $a \geq 0$, from the stability of the critical point at the origin, given by Proposition 2.1.iii and the stability of the equator of $\mathbb{S}^2$, given in Lemma 3.1, in case of existence of the limit cycle it would be internally stable and externally unstable. This gives a contradiction with its hyperbolicity. Hence no limit cycles exist for $a \geq 0$.*

Next result give us additional information to draw the phase portraits of families (I) and $(\text{II})_{c=0}$. It is a straightforward consequence of Proposition 3.2 and of the above remark.

Lemma 3.4. *When $d > 0$ and either $\Delta < 0$ or $\Delta = c = 0$, the limit cycle of system (3) exists if and only if $a < 0$.*

Collecting all the above results the proof of Theorem 1.1.i follows.

Remark 3.5. *Notice that although the phase portraits of families (I) and $(II)_{c=0}$ are topologically equivalent, there is a difference between them. After a rescalling, in family (I) there are no critical points at infinity, while in family (II) there is a cusp point on the equator.*

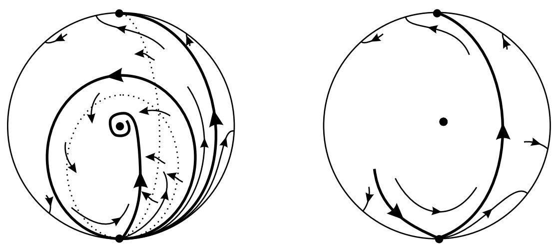
To end the proof of Theorem 1.1 let us consider family (II) when $c > 0$. To control the limit cycle in this case the following result will be useful.

Proposition 3.6. *Let us consider system (3) when $\Delta = 0$, $c > 0$ and $d = 1$. Then, for each pair of real values of b and c there exists a value, $a^* = a^*(b, c) < 0$, such that for $a = a^*$, system (3) has an infinite homoclinic saddle-node loop.*

Proof. Let us fix a pair of arbitrary b and $c > 0$ real parameters in system (3). We note that there exists a value, $a_1 = -(1 + bc)/c^2$ (corresponding to $\xi = 0$ in Figure 4) for the parameter a such that the equation (3) has $x = 1/c$, as an invariant straight line. The local chart $\mathcal{U}_2$ is introduced in the proof of Proposition 2.1. Following [12, 16], in this prove we denote by $\mathcal{V}_2$ the symmetric chart of

$\mathcal{U}_2$ with respect to the origin. The line $x = 1/c$ connects the unstable separatrix of $(0,0) \in \mathcal{V}_2$ with the stable one of $(0,0) \in \mathcal{U}_2$. This fact forces that the α-limit set of the stable separatrix of $(0,0) \in \mathcal{V}_2$ is, from Proposition 3.2, $(0,0) \in \mathbb{R}^2$ (resp. a limit cycle around $(0,0) \in \mathbb{R}^2$) if $a_1 \geq 0$ (resp. $a_1 < 0$). See Figure 5.b, for more details.

The second fact that we will prove is that there exists a value, a_2, $a_2 < a_1$, for the parameter a for which the ω-limit set of the unstable separatrix of $(0,0) \in \mathcal{V}_2$ is $(0,0) \in \mathbb{R}^2$. To prove this it is enough to take into account the following thinks. By one hand, for $\eta_1 = -(2+bc)/c^2$ and $\eta_2 = (c^4 + 4 + 4bc + b^2c^2)/(4c^3)$ and a negative enough, the parabola $y = -x^2/c + \eta_1 x + \eta_2$ is without contact with the flow associated with system (3). More specifically, the subset of $\mathbb{R}^2$ given by $\{(x,y) \,:\, y < -x^2/c + \eta_1 x + \eta_2\}$ is positively invariant with respect to this flow. Secondly, from Lemma 3.1.ii.b, if we take a negative enough parameter a in such a way that $\xi = d + bc + ac^2 < 0$, then the unstable separatrix of the saddle-node point is tangent at $(0,0) \in \mathcal{V}_2$ to the straight line $x = 1/c$ and its ω-limit set must be $(0,0) \in \mathbb{R}^2$. Let us call a_2 a negative enough parameter a for which the ω-limit set of the unstable separatrix of $(0,0) \in \mathcal{U}_2$ is $(0,0) \in \mathbb{R}^2$. See Figure 5.a, for more details. Hence, since system (3) is a complete family of rotated vector fields



(a) (b)

FIGURE 5. Phase portrait of family (II) for $c > 0$ and some value of the parameter a exhibiting: (a) that the ω-limit set of the unstable separatrix of $(0,0) \in \mathcal{U}_2$ is $(0,0) \in \mathbb{R}^2$, when a is negative enough, (b) $x = 1/c$, as an invariant straight line.

with respect the parameter a, there exists a unique value $a^* < 0$, $a_2 < a^* < a_1$, for the parameter a such that system (3) has an infinite homoclinic saddle loop connection. Whence, the proposition follows. □

Using former results it is possible to draw the phase portraits of family (II) when $c > 0$ as in Figure 2, according the bifurcation diagram depicted also in Figure 2. The bifurcation curve $\mathcal{L}$ gives the infinite homoclinic saddle loop connection and we have obtained it numerically for some values of the parameters. The other curves represented in that figure correspond with the line $\xi = ac^2 + bc + 1 = 0$, where the phase portraits have $x = 1/c$ as an invariant straight line (see Figure 2.5 and Figure 2.8) and the line $a = 0$, where the phase portraits have a Hopf bifurcation at $(0,0) \in \mathbb{R}^2$ (see Figure 2.4).

From all the above results part (ii) of Theorem 1.1 follows.

4. Perturbation of the linear center

Consider the next perturbed system

$$\left\{\begin{array}{rcl} \dot{x} & = & -y + x\sum\limits_{i\geq 1}\varepsilon^i F_i(x,y), \\ \dot{y} & = & x + y\sum\limits_{i\geq 1}\varepsilon^i F_i(x,y), \end{array}\right. \tag{7}$$

with $F_i(x,y) = a_i + b_i x + c_i y + d_i x^2 + e_i xy + f_i y^2$. Notice that it is a subfamily of systems of the form (3). Let (r,θ) be the usual polar coordinates and set $H(x,y) = (x^2+y^2)/2$. Define $\rho = H(r\cos\theta, r\sin\theta) = r^2/2$. Let $L(\rho,\varepsilon)$ be the return map of (7) associated to the OX^+-axis parameterized by the energy, ρ. It writes as

$$L(\rho,\varepsilon) = \rho + \varepsilon^k 4\pi L_k(\rho) + O(\varepsilon^{k+1}),$$

being $4\pi L_k(\rho)$ the first non-zero term of its ε-Taylor expansion at 0. The function $L_k(\rho)$ is usually called the k-th Melnikov function of system (7). It is well known that the simple positive zeroes of L_k give rise to hyperbolic limit cycles of (7) which tend, when ε goes to zero, to the level curves of H corresponding to these zeroes. Let us define the maximum number of simple positive zeroes of L_k as the *k-th Melnikov number,* $\mathbf{M}^k$, of system (7). From Poincaré's work it is well known that

$$\begin{aligned} L_1(\rho) &= \frac{1}{4\pi}\int_{H=\rho} F_1(x,y)(xdy - ydx) \\ &= \frac{1}{4\pi}\int_0^{2\pi} 2\rho F_1(\sqrt{2\rho}\cos\theta, \sqrt{2\rho}\sin\theta)d\theta = (f_1+d_1)\rho^2 + a_1\rho. \end{aligned}$$

By imposing that $L_1(\rho)\equiv L_2(\rho)\equiv\cdots\equiv L_{k-1}(\rho)\equiv 0$, the method developed in [9] allows to compute $L_k(\rho)$, for any k. For the first values of k we get

$$\begin{aligned} L_2(\rho) &= (f_2+d_2)\rho^2 + a_2\rho, \\ L_3(\rho) &= (b_1^2 f_1 - f_1 c_1^2 - c_1 b_1 e_1)\rho^3 + (f_3+d_3)\rho^2 + a_3\rho, \\ L_4(\rho) &= (-b_1 e_1 c_2 + f_2 b_1^2 - c_1 b_1 e_2 - 2c_1 f_1 c_2 + 2 f_1 b_1 b_2 - f_2 c_1^2 - e_1 c_1 b_2)\rho^3 \\ &\quad + (f_4+d_4)\rho^2 + a_4\rho. \end{aligned}$$

Making all the computations up to order 8 we obtain the main result of this section:

Theorem 4.1. *(i) For $k = 1,2,\ldots 8$, the Melnikov functions associated to (7) are polynomials of degree 3 in ρ of the form*

$$L_k(r) = m_{2,k}\rho^3 + m_{1,k}\rho^2 + m_{0,k}\rho,$$

where the coefficients $m_{i,k}$ are also polynomials in the coefficients of the polynomials $F_i(x,y)$, $i = 1,2,\ldots,8$.

(ii) The first Melnikov numbers associated to system (7) are $\mathbf{M}^1 = \mathbf{M}^2 = 1$ and $\mathbf{M}^k = 2$ for $k = 3,\ldots,8$.

Notice that the above result shows again that $\mathcal{H}(1,2) \geq 2$ and provides some more support to the fact that $\mathcal{H}(1,2)$ is 2.

5. Family (III)

Let us consider system (3) with $\Delta = e^2 - 4df > 0$. From Proposition 2.1.i, its Poincaré compactification has four critical points at infinity after a re-scaling of the variable t. Furthermore, from Proposition 2.1.ii.c, we can take $f = 0$ (therefore $e \neq 0$) and either $d = 0$ or $d = 1$. Thus we have reduced the study of the case $\Delta > 0$ to the study of family (III).

For this family we have not been able to prove that the maximum number of limit cycles is at most two. However, all our results, presented in Proposition 2.1.iii and in Section 4, give at most two limit cycles.

For a particular selection of the parameters, we show in Figure 3, an evolution of the phase portraits of family (III), in terms of the rotatory parameter a. This evolution exhibits, at least, two limit cycles and somehow is representative of the complexity the family. To depict Figure 3 we need the following results.

The first one is a technical lemma that controls the behavior of the orbits near infinity. Its proof follows from standard techniques and Proposition 2.1.

Lemma 5.1. *Consider the vector field X associated with system (3) and the local charts $\mathcal{U}_1$ and $\mathcal{U}_2$, given by (4) and (5) respectively, of $\mathcal{P}(X)$ with coordinates (z,w), when $\Delta > 0$, $f = 0$, $e \neq 0$ and either $d = 0$ or $d = 1$. Then, after a re-scaling of the variable t, $(0,0) \in \mathcal{U}_2$ and $(-d/e, 0) \in \mathcal{U}_1$ are the singularities at the equator of $\mathbb{S}^2$. Even more,*

(i) $(0,0) \in \mathcal{U}_2$ is a singularity of type: saddle if $e > 0$, unstable (resp. stable) node if $e < 0$, $c^2 + 4e \geq 0$ and $c < 0$ (resp. $c > 0$), unstable (resp. stable) focus if $e < 0$, $c^2 + 4e < 0$ and $c < 0$ (resp. $c > 0$), linear center if $e < 0$ and $c = 0$.

(ii) $(-d/e, 0) \in \mathcal{U}_1$ is a singularity of type: saddle if $e < 0$, stable (resp. unstable) focus if $e > 0$, $\delta < 0$ and $\sigma < 0$ (resp. $\sigma > 0$), linear center if $e > 0$ and $\delta < 0$ and $\sigma = 0$, stable (resp. unstable) node if $e > 0$, $\delta \geq 0$ and $\sigma < 0$ (resp. $\sigma > 0$), where $\sigma = cd - be$ and $\delta = \sigma^2 - 4e(d^2 + e^2)$.

The second result gives some properties of the phase portrait of system (3), for some selected parameters. Its proof follows from straightforward computations and from the fact that, from Proposition 2.1.iv, system (3) is a semi-complete family of rotated vector fields with respect the parameter a. In particular, the proof of part (ii) is similar to the proof of Proposition 3.6.

Lemma 5.2. *Consider the vector field $X := X_a$ associated to system (3) with $b = 1$, $c = 3$, $d = 1$, $e = 5$ and $f = 0$.*

(i) If $a = \frac{-2\mp\sqrt{29}}{25}$ then X_a has an invariant vertical straight line which equation is $x = \frac{-3\pm\sqrt{29}}{10}$. See Figures 3.2 and 3.9.

(ii) There exists an unique value a_L for the parameter a for which X_{a_L} has an infinite homoclinic saddle loop connection. See Figure 3.7.

(iii) There exists a value a_{SS} for the parameter a for which $X_{a_{SS}}$ has a semi-stable limit cycle. It is externally (resp. internally) stable (resp. unstable). See Figure 3.4.

By using again that system (3) is a semi-complete family of rotated vector fields with respect the parameter a and assuming that X_a presents at most two limit cycles, Figure 3 follows. Some numerical computations show that $a_L \simeq 0.0135$ and $a_{SS} \simeq -0.029101$.

References

[1] A. A. Andronov, E. A. Leontovich, I. I. Gordon, and A. G. Maĭer, *Qualitative theory of second-order dynamic systems*, Halsted Press (A division of John Wiley & Sons), New York-Toronto, Ont., 1973, Translated from the Russian by D. Louvish.

[2] M. A. M. Alwash, *On the center conditions of certain cubic systems*, Proc. Amer. Math. Soc. **126** (1998), no. 11, 3335–3336.

[3] C. B. Collins, *Conditions for a centre in a simple class of cubic systems*, Differential and Integral Equations **10** (1997), no. 2, 333–356.

[4] R. Conti, *Uniformly isochronous centers of polynomial systems in* $\mathbf{R}^2$, Differential equations, dynamical systems, and control science, Lecture Notes in Pure and Appl. Math., vol. 152, Dekker, New York, 1994, pp. 21–31.

[5] G. F. D. Duff, *Limit-cycles and rotated vector fields*, Ann. of Math. (2) **57** (1953), 15–31.

[6] W. W. Farr, Chengzhi Li, I. S. Labouriau, and W. F. Langford, *Degenerate Hoph bifurcation formulas and Hilbert's 16th problem*, SIAM J. Math. Anal. **20** (1989), 13–29.

[7] A. Gasull and J. Llibre, *Limit cycles for a class of Abel equations*, SIAM J. Math. Anal. **21** (1990), no. 5, 1235–1244.

[8] A. Gasull and J. Torregrosa, *A new approach to the computation of the Lyapunov constants*, Comput. Appl. Math., **20** (2001) no. 1-2 pp. 149–177.

[9] A. Gasull and J. Torregrosa, *A relation between small amplitude and big limit cycles*, Rocky Mountain J. Math. **31** (2001), no. 4, 1277–1303.

[10] A. Gasull and J. Torregrosa, *Exact number of limit cycles for a family of rigid systems*, to appear in Proc. Amer. Math. Soc. Prepublicacions UAB, no. 30, 2003.

[11] A. Gasull and J. Torregrosa, *Some results on rigid systems*, to appear in Proc. Equadiff 2003. Preprint, 2003.

[12] E. A. González, *Generic properties of polynomial vector fields at infinity*, Trans. Amer. Math. Soc. **143** (1969) pp. 201–222.

[13] A. Lins Neto, *On the number of solutions of the equation* $dx/dt = \sum_{j=0}^{n} a_j(t)x^j$, $0 \le t \le 1$, *for which* $x(0) = x(1)$, Invent. Math. **59** (1980), no. 1, 67–76.

[14] L. Mazzi and M. Sabatini, *Commutators and linearizations of isochronous centers*, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. **11** (2000), no. 2, 81–98.

[15] L. Perko, *Differential equations and dynamical systems*, third ed., Texts in Applied Mathematics, vol. 7, Springer-Verlag, New York, 2001.

[16] J. Sotomayor, *Curvas definidas por equaçoes diferenciais no plano*, 13^o Colóquio Brasileiro de Matemática. [13th Brazilian Mathematics Colloquium], Instituto de Matemática Pura e Aplicada Conselho Nacional de Desenvolvimento Científico e Tecnológico, Rio de Janeiro, 1981.

Dept. de Matemàtiques, Universitat Autònoma de Barcelona, Edifici C 08193 Bellaterra, Barcelona. Spain
E-mail address: gasull@mat.uab.es

Dept. de Matemàtiques i Informàtica, Escola Politècnica Superior, Universitat de les Illes Balears, 07071, Palma de Mallorca. Spain
E-mail address: rafel.prohens@uib.es

Dept. de Matemàtiques, Universitat Autònoma de Barcelona, Edifici C 08193 Bellaterra, Barcelona. Spain
E-mail address: torre@mat.uab.es