CENTER PROBLEM FOR SEVERAL DIFFERENTIAL EQUATIONS VIA CHERKAS METHOD

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Abstract. Cherkas method characterizes centers for analytic Liénard differential equations. We extend his method to degenerate Liénard differential equations and we apply this extension to solve the center problem for several families of polynomial differential equations. In particular we give all centers for some differential equations given by a vector field which is sum of two quasi-homogeneous ones. Finally we make some remarks for $\mathcal{C}^1$ Liénard equations.

1. Introduction.

The characterization of centers for concrete families of differential equations is a problem which has extensively been studied during the last decades. When the critical point is non degenerate (it has purely imaginary eigenvalues) the method of computing its Lyapunov constants solves theoretically the problem. In most cases the procedure to study all centers follows next idea: compute several Lyapunov constants and when you get one significative constant which is zero, try to prove that the system obtained indeed has a center. The described method has two main difficulties: How are you sure that you have computed enough Lyapunov constants? How do you prove that some system candidate to have a center, has actually a center? Anyway, this procedure has been used to study (and in some cases solve) the center problem for a lot of families of differential equations. Among other we can quote: quadratic systems [4], systems with homogeneous nonlinearities of degree 3, 4, 5 [25], [9], [10], Kukles system [14], quadratic-like cubic systems [8], [20], ... See also the paper [24].

There is a family, the Liénard differential equations, for which the second problem stated above is much more easy. To be exact, Cherkas in [11] gave a result, later developped in [13], (see also Theorem 2.6 of this paper) which characterizes when the origin of an analytic differential equation of type

$$\dot{x} = \varphi(y) - F(x), \ \dot{y} = -g(x), \tag{1}$$

is a center, where $\varphi(y) = y + O(y^2)$, $F(x) = O(x)$, and $g(x) = x + O(x^2)$.

Remember that $O(h(x))$ and $o(h(x))$ denote functions such that $\lim_{x\to 0} \frac{O(h(x))}{h(x)} = c \in \mathbb{R}$ and $\lim_{x\to 0} \frac{o(h(x))}{h(x)} = 0$, respectively.

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We remark that while most methods to prove when a system has a center give sufficient conditions (reversibility of the system, existence of an integrating factor, ...) the method developped by Cherkas gives a necessary and sufficient condition. This fact makes it very useful to study the center problem for differential equations that can be transformed into (1). This idea has been already exploited by other authors, see [12] and [21].

In this paper, first we will make a generalization of Cherkas result when $g(x) = x^{2l-1} + O(x^{2l})$ and l is a natural number. This generalization is made to apply Cherkas method to characterize the centers for the next families of differential equations,

$$\begin{aligned} \dot{x} &= -y + a_1xy + a_2x^{q+1}, \\ \dot{y} &= x^{2q-1} + b_1y^2 + b_2x^qy + b_3x^{2q}, \ q \in \mathbb{N} \end{aligned} \tag{2}$$

$$\begin{aligned} \dot{x} &= y - (a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5), \\ \dot{y} &= -(b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5). \end{aligned} \tag{3}$$

Observe that differential equation (2) is defined by a vector field which is sum of two quasi-homogeneous vector fields of degrees 1 and q ($q \geq 2$), see [6] or [7]. Furthermore, the study of the quadratic systems can be reduced (via a rotation) to equation (2) with $q = 1$. In [7] system (2) is also studied for $q \geq 2$. In that paper the authors give all families of (2) which admit a first integral which is sum of quasi-homogeneous polynomials. Of course all the systems obtained in that paper are centers. We prove that for all $q \geq 2$ there are more centers inside family (2). In fact we give all centers for (2) when q is even, see Theorem 3.3, all centers for $q = 1$ and several families of centers for $q > 1$, odd, see Theorem 3.5. We remark that we have not been able to solve even the easiest odd case $q = 3$ due to computational difficulties that will be described with more details in Section 3 of this paper.

We also note that our results give a new way to characterize all centers for quadratic systems. This new way allows to study all cases with an unified treatment, see Theorem 3.5(ii) in contrast to other methods which classify the quadratic systems in types: reversible, Hamiltonian, ...

All centers of differential equation (3) are already known when $b_1 \neq 0$, see for instance [23]. We continue the study to the case $b_1 = 0$. In this case, first we characterize the cases in which the origin can be either a focus or a center and finally we solve the center problem, see Theorem 3.8.

Most of the computations have been carried out using MAPLE V.4.

This paper is organized as follows: in Section 2 we prove the main result concerning the equivalent necessary and sufficient conditions for a degenerate (or not) Liénard differential equation to have a center at the origin, see Theorem 2.6. We stress that some of these conditions are new. This section also contains some preliminary results about Liénard systems. Section 3 has two parts. The first one deals with the system (2). First we show how to transform it into a degenerate Liénard differential equation and later we apply the results of the previous section to characterize the centers. This subsection is the one which involves more complicated computations and for some cases we just give a scheme of the proofs. Some more technical details are developped in the Appendix which is the last section of the paper. The second part of Section 3 solves the center problem for system (3), which is already in Liénard form. In Section 4 we make some remarks on equation (1) when the involved functions are just of class $\mathcal{C}^1$.

2. Liénard systems.

In this section we present a generalization of results of Cherkas [11], later developped by Christopher, Lloyd and Pearson [13], about Liénard differential equations. This generalization is needed to study systems (2) and (3), but we think that it is interesting by itself.

We will deal with the Liénard differential equation

$$\dot{x} = \frac{dx}{dt} = \varphi(y) - F(x), \;\; \dot{y} = \frac{dy}{dt} = -g(x), \tag{4}$$

where φ, F and g are analytic functions satisfying

$$\begin{aligned} \varphi(y) &= y^{2m-1} + O(y^{2m}), \\ F(x) &= a_k x^k + O(x^{k+1}), \\ G(x) &= \int_0^x g(s)\, ds = \frac{x^{2l}}{2l} + O(x^{2l+1}), \end{aligned}$$

with $m, k, l \in \mathbb{N}$ being non zero. As far as we know, the case $l > 1$ is just studied in some special situations, see for instance [28].

To study system (4) we need to introduce some notation and preliminary results.

Following the ideas of [17] for $l = 1$, we define u as

$$u := \Phi(x) := \operatorname{sign}(x) \sqrt[2l]{2lG(x)} = x \sqrt[2l]{2l\frac{G(x)}{x^{2l}}} = x \sqrt[2l]{1 + O(x)}.$$

This function is analytic at zero and has an analytic inverse which we call $\xi(u) = u(1 + O(u))$. It turns out that

$$u = \Phi(\xi(u)) = \xi(u) \sqrt[2l]{2l\frac{G(\xi(u))}{\xi(u)^{2l}}}. \tag{5}$$

Next lemma is a generalization of well-known results, see for instance [17] or [26].

Lemma 2.1. *With the change of variables*

$$u = x \sqrt[2l]{2l\frac{G(x)}{x^{2l}}}, \; y = y,$$

and the change of time $\frac{dt}{ds} = \frac{u^{2l-1}}{g(x)}$, *system (4) writes in a neighbourhood of* $(0,0)$ *as*

$$u' = \frac{du}{ds} = \varphi(y) - F(\xi(u)), \, y' = \frac{dy}{ds} = -u^{2l-1}. \tag{6}$$

Proof. Direct computations give

$$\begin{aligned}\dot{u} &= u^{1-2l}g(x)(\varphi(y)-F(x)),\\ \dot{y} &= -g(x),\end{aligned}$$

and hence

$$\begin{aligned}\frac{du}{ds}=\frac{du}{dt}\frac{dt}{ds} &= \varphi(y)-F(\xi(u)),\\ \frac{dy}{ds}=\frac{dy}{dt}\frac{dt}{ds} &= -u^{2l-1}.\end{aligned}$$

Notice that from the expressions of $g(x)=x^{2l-1}+O(x^{2l})$ and $u=x+O(x^2)$ it is clear that $\frac{dt}{ds}\neq 0$ in a neighbourhood of the origin. □

Next lemma is similar to Lemma 2.1 of [19].

Lemma 2.2. *Consider system (6) with $\varphi(y) = y^{2m-1}+d_{2m}y^{2m}+O(y^{2m+1})$, $F(\xi(u)) = f_k u^k + O(u^{k+1})$ and $mk+l-2ml>0$. Then its origin is a center if and only if $F(\xi(u))$ is an even function.*

Proof. In the first part of the proof we will show that near the origin the orbits turn around it. To do this we use the Lyapunov polar coordinates r,ϕ given by

$$u=r^m\mathrm{Cs}(\phi),\ \ y=r^l\mathrm{Sn}(\phi),$$

where the functions $z(\phi)=\mathrm{Cs}(\phi)$ and $w(\phi)=\mathrm{Sn}(\phi)$ are the solutions of the Cauchy problem

$$\begin{aligned}&\dot{z}=-w^{2m-1}, z(0)=(1/m)^{1/(2l)},\\ &\dot{w}=z^{2l-1},\quad w(0)=0,\end{aligned}$$

see [16] or [22] for more details. In particular we will use that $m\mathrm{Cs}^{2l}(\phi)+l\mathrm{Sn}^{2m}(\phi)=1$.

In these new coordinates and with a new time variable τ given by $\frac{dt}{d\tau}=r^{l+m-2ml}$ the expression of (6) is

$$\begin{aligned}\frac{dr}{d\tau} &= r^{l+m+1-4ml}[u^{2l-1}\dot{u}+y^{2m-1}\dot{y}],\\ \frac{d\phi}{d\tau} &= r^{-2ml}[m\dot{y}u-l\dot{u}y],\end{aligned}$$

that is

$$\begin{aligned}\frac{dr}{d\tau} &= O(r^{p+1}),\\ \frac{d\phi}{d\tau} &= -1+Ar^p+O(r^{p+1}),\end{aligned}$$

where

$$p=\begin{cases} l & \text{when } k-2l\geq 0,\\ mk+l-2ml & \text{when } k-2l<0,\end{cases}$$

and

$$A = \begin{cases} -ld_{2m}\mathrm{Sn}^{2m+1}(\phi) & \text{when } k-2l>0, \\ -ld_{2m}\mathrm{Sn}^{2m+1}(\phi)+lf_k\mathrm{Sn}(\phi)\mathrm{Cs}^k(\phi) & \text{when } k-2l=0, \\ lf_k\mathrm{Sn}(\phi)\mathrm{Cs}^k(\phi) & \text{when } k-2l<0. \end{cases}$$

In any case, by using that $mk+l-2ml>0$ we have proved that the origin is either a focus or a center. At this point the proof follows like the proof of Lemma 2.1 of [13]. In fact it suffices to compare the orbits of (6) taking instead of $F(\xi(u))$, its even part $\frac{F(\xi(u))+F(\xi(-u))}{2}$. Observe that this latest case has a center at the origin. □

Remark 2.3. (i) *Observe that when in the statement of Lemma 2.2, $mk+l-2ml<0$, the origin cannot be either a focus or a center because the origin has some trajectories which reach or leave it with directions ϕ satisfying $Sn(\phi)=0$.*

(ii) *The case $mk+l-2ml=0$ is much more delicate. The type of critical point depends on the value f_s. In fact if the function $H(\phi):=-1+lf_kSn(\phi)\,Cs^k(\phi)$ is always negative the origin is either a focus or a center and the same results of Lemma 2.2 hold. When the function H changes sign we are in the same situation than in (i). The case $H(\phi)\le 0$ is more complicated. We just study the case $m=1$ in Lemma 2.5.*

Remark 2.4. *The results of Lemma 2.2 also follow from Theorem A.(a) of* [15]. *Finally note that these results, when $l=1$ and $\varphi(y)=y$, are also a consequence of the fact that the $(2k+1)$-th Lyapunov constant of system (6) is a multiple of f_{2k+1} for $k\ge 1$, see* [5] *or* [29].

Lemma 2.5. *Consider equation (4) with $m=1$ and a more general $g(x)$ of the form $g(x)=b_nx^n+O(x^{n+1})$. Then its origin is either a center or a focus if and only if one of the following three conditions is satisfied*

(i) $n=1$, $a_1^2-4b_1<0$,
(ii) $n>1$, *odd*, $a_1=0$, $n<2k-1$, $b_n>0$,
(iii) $n>1$, $n=2k-1$, $ka_k^2-4b_n<0$.

Proof. Let X denote the vector field associated to (4). When $n=1$, $DX(0)=\begin{pmatrix} -a_1 & 1 \\ -b_1 & 0 \end{pmatrix}$. By imposing that the eigenvalues of the matrix are complex we are done. When $n>1$, $DX(0)=\begin{pmatrix} -a_1 & 1 \\ 0 & 0 \end{pmatrix}$. If $a_1\neq 0$, the theorem of classification of such kind of critical points (Theorem 65 of [2]) implies that there are neither foci nor centers. When $a_1=0$, the results of [1] characterize the points which are either center or focus. Straightforward computations prove the lemma. □

Next theorem is the main result of this section. The first three points are a generalization of known results for $l=1$, see [11] and [13]. As far as we know points (iv-v) are new equivalent characterizations of centers. As we will see in the last section, the most useful of the given characterizations turns out to be (v).

Theorem 2.6. *Consider system (4), that is*

$$\dot{x} = \varphi(y) - F(x),\ \dot{y} = -g(x), \tag{7}$$

where all the involved functions are analytic and satisfy

$$\begin{aligned} \varphi(y) &= y^{2m-1} + O(y^{2m}), \\ F(x) &= a_k x^k + O(x^{k+1}), \\ G(x) &= \int_0^x g(s)\,ds = \frac{x^{2l}}{2l} + O(x^{2l+1}), \end{aligned}$$

with $m, k, l \in \mathbb{N}$ being non zero. Consider

$$F(\xi(u)) = f_k u^k + O(u^{k+1}),$$

where $\xi(u)$ is defined as the inverse of $u = \Phi(x) = x\sqrt[2l]{2l\frac{G(x)}{x^{2l}}}$, and assume that $k > l(2m-1)/m$. Then it has a center at the origin if and only if one of the following conditions is satisfied:

(i) *$F(\xi(u))$ is an even function (including $F(\xi(u)) \equiv 0$).*

(ii) *There exists an analytic function Ψ, such that $\Psi(0) = 0$ and satisfying*

$$F(x) = \Psi(G^{1/l}(x)).$$

(iii) *For any small enough x the system*

$$F(x) = F(z),\ G(x) = G(z),$$

has a unique solution $z(x)$ satisfying $z(0) = 0$ and $z'(0) < 0$.

(iv) *There exist analytic functions α, β and h satisfying $\alpha(0) = 0$, $\beta(x) = bx + O(x^2)$, with $b \neq 0$, and such that*

$$F(x) = \alpha(h(x)),\ G^{1/l}(x) = \beta(h(x)).$$

(v) *There exist analytic functions α, γ and h satisfying $\alpha(0) = 0$, $\gamma(x) = cx^l + O(x^{l+1})$, with $c \neq 0$, and such that*

$$F(x) = \alpha(h(x)),\ G(x) = \gamma(h(x)).$$

Proof. First notice that the fact that equation (7) has a center is equivalent to condition (i). This fact is a direct consequence of Lemmas 2.1 and 2.2.

The proof that (i) is equivalent to (ii) follows from next chain of equalities

$$F(x) = F(\xi(u)) = \hat{\Psi}(u^2) = \hat{\Psi}([2lG(x)]^{1/l}) = \Psi(G^{1/l}(x)),$$

where $\hat{\Psi}$ is defined from the fact that $F(\xi(u))$ is an even function of u.

Here we want to prove that (i) $\iff$ (iii). First consider equation $G(x) = G(z)$. Observe that it writes as

$$x^{2l} - z^{2l} + G_{2l+1}(x, z) = 0, \tag{8}$$

for some analytic function G_{2l+1} starting at least with terms of degree $2l + 1$ in x and y. From the Weirstrass' preparation Theorem and the results of Chapter XIII, Section 32 of [3] we can assume that the above equation has only two real solutions in a neighbourhood of $(0, 0)$, $z_1(x) = x$ and $z_2(x) = -x + O(x^2)$.

Let us prove that (i) implies (iii). Since $F(\xi(u)) = F(\xi(-u))$ and $u^{2l} = 2lG(\xi(u)) = 2lG(\xi(-u))$, we have that $z_2(x) = \xi(-u(x))$ is the only solution of equation (8) satisfying $z_2(0) = 0$ and $z_2'(0) < 0$. The converse follows in a similar way.

Let us prove that (ii) $\iff$ (iv). The fact that (iv) implies (ii) follows just taking $\alpha(x) = \Psi(x)$ and $\beta(x) = x$. The converse follows by taking $\Psi = \alpha \circ \beta^{-1} \circ G$, where we have used that $b \neq 0$.

Finally we will prove that (iv) $\iff$ (v). We see that (iv) implies (v) just by taking $\gamma = \beta^l$. The other implication follows from the fact that $\gamma(x) = cx^l + O(x^{l+1})$ implies that $\gamma^{1/l}$ is also an analytic function and we can take $\alpha = \gamma^{1/l}$. □

Corollary 2.7. *Let $F(x) = a_{2k}x^{2k} + O(x^{2k+1})$, and φ analytic functions satisfying $\varphi(0) = 0$ and $k \in \mathbb{N}$. If $G = \varphi(F^{\frac{1}{k}})$ then the system (4) has a center at the origin.*

Proof. We define the next analytic functions,

$$\alpha(x) = a_{2k}x^k, \ \ \gamma(x) = \varphi(\sqrt[k]{a_{2k}}x) \text{ and } h(x) = \sqrt[k]{\frac{F(x)}{a_{2k}}}.$$

It is clear that $G(x) = \varphi(\text{sign}(a_{2k}) \sqrt[k]{|a_{2k}|} h(x))$ and $F(x) = \alpha(h(x))$ and from Theorem 2.6(v) the system (4) has a center at the origin. □

When $l = m = 1$, the characterization (iii) of the above theorem gives light to a nice result proved in [13] which allows compute the order of a weak focus for the Liénard differential equation (4) in terms of the multiplicity the map $((F(x) - F(y))/(x-y), (G(x) - G(y))/(x-y))$. See also [18], for an extension to general l and m.

Remark 2.8. *Consider system (4). Associated to it, we can consider the analytic function*

$$F(\xi(u)) = \sum_{i=1}^{\infty} f_i u^i,$$

defined in the statement of Theorem 2.6. Result (i) of this theorem can be understood in the following way: all the centers of (4) are characterized by the conditions

$$f_{2i-1} = 0, \ \ i = 1, 2, \ldots$$

Therefore, for all Liénard equations (degenerate or not) there is a way to obtain all centers consisting of the following steps:

(i) *Compute the function $\xi(u)$ satisfying $u = \xi(u)\sqrt[2l]{2l\frac{G(\xi(u))}{\xi(u)^{2l}}}$. This can be done by a formal substitution and then by solving recursively the linear system obtained.*
(ii) *Substitute $\xi(u)$ in $F(x)$ to obtain $F(\xi(u)) = \sum_{i=1}^{2M-1} f_i u^i$, until some order $2M-1$.*
(iii) *Denote by f_{2i-1}^* a simplified expression of f_{2i-1} taking into account that $f_1^* = f_3^* = \cdots = f_{2i-3}^* = 0$, and $f_1^* := f_1$. Solve the nonlinear system $f_1^* = f_3^* = \cdots = f_{2M_1-1}^* = 0$, for some $M_1 < M$. Verify that $f_{2i-1}^* = 0$ for $i = M_1 + 1, \ldots, M$. If this is not the case return to item (ii) and enlarge M.*

(iv) *Prove that the Liénard systems associated to the solutions of* $f_1^* = f_3^* = \cdots = f_{2M_1-1}^* = 0$ *are centers by using one of the characterizations given in Theorem 2.6.*

Our computations for $l = m = 1$ *show that this new algorithm is simpler than the algorithms to compute Liapunov constants. Furthermore, observe that while the methods for computing the Liapunov constants work just for* $l = m = 1$, *the above algorithm works for all* l *and* m.

3. Applications

3.1. **Sum of two quasi-homogeneous vector fields.** This section has two parts. In the first part we solve the center problem for system (2) when q is even. In the second part we consider the case q odd.

We need some preliminary results.

Lemma 3.1 ([27])**.** *Let* $p_0(x)$, $p_1(x)$, $q_0(x)$, $q_1(x)$ *and* $q_2(x)$ *be* $\mathcal{C}^1$ *functions, such that* $p_1(0) \neq 0$, *then the system*

$$\begin{aligned} \dot{x} &= p_0(x) + yp_1(x) \\ \dot{y} &= q_0(x) + yq_1(x) + y^2 q_2(x) \end{aligned} \tag{9}$$

can be transformed to a Liénard system.

Proof. We define the functions f, g and ψ as

$$\begin{aligned} f &= -(p_0' - p_0 p_1' p_1^{-1} + q_1 - 2p_0 p_1^{-1} q_2)\psi, \\ g &= (p_0 q_1 - p_1 q_0 - p_0^2 q_2^{-1} p_1)\psi^2, \end{aligned}$$

and

$$\psi(x) = (p_1(x))^{-1} \exp\left(- \int_0^x q_2(s)(p_1(s))^{-1} ds \right).$$

Under the transformation

$$(x, y, t) \longrightarrow (x, (p_0(x) + yp_1(x))\psi(x), \tau),$$

where $\frac{dt}{d\tau} = \psi$, the system (9) becomes

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -g(x) - f(x)y \end{aligned} \tag{10}$$

which is already in Liénard form. ☐

If we take $p_0(x) = a_2 x^{q+1}$, $p_1(x) = -1 + a_1 x$, $q_0(x) = x^{2q+1} + b_3 x^{2q}$, $q_1(x) = b_2 x^q$, $q_2(x) = b_1$, in the above lemma we obtain the next result.

Lemma 3.2. *The system (2) can be transformed into a Liénard system (4), and the functions* f, *and* g *have the expressions*

$$\begin{aligned} f &= x^q\big(- a_2 q - a_2 - b_2 + (a_1 a_2 q + a_1 b_2 - 2b_1 a_2)x\big)(1 - a_1 x)^{-\frac{b_1}{a_1} - 2}, \\ g &= x^{2q-1}\big(1 + (-2a_1 + b_3)x + (-2a_1 b_3 + a_1^2 + a_2 b_2)x^2 + \\ &\quad (a_1^2 b_3 - a_1 a_2 b_2 + a_2^2 b_1)x^3\big)(1 - a_1 x)^{-2\frac{b_1}{a_1} - 3}, \end{aligned}$$

for the case $a_1 \neq 0$ *and*

$$\begin{aligned} f &= -x^q(a_2(q+1)+b_2+2b_1a_2x)e^{2xb_1}, \\ g &= x^{2q-1}(1+b_3x+a_2b_2x^2+b_1a_2^2x^3)e^{2xb_1}, \end{aligned}$$

for the case $a_1 = 0$.

3.1.1. *The even case.* Next theorem solves the center problem for system (2) when q is even. Notice that the family (iii) is not studied in [7].

Theorem 3.3. *The system (2), for* q *even, has the next families of centers at the origin,*

(i) $a_2 = b_2 = 0$,
(ii) $a_1 + 2b_1 = b_2 + (q+1)a_2 = 0$,
(iii) $2a_1 + b_3 = b_2 + (q+1)a_2 = 2b_1 + (q+1)b_3 = 0$.

For $q = 2$*, there is a new family of centers,*

(iv) $a_1 - b_1 - b_3 = 6a_2^2 + b_1b_3 + 2b_3^2 = b_2 + 3a_2 = 0$, *and* $a_2 \neq 0$.

Furthermore, the previous four families are all the centers of (2) at the origin for q *even.*

Proof. The cases (i) and (ii) are centers because $F = 0$. For the third case we can apply Theorem 2.6(v) by taking the analytic functions α, γ and h defined as

$$\begin{aligned} \alpha(x) &= -\frac{a_1a_2(3+2q)}{2+q}x^{\frac{q+2}{2}}, \\ \gamma(x) &= \left(\frac{1}{2q} - \frac{1}{2(q+1)}(a_2^2q + a_1^2 + a_2^2)x + \frac{(q+1)a_1^2a_2^2}{2(q+2)}x^2\right)x^q, \\ h(x) &= \left(\frac{x}{1-a_1x}\right)^2, \end{aligned}$$

and therefore the origin has a center.

For the case (iv), with $q = 2$, we can assume that b_3 and b_2 are non zero constants, and apply again Theorem 2.6.(v) with the functions

$$\begin{aligned} \alpha(x) &= -\frac{b_2\left(5b_3^2+2b_2^2\right)}{12b_3}x^2, \\ \gamma(x) &= -\frac{3b_3}{b_2\left(5b_3^2+2b_2^2\right)}x^2 + \frac{\left(4b_2^2+9b_3^2\right)\left(b_2^2+3b_3^2\right)}{9\left(5b_3^2+2b_2^2\right)^2}x^4, \\ h(x) &= \left(\frac{6\left(\left(27b_3^2+\left(4b_2^2+9b_3^2\right)\left(\left(b_2^2+3b_3^2\right)(x^3+3x^2)+9b_3x\right)\right)h_1(x)-162b_3^2\right)}{\left(4b_2^2+9b_3^2\right)\left(b_2^2+3b_3^2\right)}\right)^{\frac{1}{2}}, \\ h_1(x) &= \left(1+\frac{2b_2^2+3b_3^2}{3b_3}x\right)^{-\frac{4b_2^2+9b_3^2}{2b_2^2+3b_3^2}}. \end{aligned}$$

Notice that $h(x)$ is an analytic function because $h_1(x)$ and $h(x)^2 = x^4 + O(x^5)$ are analytic functions. Then the origin has a center.

Let us prove that the centers obtained above are all the centers for q even.

From the method described in the Appendix, we can obtain the values

$$\begin{aligned} f^*_{q+1} &= -\frac{1}{q+1}b_2 - a_2, \\ f^*_{q+3} &= \frac{(3qa_1 - qb_3 - a_1 - 3b_3 - 5b_1)(a_1+2b_1)b_2}{(2q+1)(q+3)(q+1)}, \end{aligned}$$

To obtain all the centers, we can solve the system $\{0 = f^*_{q+1} = f^*_{q+3} = \dots\}$, but it is easier to study the three subcases $f^*_{q+3} = 0$, namely: $b_2 = 0$, for which we obtain the family (i); $a_1 + 2b_1 = 0$ for which we obtain the family (ii); and finally, the case $3qa_1 - qb_3 - a_1 - 3b_3 - 5b_1 = 0$ and $b_2 + (q+1)a_2 = 0$ which we nwxt study in more detail under the assumptions $b_2 \neq 0$ and $a_1 + 2b_1 \neq 0$. For this case we compute the next values of f^*_{2i+1} and we obtain:

$$\begin{aligned} f^*_{q+5} &= \frac{2a_2(b_3+2a_1)(3(2q+1)a_1 - 2(q+3)b_3)}{1875(2q+3)(q+5)}(75q^2a_2^2 - 3qa_1^2 + 32qa_1b_3 + \\ &\quad 375a_2^2q + 23qb_3^2 + 6a_1^2 + 111a_1b_3 + 129b_3^2), \\ f^*_{q+7} &= \frac{8a_2(b_3+2a_1)(3(2q+1)a_1 - 2(q+3)b_3)}{703125(2q+5)(2q+3)(q+7)(q+1)}(-387000b_3^2a_2^2 + 112q^2b_3^4 - \\ &\quad 16q^3b_3^4 - 564qb_3^4 - 1125000a_2^4q + 90750qa_1b_3a_2^2 + 52308b_3^4 + 25092a_1^2b_3^2 + \\ &\quad 172q^3a_1b_3^3 - 10494b_3a_1^3 - 972a_1^4 + 85914a_1b_3^3 - 334q^3a_1^2b_3^2 + 609q^2b_3a_1^3 - \\ &\quad 108q^2a_1^4 - 162q^3b_3a_1^3 + 144q^3a_1^4 - 737q^2a_1^2b_3^2 - 154q^2a_1b_3^3 + 2514qa_1^2b_3^2 + \\ &\quad 1038qa_1b_3^3 + 126qa_1^4 - 1698qb_3a_1^3 - 18000a_1^2a_2^2 - 333000a_1b_3a_2^2 + \\ &\quad 160500b_3^2a_2^2q - 29250qa_1^2a_2^2). \end{aligned}$$

From the expression of f^*_{q+5}, it is clear that it is necessary to study the next three subcases:

- $b_3 + 2a_1 = 0$, which lead us to the family (iii) of the statement,
- $3(2q+1)a_1 - 2(q+3)b_3 = 0$, which is incompatible with the assumption $a_1 + 2b_1 \neq 0$, and
- $75q^2a_2^2 - 3qa_1^2 + 32qa_1b_3 + 375a_2^2q + 23qb_3^2 + 6a_1^2 + 111a_1b_3 + 129b_3^2 = 0$.

If we consider the lastest subcase, we only obtain four solutions of the system $\{f^*_{q+5} = f^*_{q+7} = 0\}$:

- $q = 2$, $6a_2^2 + a_1b_3 + b_3^2$,
- $a_1 = \frac{1}{3}b_3$, and $a_2 = \frac{2}{3}\alpha b_3$ with $q\alpha^2 + 1 = 0$,
- $a_1 = 2b_3$, and $a_2 = \frac{2}{3}\alpha b_3$ with $q\alpha^2 + 1 = 0$,
- $a_1 = \alpha b_3$ and $a_2 = \beta b_3$, with α and β satisfying

$$\begin{aligned} (40q^3 + 350q^2 + 700q)\beta^2 + (19q^2 + 132q + 180)\alpha + 12q^2 + 116q + 240 &= 0, \\ (24q^2 + 210q + 420)\alpha^2 + (29q^2 + 1020 + 335q)\alpha - 4q^2 - 10q + 30 &= 0. \end{aligned}$$

The first family is the family (iv) of the statement. The other three families never give rise to real solutions of the system considered previously. Hence the theorem follows. □

3.1.2. *The odd case.* In this subsection we solve the center problem for $q = 1$ and we give some families of centers for q odd, $q \geq 3$.

Proposition 3.4. *The system (2), for q odd, has the next families of centers at the origin,*

(i) $a_2 = b_2 = 0$,
(ii) $a_2 = b_1 + b_3 = a_1 - b_3 = 0$ *and* $b_2 \neq 0$,
(iii) $a_2 = 2b_1 + (q+1)b_3 = 2a_1 + b_3 = 0$ *and* $b_2 \neq 0$,
(iv) $a_2 = b_1 + qb_3 = a_1 + 2b_3 = 0$ *and* $b_2 \neq 0$,
(v) $a_1 + 2b_1 = b_2 + (q+1)a_2 = 0$,
(vi) $b_1 = b_3 = a_1 = 0$.

Furthermore, for $q = 1$, *there are six new families of centers,*

(vii) $a_1 + b_3 = b_2 - 3a_2 = b_1 + 2b_3 = 0$ *and* $a_2 \neq 0$,
(viii) $b_3 = a_1 - 3b_1 = b_2 - 3a_2 = 0$ *and* $b_1 a_2 \neq 0$,
(ix) $b_3 = a_1 + b_2 = a_2 + b_1 = 0$ *and* $a_1 + 2b_1 \neq 0$,
(x) $b_3 = a_1 - b_2 = a_2 - b_1 = 0$ *and* $a_1 + 2b_1 \neq 0$,
(xi) $a_2^2(2b_3 - a_1 + b_3) - (-a_1 + b_1)(b_1 + b_3)^2 = a_2(a_1 - 2b_3) - b_2(b_1 + b_3) = 0$ *and* $(a_1 + 2b_1)(a_1 - b_1) \neq 0$,
(xii) $a_2 = b_1 + b_3 = 0$ *and* $b_2 \neq 0$.

In the next theorem we characterize all the centers for some particular cases. Observe that it provides an unified treatment of all quadratic systems.

Theorem 3.5. (i) *For* $q \geq 3$, *the families (i-iv) of Proposition 3.4 are all the centers of system (2) when* $a_2 = 0$.

(ii) *For* $q = 1$, *the families (i-xii) of Proposition 3.4 are all the centers of system (2). Furthermore, if we transform it to a Liénard equation, there exist* A, B, C *and* D *real numbers such that either* $F = 0$ *or* $G = AF + BF^2 + C(\sqrt{1 - DF} - 1)$.

Proof of Proposition 3.4. The cases (i) and (v) are centers because $F = 0$.

The other cases, (ii), (iii), (iv) and (vi), are centers because there exist analytic functions $\alpha(x)$, $\gamma(x)$ and $h(x)$ which allow us to apply Theorem 2.6(v). Those functions are

$$\alpha(x) = -\frac{b_2}{q+1} x^{\frac{q+1}{2}}, \ \gamma(x) = \frac{1}{2q} x^q - \frac{1}{2(q+1)} b_3^2 x^{q+1} \text{ and } h(x) = x^2,$$

for the case (ii);

$$\alpha(x) = -\frac{b_2 2^{q+1}}{q+1} x^{\frac{q+1}{2}}, \ \gamma(x) = 2^{2q}\left(\frac{1}{q} x^q - \frac{b_3}{q+1} x^{q+1}\right) \text{ and } h(x) = \left(\frac{x}{2 + b_3 x}\right),$$

for the case (iii);

$$\begin{aligned} \alpha(x) &= \frac{-b_2}{2^q b_3^{q+1}} \sum_{i=0}^{\frac{q-1}{2}} \binom{\frac{q-1}{2}}{i} \frac{4^i}{q-i} x^{q-i}, \ \gamma(x) = \frac{1}{2q}\left(\frac{x^2 + 4x}{4b_3^2}\right)^q \text{ and} \\ h(x) &= 2\left(\frac{1 + b_3 x}{\sqrt{1 + 2b_3 x}} - 1\right), \end{aligned}$$

	A	B	C	D
(vi)	$\dfrac{-1}{2a_2+b_2}$	$\dfrac{a_2b_2}{(2a_2+b_2)^2}$	0	0
(vii)	$\dfrac{-1}{5a_2}$	$\dfrac{3}{25}$	0	0
(viii)	$\dfrac{-a_2}{5b_1^2}$	$\dfrac{2}{25}$	$\dfrac{b_1^2-a_2^2}{2b_1^4}$	$\dfrac{4b_1^2}{5a_2}$
(ix)	$\dfrac{1}{a_1+2b_1}$	$\dfrac{(b_1+a_1)b_1}{2(a_1+2b_1)^2}$	0	0
(x)	$\dfrac{-1}{a_1+2b_1}$	$\dfrac{(b_1+a_1)b_1}{2(a_1+2b_1)^2}$	0	0
(xi)	$-\mathrm{sgn}\,(\alpha)\sqrt{\dfrac{\lvert\alpha\rvert}{(a_1+2b_1)^2}}$	$\dfrac{b_1(a_1+b_1)}{2(a_1+2b_1)^2}$	0	0
(xii)	$\dfrac{-1}{b_2}$	$\dfrac{-b_3(a_1+a_3)}{b_2^2}$	0	0

TABLE 1. The coefficients A, B, C and D of (11), where $\alpha=\frac{a_1-b_1-2b_3}{a_1-b_1}$.

for the case (iv); and

$$\alpha(x)=-a_2x^{\frac{q+1}{2}},\ \gamma(x)=\frac{1}{2q}x^q \text{ and } h(x)=x^q,$$

for the case (vi).

Now, we study the case $q=1$. Notice that, the cases (ii), (iii) and (iv) are included in the case (xii). For the case (vi) we consider two subcases, $2a_2+b_2=0$ and $2a_2+b_2\neq 0$, in the first subcase $F=0$ and for the second one there exist A and B such that $G=AF+BF^2$, see Table 1.

For all the new cases, (vii)-(xii), there exist A, B, C, and D real constants such that the functions F and G satisfy

$$G=AF+BF^2+C(\sqrt{1-DF}-1), \tag{11}$$

and $F=-\frac{2a_2+b_2}{2}x^2+O(x^3)$, see the Table 1. Then, from Corollary 2.7, the origin has a center. Notice that all values A and B of Table 1 are well defined.

□

Proof of Theorem 3.5. From the algorithm described in the Appendix we obtain, for $a_2=0$ and $q>1$ odd, the next values of f^*_{2i-1}:

$$f^*_{q+2} = -\frac{(a_1q-b_3q-3b_1-a_1-2b_3)b_2}{(q+2)(2q+1)},$$

$$
\begin{aligned}
f^*_{q+4} &= \frac{2b_2(b_1+b_3)}{3(2q+3)(q+4)(q+1)(2q+1)^3}(16b_1^2q^2+164b_1^2+72b_1^2q+24b_3b_1q^3+\\
&\quad 282b_3b_1q+116b_3b_1q^2+397b_3b_1+90a_1b_1+8b_3^2q^4+217b_3^2q+44b_3^2q^3+\\
&\quad 12a_1^2+192b_3^2+120a_1b_3+118b_3^2q^2),\\
f^*_{q+6} &= 0.
\end{aligned}
$$

By solving the system $\{f^*_{q+2}=f^*_{q+4}=0\}$ we just obtain the cases (i-iv) of the statement (i) of the theorem.

(ii) For the particular case $q=1$, and by using again the method described in Remark 2.8, we obtain

$$
\begin{aligned}
f^*_3 &= \frac{1}{3}\left(-a_1a_2+2a_2b_3+b_2b_1+b_2b_3\right),\\
f^*_5 &= \frac{1}{15}(-b_2b_1a_1^2-b_2a_1^2b_3+2b_2b_1^2a_1-b_3a_2b_2^2+b_2^3b_1+b_2^3b_3+12a_2b_3^3+\\
&\quad 15b_2b_1b_3^2+6b_2b_3^3+18a_2b_1b_3^2+14b_2b_1^2b_3+5b_2b_1^3-6b_1^3a_2-6b_3a_2^2b_2+\\
&\quad 5b_2b_1a_1b_3-2b_2^2b_1a_2+3b_2a_1b_3^2+6b_1a_2^3-5a_2^2b_2b_1),\\
f^*_7 &= \frac{2b_3}{7}\left(2b_3+b_1\right)(-2b_1^3a_2+b_2b_1^3+4b_2b_1^2b_3+b_2b_1^2a_1+6a_2b_1b_3^2+5b_2b_1b_3^2+\\
&\quad 2b_2b_1a_1b_3+2b_1a_2^3-a_2^2b_2b_1-b_2^2b_1a_2+4a_2b_3^3+2b_2b_3^3+b_2a_1b_3^2-\\
&\quad 2b_3a_2^2b_2-b_3a_2b_2^2),\\
f^*_9 &= 0.
\end{aligned}
$$

To characterize the centers of this family, it is useful to consider also $f_2=2a_2+b_2$ and solve the two systems of equations, $\{f_2=f^*_3=f^*_5=f^*_7=0\}$ and $\{f_2\neq 0, f^*_3=f^*_5=f^*_7=0\}$. The first system has only two solutions, which are the ones given in cases (i) and (v) of Proposition 3.4 for all q. The rest of cases are the solutions of the second system. Hence, the proof is done. □

To end this subsection we try to show the difficulties that the study of the rest of the cases $q>1$ and odd presents. From the method described in the Appendix, we can obtain expressions for $f^*_{q+2}, f^*_{q+4}, \ldots$, but have not been able to solve the system $\{f^*_{q+4}=f^*_{q+6}=\ldots=0\}$(we mean that Maple V.4 has not been able to solve the system). For this reason we study it step by step.

We can restrict the problem to two cases:

- $f^*_{q+1}=-\frac{1}{q+1}b_2-a_2=0$. In this case the system $\{f^*_{q+2}=f^*_{q+4}=f^*_{q+6}=0\}$ has only two solutions, $a_2=b_2=0$ and $a_1+2b_1=b_2+(q+1)a_2=0$. The two solutions give rise to cases (i) and (v) of Proposition 3.4. These are the families described in [7].
- $f^*_{q+1}=-\frac{1}{q+1}b_2-a_2\neq 0$. In this situation there exists a trivial case, $b_1=b_3=0$ and $a_1=0$, which corresponds to the case (vi) of Proposition 3.4. The particular case $a_2=0$ has been studied in Proposition 3.4. The general case $a_2\neq 0$ is more complicated. We have a system of five equations ($\{f^*_{q+2}=f^*_{q+4}=f^*_{q+6}=f^*_{q+8}=f^*_{q+10}=0\}$) and five variables of homogeneous degree

2, 4, 6, 8, 10, respectively. Maple V.4 has not been able to solve it, even after fixing $q = 3$.

As far as we know, for odd values of $q > 1$, the complete characterization of centers of system (2) is an open question.

3.2. **A polynomial Liénard system.** In this section we characterize all centers for system (3),

$$\begin{aligned} \dot{x} &= y - (a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5), \\ \dot{y} &= -(b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5). \end{aligned} \tag{12}$$

As a first step, in next lemma we classify all cases for which the origin is either a center or a focus for the above system. Its proof is a straightforward corollary of Lemma 2.5.

Lemma 3.6. *A system of type (12) has either a center or a focus at the origin if and only if one of the following three conditions hold.*

(i) $a_1^2 - 4b_1 < 0$,
(ii) $a_1 = b_1 = b_2 = 0$, *and* $2a_2^2 - 4b_3 < 0$,
(iii) $a_1 = a_2 = b_1 = b_2 = b_3 = b_4 = 0$, *and* $3a_3^2 - 4b_5 < 0$.

Remark 3.7. *From the above lemma it is clear that the only cases for which the origin of (12) can be a center are the cases in which* $g(x) = b_{2i-1}x^{2i-1} + O(x^{2i}), b_{2i-1} \neq 0$ *for* $i = 1, 2, 3$. *From now on, and for the sake of simplicity, we just will consider the cases either* $b_1 = 1$, *or* $b_1 = 0$, $b_3 = 1$, *or* $b_1 = b_3 = 0$, $b_5 = 1$.

Theorem 3.8. *A system of type (12) (taking into account the simplifications of Remark 3.7 has a center at the origin if and only if one of the following conditions holds:*

(i) $b_1 = 1$ *and*
  (a) $a_3 = a_5 = b_2 = b_4 = 0$,
  (b) $a_2 = a_3 = a_4 = a_5 = 0$,
  (c) $a_4 = a_5 = 0$, $a_3 = 2a_2b_2/3$, $b_5 = 2b_2^2b_3/3$ *and* $b_4 = 5b_3b_2/3$,
  (d) $b_5 = 0$, $a_3 = 2a_2b_2/3$, $a_4 = a_2b_3/2$ *and* $a_5 = 2a_2b_4/5$.
(ii) $b_1 = b_2 = 0$, $b_3 = 1$, $a_2^2 < 2$, *and*
  (a) $a_3 = a_5 = 0$ *and* $b_4 = 0$,
  (b) $a_2 = a_3 = a_4 = a_5 = 0$,
  (c) $a_2 = a_3 = 0$, $a_5 = 4a_4b_4/5$ *and* $b_5 = 0$,
  (d) $a_4 = a_5 = 0$, $a_3 = 2a_2b_4/5$ *and* $b_5 = 6b_4^2/25$.
(iii) $b_1 = b_2 = b_3 = b_4 = 0$, $b_5 = 1$, *and* $a_1 = a_2 = a_3 = a_5 = 0$.

Proof. (i) By Lemma 3.6 and by using the algorithm described in Remark 2.8 we obtain the following necessary conditions to have a center:

$$
\begin{aligned}
f_1^* &= a_1,\\
f_3^* &= -\tfrac{2}{3}a_2b_2 + a_3,\\
f_5^* &= -\tfrac{4}{3}a_4b_2 + a_3b_3 + a_5 - \tfrac{2}{5}a_2b_4,\\
f_7^* &= \tfrac{2}{3}a_5b_2^2 - \tfrac{2}{5}a_3b_4b_2 + a_5b_3 + a_3b_5 - \tfrac{4}{5}a_4b_4,\\
f_9^* &= \tfrac{11}{9}a_5b_3^2 + \tfrac{22}{45}a_5b_4b_2 - \tfrac{44}{45}a_4b_4b_3 - \tfrac{22}{75}a_3b_4^2 + \tfrac{11}{9}a_3b_3b_5 + a_5b_5,\\
f_{11}^* &= -\tfrac{26}{45}a_5b_4b_3b_2 + \tfrac{26}{75}a_3b_4^2b_3 - \tfrac{52}{375}a_2b_4^3 - \tfrac{208}{495}a_3b_5b_2b_4 + \tfrac{26}{75}a_5b_4^2 +\\
&\quad \tfrac{221}{99}a_5b_3b_5 - \tfrac{416}{495}a_4b_4b_5 + \tfrac{104}{99}a_3b_5^2,\\
f_{13}^* &= -\tfrac{34}{75}a_5b_4^2b_3 + \tfrac{136}{375}a_4b_4^3 - \tfrac{782}{495}a_5b_5b_2b_4 + \tfrac{68}{55}a_4b_4b_3b_5 + \tfrac{102}{275}a_3b_4^2b_5 -\\
&\quad \tfrac{17}{11}a_3b_3b_5^2 - \tfrac{175}{121}a_5b_5^2,\\
f_{15}^* &= \tfrac{19}{2475}a_5b_5(75b_3b_5 - 34b_4^2).
\end{aligned}
$$

By solving the non linear system $f_1^* = \cdots = f_{15}^* = 0$ we exactly obtain the 4 cases given in the statement. To end the proof of this case we need to prove that in the four cases the origin is a center. This fact follows from Theorem 2.6 and next claims which can be easily verified:

(i.a) $z(x) = -x$ is the only solution of system proposed in Theorem 2.6. (iii).
(i.b) $F = 0$.
(i.c) The functions defined in Theorem 2.6(v) are

$$\alpha(x) = \frac{1}{3}a_2x,\ \gamma(x) = \frac{x(6+b_3x)}{36} \text{ and } h(x) = x^2(3+2b_2x).$$

(i.d) $F(x) = 2a_2G(x)$.

Now let us prove part (ii) of the theorem. As in the previous case and in order to obtain necessary conditions to have a center, we will use again the algorithm described in Remark 2.8 and Lemma 3.6. We get the conditions

$$
\begin{aligned}
f_3^* &= -\frac{2}{5}a_2b_4 + a_3,\\
f_5^* &= a_3b_5 - \frac{6}{25}a_3b_4^2 + a_5 - \frac{4}{5}a_4b_4,\\
f_7^* &= \frac{88}{375}a_4b_4^3 + a_5b_5 - \frac{22}{75}a_5b_4^2,\\
f_9^* &= -\frac{13}{825}a_5b_5(-75b_5 + 34b_4^2),\\
f_{11}^* &= \frac{912}{3179}a_5b_5^3.
\end{aligned}
$$

Arguing also as in the precedent case we prove that the first two cases, (ii.a) and (ii.b), give centers for the degenerate Liénard equation. For third and fourth cases, (ii.c) and (ii.d), the analytic functions α, γ and h are,

$$\alpha(x) = a_4x^2,\ \gamma(x) = \frac{1}{4}x^2, \text{ and } h(x) = x^2\sqrt{1+\frac{4}{5}b_4x},$$

and

$$\alpha(x) = \frac{a_2}{5}x,\ \gamma(x) = \frac{1}{100}x^2,\ \text{ and } h(x) = x^2(5 + 2b_4x),$$

respectively. Therefore, again by Theorem 2.6(v) we are done.

For the case (iii), we obtain that

$$f_3^* = a_3,\ f_5^* = a_5,$$

and in this case the solution of the system $\{f_3^* = f_5^* = 0\}$ coincides with that of the statement. Furthermore, if we take

$$\alpha(x) = a_4x^2,\ \gamma(x) = \frac{1}{6}x^3 \text{ and } h(x) = x^2,$$

from Theorem 2.6(v) this family has a center at the origin. Hence the theorem is proved. ☐

Remark 3.9. *Observe that the algorithm described in Remark 2.8 is also useful to compute the maximum order of degeneracy of a weak focus for a Liénard equation when we fix the set of functions in which φ, F, g vary. For instance, if we fix $\varphi(y) \equiv y$ and F, G polynomials of degree less or equal than 5, the computations made to prove (i) of the previous theorem show that this number is 6 (we remark that condition $f_{15}^* = 0$ is redundant to find all centers). This result coincides with the results obtained in* [23] *by other methods. In fact, the algorithm of Remark 2.8 could be applied to try to enlarge the values of the degrees of F and G for which the maximum order of degeneracy of the origin is known (see again* [23]*). Anyway, we think that the algorithm developed in* [18] *for $l = m = 1$ and suggested by the results of the paper* [13] *is more powerful.*

4. The $\mathcal{C}^1$-case

In this Section we make some remarks on the center problem in the case in which the functions, φ, F and g, involved in the Liénard equation (1) are just $\mathcal{C}^1$ functions. We have started to study this natural problem motivated by a question formulated to us by our colleague J.J. Nieto after a talk given by the first author about the first part of this paper.

For sake of simplicity, we just consider the nondegenerate case and we will assume that φ, F and g are $\mathcal{C}^1$ functions that satisfy

$$\begin{aligned} \varphi(y) &= y + o(y), \\ F(x) &= o(x), \\ g(x) &= x + o(x). \end{aligned}$$

As in (5) we can define $u = \phi(x) = x\sqrt{\frac{2G(x)}{x^2}} = x\sqrt{1 + o(1)}$, which has a local $\mathcal{C}^1$ inverse called $x = \xi(u) = u + o(u)$. Hence $F(\xi(u))$ is also a $\mathcal{C}^1$ function, that we denote as $F^*(u)$.

Following the proofs of Lemmas 2.1 and 2.2 we can conclude that in the $\mathcal{C}^1$-case:

(i) Equation (1) has a center at the origin if $F^*(u) = F^*(-u)$ in a neighbourhood of $u = 0$.

(ii) Equation (1) has not a center at the origin if $\left(\frac{F^*(u)-F^*(-u)}{2}\right) u$ keeps sign fixed in a neigbourhood of 0 for $u \neq 0$.

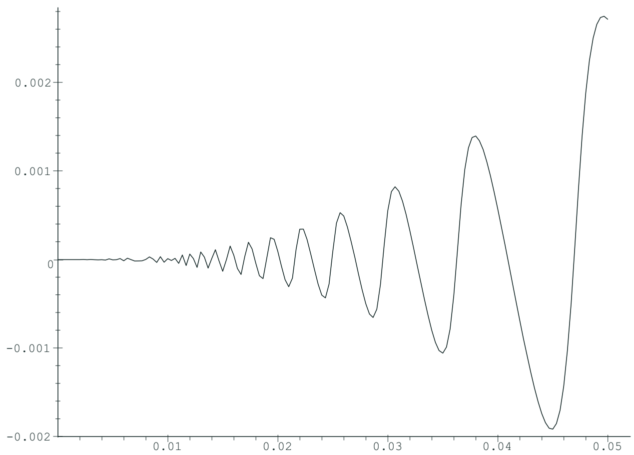

FIGURE 1. $\Pi(x) - x$ for system (13)

Although the above two statements are enough to give necessary and sufficient conditions to have a center for equation (1) when all the involved functions are analytic, they are not enough for the $\mathcal{C}^1$ case. Take for instance the equation

$$\begin{cases} \dot{x} & = y - x^2 \sin \frac{1}{x} \\ \dot{y} & = -x \end{cases}, \tag{13}$$

for which $F^*(u) = \frac{F^*(u)-F^*(-u)}{2} = u^2 \sin \frac{1}{u}$ and note that it is not either under condition (i) or under condition (ii). Therefore, the techniques developped in this paper are not enough to solve the center problem for equation (13). Our numeric simulations seem to show that (13) has at the origin a critical point of center-focus type with infinitely many limit cycles accumulating into it. See in Figure 1 the plot of $\Pi(x) - x$, where Π is the Poincaré return map associated to the OX^+ axis obtained numerically. We do not study this problem here.

APPENDIX A. THE GENERAL EXPRESSIONS OF f_k^*

From the method described in Remark 2.8, we can obtain the results of the Section 3 if we are able to calculate the values of $f_{q+1}^*, \ldots, f_{q+7}^*$. From the expression of

$$u^{2q} = 2qG(x) = x^{2q} + 2qG_{2q+1}x^{2q+1} + \ldots + 2qG_{2q+6}x^{2q+6} + O(x^{2q+7}), \tag{14}$$

and in order to get the function $\xi(u)$, it is necessary to use the Taylor expansion of $\xi(u) = u + u_2u^2 + \ldots, u_7u^7 + O(u^8)$. If we denote by $U_k(l)$ the expressions of degree k of $(\xi(u))^l$ we obtain

$$\begin{aligned}
U_0(l) &= 1,\\
U_1(l) &= lu_2,\\
U_2(l) &= lu_3 + \tfrac{l(l-1)}{2}u_2^2,\\
U_3(l) &= lu_4 + \tfrac{l(l-1)}{2}(2u_3u_2) + \tfrac{l(l-1)(l-2)}{3!}u_2^3,\\
U_4(l) &= lu_5 + \tfrac{l(l-1)}{2}(2u_4u_2 + u_3^2) + \tfrac{l(l-1)(l-2)}{3!}(3u_3u_2^2) + \tfrac{l(l-1)(l-2)(l-3)}{4!}u_2^4,\\
U_5(l) &= lu_6 + \tfrac{l(l-1)}{2}(2u_5u_2 + 2u_4u_3) + \tfrac{l(l-1)(l-2)}{3!}(3u_3^2u_2 + 3u_4u_2^2) +\\
&\quad \tfrac{l(l-1)(l-2)(l-3)}{4!}(4u_3u_2^3) + \tfrac{l(l-1)(l-2)(l-3)(l-4)}{5!}u_2^5,\\
U_6(l) &= lu_7 + \tfrac{l(l-1)}{2}(2u_6u_2 + 2u_5u_3 + u_4u_4) + \tfrac{l(l-1)(l-2)}{3!}(3u_5u_2^2 + 32u_4u_3u_2 + u_3^3) +\\
&\quad \tfrac{l(l-1)(l-2)(l-3)}{4!}(4u_4u_2^3 + 6u_3^2u_2^2) + \tfrac{l(l-1)(l-2)(l-3)(l-4)}{5!}(5u_3u_2^4) +\\
&\quad \tfrac{l(l-1)(l-2)(l-3)(l-4)(l-5)}{6!}(u_2^6).
\end{aligned}$$

From a formal substitution of $\xi(u)$ in (14) we can obtain the values of $u_2, \ldots, u_7$ from the solution of the linear system defined by the coefficients of u^k for $k = 1, \ldots, 6$, which are

$$\sum_{i=0}^{k} G_{2q+i}U_{k-i}(2q+i) = 0,\ k = 1, \ldots, 6,$$

and where $G_{2q} := 1$.

Then if we substitute $\xi(u) = u + u_2u^2 + \ldots + u_7u^7$ in the Taylor expansion of $F(x)$ of degree $q+7$, $F_{q+1}x^{q+1} + \ldots + F_{q+7}x^{q+7}$, we obtain the values of f_i for $i = q+1, \ldots, q+7$,

$$f_i = \sum_{j=q+1}^{i} F_jU_{i-j}(j).$$

From the above expressions we obtain the f_i^*, for $i = q+1, \ldots, q+7$ just by simplification.

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