EULER-JACOBI FORMULA FOR DOUBLE POINTS AND APPLICATIONS TO QUADRATIC AND CUBIC SYSTEMS

ARMENGOL GASULL AND JOAN TORREGROSA

ABSTRACT. We prove a generalization of the Euler-Jacobi formula for double points. We apply it to study the distribution of the critical points for quadratic and cubic systems, when one of these points is double.

1. INTRODUCTION.

Consider a planar polynomial vector field $X = (P, Q)$, with $\deg P = n$, $\deg Q = m$. Assume that X has exactly nm simple critical points (real or complex). In this situation the Euler-Jacobi formula gives an algebraic relation between the critical points of X and their indices, see Section 3. In this paper we prove a generalization of the formula, allowing X to have double points. In fact in Section 2 we give necessary and sufficient conditions for a critical point be double and in Section 3 we give a proof of this new Euler-Jacobi formula. The method used could be applied to get a general formula for more degenerate critical points, but the computations involved would increase very much. We have not made these computations here.

Finally, Section 4 deals with applications of the formula obtained to study the distribution of the critical points of quadratic vector fields ($n = m = 2$) and a sub class of cubic vector fields ($n = 2, m = 3$). The result obtained can be considered as a continuation of the paper [3]. For instance Theorem 4.1 is a generalization of the well-known Berlinskii's Theorem, see [2, 5], and can be interpreted as the limit case of that result.

Partially supported by the DGICYT grant number PB96-1153.

2. Study of double points.

Definition 2.1. *Let f and g be planar analytic functions and let p be a common zero of f and g. The multiplicity of $X = (f, g)$ at p is defined by the formula*

$$\mu_p[(f,g)] = \dim_{\mathbb{C}} \frac{C\{x,y\}_p}{(f,g)},$$

where $C\{x,y\}_p$ is the ring of germs at p of holomorphic functions of $\mathbb{C}^2$ and (f,g) is the ideal generated by the components of f and g.

We say that p is a simple point if it has multiplicity one, and a double point if it has multiplicity two.

The following Lemma characterizes double points.

Lemma 2.2. *Let $f = 0, g = 0$ be two analytic curves, and p such that $f(p) = g(p) = 0$. Then*

(i) *p is a simple point if and only if $J_1(f(x,y), g(x,y)) \neq 0$,*

(ii) *p is a double point if and only if*

$$J_1(f(x,y), g(x,y)) = 0,$$

and

$$\begin{aligned} M_2(x,y) = \{&J_2(f(x,y),g(x,y)), J_2(f(y,x),g(y,x)), \\ &J_2(g(x,y),f(x,y)), J_2(g(y,x),f(y,x))\} \neq \{0\}, \end{aligned}$$

where

$$J_1(f(x,y),g(x,y)) = \left.\left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}\right)\right|_p,$$

and

$$\begin{aligned} J_2(f(x,y),g(x,y)) = \Bigg[&\left(\frac{\partial f}{\partial y}\right)^2 \left(\frac{\partial f}{\partial x}\frac{\partial^2 g}{\partial x^2} - \frac{\partial g}{\partial x}\frac{\partial^2 f}{\partial x^2}\right) - \\ &2\frac{\partial f}{\partial x}\frac{\partial f}{\partial y}\left(\frac{\partial f}{\partial x}\frac{\partial^2 g}{\partial x \partial y} - \frac{\partial g}{\partial x}\frac{\partial^2 f}{\partial x \partial y}\right) + \left(\frac{\partial f}{\partial x}\right)^2 \left(\frac{\partial f}{\partial x}\frac{\partial^2 g}{\partial y^2} - \frac{\partial g}{\partial x}\frac{\partial^2 f}{\partial y^2}\right)\Bigg]\Bigg|_p . \end{aligned}$$

Proof. It is not restrictive to assume that $p = (0,0)$. In a neighborhood of $(0,0)$ the functions $f(x,y)$ and $g(x,y)$ can be written as

$$\begin{aligned} f(x,y) &= f_{10}x + f_{01}y + f_{20}x^2 + f_{11}xy + f_{02}y^2 + \cdots, \\ g(x,y) &= g_{10}x + g_{01}y + g_{20}x^2 + g_{11}xy + g_{02}y^2 + \cdots \end{aligned}$$

If $f_{10} = f_{01} = g_{10} = g_{01} = 0$ it is obvious that $\mu_p[(f,g)] \geq 3$ because $1, x$ and y are linearly independent elements of $\frac{C\{x,y\}_p}{(f,g)}$. Hence, to study simple or double points we can assume that $\{f_{10}, f_{01}, g_{10}, g_{01}\} \neq \{0\}$.

In this situation it is sufficient to study the case $f_{10} \neq 0$ because all the other cases can be reduced to it interchanging f and g or x and y.

Note that if $f_{10} \neq 0$, then the change of variables

$$(z, w) \quad = \quad (f(x,y), y), \tag{1}$$

is well-defined in a neighborhood of $(0,0)$. The inverse change of variables is

$$(x, y) = (x_{10}z + x_{01}w + x_{20}z^2 + x_{11}zw + x_{02}w^2 + \cdots, w).$$

Recall that the multiplicity is invariant by regular changes of coordinates, see [1, Sec.4]. Let $\widetilde{f}(z,w)$ and $\widetilde{g}(z,w)$ be the expressions of $f(x,y)$ and $g(x,y)$ in the new coordinates. We have that $\widetilde{f}(z,w) = z$. Hence we get that, when the multiplicity is infinite, $\widetilde{g}(0,w) \equiv 0$ and that, when the multiplicity is μ, $\widetilde{g}(z,w)$ is such that $\widetilde{g}(0,w) = q_\mu w^\mu + \cdots$, with $q_\mu \neq 0$.

Therefore, to prove the lemma it is sufficient to compute the Taylor expansion of $\widetilde{g}(0,w)$. Direct computations give $x_{10} = \frac{1}{f_{10}}$, $x_{01} = -\frac{f_{01}}{f_{10}}$, and

$$\widetilde{g}(0,w) = \left(\frac{-g_{10}f_{01}}{f_{10}} + g_{01} \right) w + \cdots = \left. \frac{J_1(f(x,y), g(x,y))}{f_{10}} \right|_{(0,0)} w + \cdots$$

Hence (i) follows.

If $J_1(f(x,y), g(x,y))|_{(0,0)} = 0$ we obtain $x_{10} = \frac{1}{f_{10}}$, $x_{01} = -\frac{f_{01}}{f_{10}}$, $x_{20} = -\frac{f_{20}}{f_{10}^3}$, $x_{11} = \frac{2f_{20}f_{01}}{f_{10}^3} - \frac{f_{11}}{f_{10}^2}$ and $x_{02} = -\frac{f_{20}f_{01}^2}{f_{10}^3} + \frac{f_{11}f_{01}}{f_{10}^2} - \frac{f_{02}}{f_{10}}$. A straightforward computation shows that $\widetilde{g}(0,w)$ is

$$\left(-\frac{g_{10}f_{20}f_{01}^2}{f_{10}^3} + \frac{g_{10}f_{11}f_{01}}{f_{10}^2} - \frac{f_{02}g_{10}}{f_{10}} + \frac{g_{20}f_{01}^2}{f_{10}^2} - \frac{g_{11}f_{01}}{f_{10}} + g_{02} \right) w^2 + \cdots$$

$$= \frac{J_2(f(x,y), g(x,y))|_{(0,0)}}{2f_{10}^3} w^2 + \cdots ,$$

and so the lemma is proved. □

Remark 2.3. *The condition $J_1(f(x,y), g(x,y)) \neq 0$ of (i) can be written also as:*

$$\{J_1(f(x,y), g(x,y)), J_1(f(y,x), g(y,x)),$$
$$J_1(g(x,y), f(x,y)), J_1(g(y,x), f(y,x))\} \neq \{0\},$$

because in the above set the four numbers have the same absolute value.

Remark 2.4. *Recall that if $f = 0$ and $g = 0$ are planar polynomial curves, the multiplicity and the intersection number of $f \cap g$ at a common zero coincide (see* [4, Chap.6]*).*

Definition 2.5. *Let f and g be planar analytic functions and let p be a common isolated zero of f and g. The index of $X = (f, g)$ at p,* $\mathrm{ind}_p[X]$*, is defined as the degree of the map*

$$\frac{X}{\| X \|} : S^1_\varepsilon \longrightarrow S^1,$$

where $S^1_\varepsilon = \{(x, y) \in \mathbb{R}^2 : \| (x, y) - p \| = \varepsilon\}$ and ε is sufficiently small.

The following result, proved in [6], gives a relation between index and multiplicity.

Proposition 2.6. *In the notation of Definitions 2.1 and 2.5,*

$$\begin{aligned} |\mathrm{ind}_p[(P, Q)]| &\leq (\mu_p[(P, Q)])^{\frac{1}{2}}, \\ \mathrm{ind}_p[(P, Q)] &\equiv \mu_p[(P, Q)] \pmod 2. \end{aligned}$$

Corollary 2.7. (i) *For planar analytic vector fields, simple points have index $+1$ (-1) if and only if $J_1(f, g) > 0$ (< 0).*
(ii) *For planar analytic vector fields, double points have index 0.*

We remark that, of course, the above corollary holds for less regular vector fields.

3. The Euler-Jacobi formula for simple and double points.

Consider a system of two polynomial equations $P(x, y) = Q(x, y) = 0$ with $\deg P = n$ and $\deg Q = m$. If it has nm intersection points (real or complex) then for all polynomial R with $\deg R < n + m - 2$, the Euler-Jacobi formula says:

$$\sum_p \frac{R(p)}{J(p)} = 0, \tag{2}$$

where $J(p)$ is the Jacobian of $X = (P, Q)$ at the point p and the summation is extended to all nm intersection points. Observe that if P and Q have one double point $p = (p_1, p_2)$, at this point $J(p) = 0$ and formula (2) has no sense. Our objective in this section is to give a formula that generalizes (2) allowing double points. As we will see, the expression that substitutes $\frac{R(p)}{J(p)}$ in (2) is very complicated and this is the reason for which we will not deal with triple or more degenerate points.

First we recall how to prove (2). Consider the three polynomial given above, P, Q and R. Following [7, chap. 5], define

$$\begin{aligned}&\mathrm{Res}_p\left(\frac{R(x,y)}{P(x,y)Q(x,y)}dx\wedge dy\right)\\&=\left(\frac{1}{2\pi i}\right)^2\int_\Gamma\frac{R(x-p_1,y-p_2)}{P(x-p_1,y-p_2)Q(x-p_1,y-p_2)}dx\wedge dy,\end{aligned}$$

where ε is small enough, and the set $\Gamma=\{|P(x-p_1,y-p_2)|=\varepsilon,|Q(x-p_1,y-p_2)|=\varepsilon\}$ is oriented in such a way that $d(\arg P)\wedge d(\arg Q)\geq 0$. Hence, since $\deg R<n+m-2$ the Global Residue Theorem asserts that

$$\sum_p \mathrm{Res}_p\left(\frac{R(x,y)}{P(x,y)Q(x,y)}dx\wedge dy\right)=0,$$

when we assume that P and Q have nm intersection points (real or complex, taking into account their multiplicity) and the summation is extended to all the intersection points.

Hence to get a generalization of the Euler-Jacobi formula for simple and double points it suffices to have an expression of the residue at these points. The next lemma solves this problem.

Lemma 3.1. *Let P, Q and R be planar polynomials. Let p be a common zero of P and Q. Then in the notation of Lemma 2.2 we have:*

(i) *If $J_1(P(x,y),Q(x,y))\neq 0$ then*

$$\mathrm{Res}_p\left(\frac{R(x,y)}{P(x,y)Q(x,y)}dx\wedge dy\right)=\frac{R(p)}{J_1(P,Q)}.$$

(ii) *Assume that $p=(0,0)$ has multiplicity 2, (then $J_1(P,Q)(0,0)=0$ and $M_2(0,0)\neq\{0\}$).*

(ii.a) *Suppose, for instance, that $J_2=J_2(P(x,y),Q(x,y))|_{(0,0)}\neq 0$. Then, writing P, Q and R as*
$P(x,y)=P_{10}x+P_{01}y+P_{20}x^2+P_{11}xy+P_{02}y^2+\cdots,$
$Q(x,y)=Q_{10}x+Q_{01}y+Q_{20}x^2+Q_{11}xy+Q_{02}y^2+\cdots,$
$R(x,y)=R_{00}+R_{10}x+R_{01}y+R_{20}x^2+R_{11}xy+R_{02}y^2+\cdots,$
we have that

$$\begin{aligned}&\mathrm{Res}_{(0,0)}\left(\frac{R(x,y)}{P(x,y)Q(x,y)}dx\wedge dy\right)\\&=\left[\frac{4P_{10}R_{00}N}{(J_2)^2}+\frac{2P_{10}(P_{10}R_{01}-P_{01}R_{10})}{J_2}\right],\end{aligned}$$

where $N = P_{10}^3(Q_{10}P_{03} - P_{10}Q_{03}) - P_{10}^2P_{01}(Q_{10}P_{12} - P_{10}Q_{12}) + P_{10}P_{01}^2(Q_{10}P_{21} - P_{10}Q_{21}) - P_{01}^3(Q_{10}P_{30} - P_{10}Q_{30}) + P_{10}^3(Q_{11}P_{02} - P_{11}Q_{02}) - 2P_{10}^2P_{01}(Q_{20}P_{02} - P_{20}Q_{02}) + P_{10}P_{01}^2(Q_{20}P_{11} - P_{20}Q_{11})$.

(ii.b) *If* $J_2(P(x,y),Q(x,y))|_{(0,0)} = 0$*, then one of the other three elements of* $M_2(0,0)$ *is not zero. Interchanging either* P *and* Q *or* x *and* y*, we can obtain the expression of the residue using (ii.a).*

Proof. (i) This result is proved in [7, p. 671] to deduce the Euler-Jacobi formula.

(ii) If $J_2(P(x,y),Q(x,y))|_{(0,0)} \neq 0$ we have that $P_{10} \neq 0$. Hence we can consider the local change of variables (1). Denote by $\widetilde{P}(z,w)$, $\widetilde{Q}(z,w)$ and $\widetilde{R}(z,w)$ the expressions of $P(x,y)$, $Q(x,y)$ and $R(x,y)$ in these new coordinates. Hence, $\widetilde{P}(z,w) = z$ and $\widetilde{Q}(0,w) = q_2w^2 + \cdots$ Then

$$
\begin{aligned}
&\operatorname{Res}_{(0,0)}\left(\frac{R(x,y)}{P(x,y)Q(x,y)}dx\wedge dy\right)\\
&= \left(\frac{1}{2\pi i}\right)^2 \int_{\substack{|P(x,y)|=\varepsilon\\|Q(x,y)|=\varepsilon}} \frac{R(x,y)}{P(x,y)Q(x,y)}dx\wedge dy\\
&= \begin{bmatrix} z=P(x,y) & dz=\frac{\partial P}{\partial x}dx+\frac{\partial P}{\partial y}dy & dz\wedge dw=\frac{\partial P}{\partial x}dx\wedge dy\\ w=y & dw=dy & dx\wedge dy=\left(\frac{\partial P}{\partial x}\right)^{-1}dz\wedge dw\end{bmatrix}\\
&= \left(\frac{1}{2\pi i}\right)^2 \int_{\substack{|z|=\varepsilon\\|\widetilde{Q}(z,w)|=\varepsilon}} \frac{\widetilde{R}(z,w)}{z\widetilde{Q}(z,w)\left(\frac{\partial P}{\partial x}\right)\big|_{(z,w)}}dz\wedge dw\\
&= \left(\frac{1}{2\pi i}\right)^2 \int_{|z|=\varepsilon}\frac{1}{z}\left(\int_{|\widetilde{Q}(z,w)|=\varepsilon}\frac{\widetilde{R}(z,w)}{\widetilde{Q}(z,w)\left(\frac{\partial P}{\partial x}\right)\big|_{(z,w)}}dw\right)dz.
\end{aligned}
$$

This last expression, using the Cauchy integral formula, is equal to

$$
\left(\frac{1}{2\pi i}\right)\int_{|\widetilde{Q}|=\varepsilon}\frac{\widetilde{R}(0,w)}{\widetilde{Q}(0,w)\left(\frac{\partial P}{\partial x}\right)\big|_{(0,w)}}dw.
$$

Then $\operatorname{Res}_{(0,0)}\left(\frac{R(x,y)}{P(x,y)Q(x,y)}dx\wedge dy\right)$ is the coefficient of $\frac{1}{w}$ of the Laurent's serie of $\frac{\widetilde{R}(0,w)}{\widetilde{Q}(0,w)\left(\frac{\partial P}{\partial x}\right)\big|_{(0,w)}}$.

If $\widetilde{R}(0,w) = r_0 + r_1 w + r_2 w^2 + \cdots$, $\widetilde{Q}(0,w) = q_2 w^2 + q_3 w^3 + \cdots$ and $\frac{\partial P}{\partial x}\big|_{(0,w)} = p_0 + p_1 w + \cdots$, then

$$\frac{\widetilde{R}(0,w)}{\widetilde{Q}(0,w)\left(\frac{\partial P}{\partial x}\right)\big|_{|(0,w)}} = \frac{1}{w^2}\frac{r_0 + r_1 w + r_2 w^2 + \cdots}{(q_2 w^2 + q_3 w^3 + \cdots)(p_0 + p_1 w + \cdots)}$$

$$= \frac{1}{w^2 p_0 q_2}\left(r_0 + r_1 w + r_2 w^2 + \cdots\right)\left(1 - \frac{q_3}{q_2} w + \cdots\right)\left(1 - \frac{p_1}{p_0} w + \cdots\right),$$

and the expression of the residue is

$$\frac{r_1}{p_0 q_2} - \frac{r_0}{p_0 q_2}\left(\frac{q_3}{q_2} + \frac{p_1}{p_0}\right).$$

If we replace in the above formula the values of r_0, r_1, p_0, p_1, q_2 and q_3 by their expression in terms of the Taylor's expansion of P, Q and R, tedious but straightforward computations give the proof of $(ii.b)$. □

Theorem 3.2 (Generalized Euler-Jacobi formula). *Consider a system of two polynomial equations of degrees n, m in two complex unknowns $P(x,y) = Q(x,y) = 0$. Assume that the set of roots A of the system has nm elements (real or complex, taking into account their multiplicity), and that all them are either simple $A_S \subset A$, or double $A_D \subset A$. Then, for every polynomial R of degree less than $n + m - 2$, we have that*

$$\sum_{a \in A_S} \frac{R(a)}{J(a)} + \sum_{a \in A_D} S(a) = 0,$$

where $J(a)$ is the Jacobian of (P,Q) at a and $S(a)$ has to be computed in the way showed in (ii) of Lemma 3.1.

4. Distribution of critical points.

Consider a quadratic vector field $X = (P,Q)$, that is a vector field on the plane with polynomial components $P(x,y)$ and $Q(x,y)$ with $\deg(P^2 + Q^2) = 4$. The Berlinskii's Theorem gives a relation between the distribution of its critical points (when there are four) and their indices, see [2, 5]. Here we prove a generalization of this relation when the vector field X only has three real critical points. Our proof will use the generalized Euler-Jacobi formula, stated in Section 3. Another different approach to the problem would be to consider it as a limit situation of the case in which X has four critical points.

Theorem 4.1 (Generalized Berlinskii's Theorem). *Let (P,Q) be a quadratic vector field with a double point at $(0,0)$ and two simple real points.*

(i) *Consider the straight line through the origin*

$$L = DP(0,0)\begin{pmatrix} x \\ y \end{pmatrix} = P_{10}x + P_{01}y = 0,$$

or

$$\overline{L} = DQ(0,0)\begin{pmatrix} x \\ y \end{pmatrix} = Q_{10}x + Q_{01}y = 0,$$

if $L \equiv 0$*. Then the two simple points are in different connected components of* $\mathbb{R}^2 \setminus L$ *if and only if their indices coincide.*

(ii) $\mathrm{ind}_{(0,0)}[(P,Q)] = 0$.

Remark 4.2. (i) *Observe that if* $L \not\equiv 0$ *and* $\overline{L} \not\equiv 0$*, then* $L = \overline{L}$*, because* $J_1(0,0) = 0$*.*

(ii) *When the double point of* $X = X_0$ *comes from two real simple points of a quadratic vector field* X_ε*,* L *can be interpreted as the limit (when* ε *goes to zero) of the line joining these two simple points.*

Proof. (i) We are going to prove the theorem in the case $P_{10} \neq 0$. The other cases can be reduced to it.

Let $(x_0, y_0), (x_1, y_1)$ be the two simple points. Take the straight line through the origin and the point (x_0, y_0), $R(x,y) = y_0x - x_0y = 0$. By Theorem 3.2 we get:

$$\frac{y_0x_1 - x_0y_1}{J_1(x_1,y_1)} + \frac{2P_{10}(-P_{10}x_0 - P_{01}y_0)}{J_2} = 0,$$

because in this case $R_{00} = 0$.

Taking (x_1, y_1) instead of (x_0, y_0) we have

$$\frac{y_1x_0 - x_1y_0}{J_1(x_0,y_0)} + \frac{2P_{10}(-P_{10}x_1 - P_{01}y_1)}{J_2} = 0,$$

where the denominator J_2 is the same in both expressions.

Thus, we have

$$-(y_0x_1 - x_0y_1)J_2 = 2P_{10}(-P_{10}x_0 - P_{01}y_0)J_1(x_1,y_1),$$
$$-(y_1x_0 - x_1y_0)J_2 = 2P_{10}(-P_{10}x_1 - P_{01}y_1)J_1(x_0,y_0),$$

and therefore,

$$\frac{L(x_1,y_1)}{L(x_0,y_0)} = \frac{P_{10}x_1 + P_{01}y_1}{P_{10}x_0 + P_{01}y_0} = -\frac{J_1(x_1,y_1)}{J_1(x_0,y_0)},$$

by definition of L. Then (i) follows.

(ii) Follows from Corollary 2.7. □

Now we study the configuration of critical points for vector fields $X = (P, Q)$ with $\deg P = 2$, $\deg Q = 3$. We consider the definition of configuration used in [3]. Given a finite subset of $\mathbb{R}^2$, A, we denote by $\widehat{A}$ its convex hull and by ∂A its boundary. We will say that the set A has configuration $(K_0; \cdots ; K_p)$ if K_i are the natural positive numbers defined by $K_i = \#(A_i \cap \partial \widehat{A}_i)$, where $A_0 = A$ and $A_i = A_{i-1} - (A_{i-1} \cap \partial \widehat{A}_{i-1})$. If we want to be more precise, we substitute K_i by the indices of the points of $A_i \cap \partial \widehat{A}_i$, $K_i^1, K_i^2, \dots , K_i^{l_i}$, where $K_i^j \in \{+, -, 0\}$. When $A_i \cap \partial \widehat{A}_i$ is a polygon, these symbols are taken following the polygon counterclockwise from a certain start point.

Proposition 4.3. *Let $X = (P, Q)$ be a polynomial vector field such that the degrees of P and Q are 3 and 2 respectively. Assume that X has a double point and four simples points. Then, the only possible configurations are:*

(i) $(5) = (0, +, -, +, -)$ *or* $(0, -, +, -, +)$,
(ii) $(4; 1) = (0, +, +, -; -)$ *or* $(0, -, -, +; +)$ *or* $(+, +, -, -; 0)$,
(iii) $(3; 2) = (0, +, -; +, -)$ *or* $(+, +, -; -, 0)$ *or* $(-, -, +; +, 0)$.

Furthermore there are vector fields X with the above configurations.

Proof. We have that X has five critical points. Then, they can present only three configurations: (5), $(4; 1)$ and $(3; 2)$. Assume that they have configuration (5). We only consider the non degenerate case in which three critical points are never aligned, because this is the most difficult one. The degenerate case follows by similar arguments.

Put x_0 for the double point and x_1, x_2, x_3, x_4 for the simple points. Let L_i be the straight lines through x_0 and x_i for $i = 1, \dots , 4$, see Figure 1.

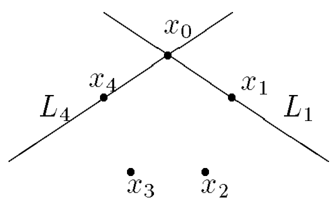

FIGURE 1. Distribution (5).

It is not restrictive to assume that $x_0 = (0, 0)$. From Corollary 2.7, x_0 has index 0, and the other critical points have index ± 1.

Now we will apply Theorem 3.2 for $R = L_1L_4$. Since $R_{00} = R_{10} = R_{01} = 0$ we get

$$\frac{R(x_2,y_2)}{J_1(x_2,y_2)} + \frac{R(x_3,y_3)}{J_1(x_3,y_3)} = 0,$$

and hence x_2 and x_3 have opposite indices since they are in the same connected component of $\mathbb{R}^2 \setminus R$. Using $R = L_2L_4$ (L_1L_3), we get that the indices of x_1 and x_3 (x_2 and x_4) coincide. Then, the configuration of the critical points are either $(0,+,-,+,-)$ or $(0,-,+,-,+)$.

The cases $(4;1)$ and $(3;2)$ follow doing a similar study.

Examples showing the configurations given are not difficult to construct. □

References

[1] V. I. Arnol'd, A. Varchenko, and S. Goussein-Zadé. *Singularités des applications différentiables*. Mir, Moscow, 1982.

[2] A. N. Berlinskiĭ. On the behavior of the integral curves of a differential equation. *Izv. Vysš. Učebn. Zaved. Matematika*, 1960(2 (15)):3–18, 1960.

[3] A. Cima, A. Gasull, and F. Mañosas. Some applications of the Euler-Jacobi formula to differential equations. *Proc. Amer. Math. Soc.*, 118(1):151–163, 1993.

[4] A. Cima. *Indices of polynomial vector fields with applications*. PhD thesis, Universitat Autònoma de Barcelona, 1987.

[5] W. A. Coppel. A survey of quadratic systems. *J. Differential Equations*, 2:293–304, 1966.

[6] D. Eisenbud and H. I. Levine. An algebraic formula for the degree of a C^∞ map germ. *Ann. Math. (2)*, 106(1):19–44, 1977. With an appendix by Bernard Teissier, "Sur une inégalité à la Minkowski pour les multiplicités".

[7] P. Griffiths and J. Harris. *Principles of algebraic geometry*. Wiley-Interscience [John Wiley & Sons], New York, 1978. Pure and Applied Mathematics.

Dept. de Matemàtiques, Universitat Autònoma de Barcelona, Edifici Cc 08193 Bellaterra, Barcelona. Spain

E-mail address: gasull@mat.uab.es, torre@mat.uab.es