SMALL-AMPLITUDE LIMIT CYCLES IN LIÉNARD SYSTEMS VIA MULTIPLICITY

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Abstract. A recent paper of Christopher and Lloyd reduces the computation of the order of degeneracy of a weak focus for a polynomial Liénard system $\dot{x} = y - F(x), \dot{y} = -g(x)$ to the computation of the multiplicity of a polynomial map. In this paper, we first take advantage of that approach to obtain new lower and upper bounds for the maximum order of degeneracy of the origin in terms of the degrees of F' and g. Later on, we implement an algorithm to compute this maximum order for concrete values of these degrees. As far as we know we enlarge the set of degrees for which this maximum order was known. Finally we extend the Christopher and Lloyd's result to analytic degenerate (or not) Liénard equations.

1. Introduction.

One of the second order differential equations that has attracted the interest of the mathematicians during these last decades is the equation of Liénard

$$\ddot{x} + f(x)\dot{x} + g(x) = 0, \tag{1}$$

or its equivalent first order system

$$\dot{x} = y - F(x),\ \dot{y} = -g(x), \tag{2}$$

where $y = \dot{x} + F(x)$, and $F(x) = \int_0^x f(s)ds$. Apart from the fact that (1) frequently appears in applications, it is studied because many other systems can be transformed into this form, see for instance [11] and [16].

One of the most studied problems concerns the number of limit cycles that (2) can have in terms of some properties of F and g.

By one side, a large number of criteria for the non existence, existence, uniqueness, ... of periodic orbits have been found, see [7], [16] and [17]. On the other side, after fixing some class of functions F and g, lower bounds for its number of limit cycles are given. These bounds are obtained either by perturbing weak foci (see for instance [5] and [18]) or by perturbing centers (see [12]).

Assume that the origin of (2) is a weak focus. In this paper we contribute to the study of its maximum order of degeneracy when $F(x)$ and $g(x)$ are polynomials of fixed degree. Our results include some of the previous ones obtained by Lloyd and Lynch in [13], [14], [15] and also by Chistopher and Lloyd in [9]. Our technique is based on a nice theorem of these last authors which we enunciate in an equivalent

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form below, after some preliminary definitions. We also have been able to extend that theorem to more general F and g.

Remember that, given a system of the form

$$\dot{x} = \lambda x + y + p(x, y),\ \dot{y} = -x + \lambda y + q(x, y), \tag{3}$$

where p and q are analytic functions beginning with at least second order terms, it is said that the origin is a *weak focus* if $\lambda = 0$. It is known that there exists an analytic function $V(x, y)$ defined in a neighbourhood of the origin such that $\frac{d}{dt}V(x(t), y(t)) = \dot{V}$ has the form $\eta_2 r^2 + \eta_4 r^4 + \ldots$ where $r^2 = x^2 + y^2$. The order of the weak focus is k if and only if

$$\eta_{2l} = 0 \text{ for } l \leq k \text{ and } \eta_{2k+2} \neq 0.$$

It is also known that at most k limit cycles can bifurcate from a weak focus of order k. The values η_{2k} defined previously are called *Lyapunov constants.*

When p and q are constrained to be polynomials of a fixed degree it is well-known that the origin of system (3) is either a center or a weak focus of bounded order.

In the case of Liénard system, following [9], we denote by ∂f the degree of a polynomial $f(x)$ and by $\mathcal{L}(n, m)$ the class of maps

$$\begin{aligned} \mathcal{L}(n, m) \ &= \ \{(f, g) : f \text{ and } g \text{ are polynomials,} \\ &\quad \partial f = n, \partial g = m, f(0) = g(0) = 0 \text{ and } g'(0) > 0\}. \end{aligned}$$

When (2) has a weak focus we denote by $\phi(f, g)$ its order. When it has a center we say that $\phi(f, g) = \infty$.

We also put

$$H(n, m) := \max_{\{(f,g)\in\mathcal{L}(n,m),\phi(f,g)<\infty\}} \phi(f, g), \tag{4}$$

and define $F(x) = \int_0^x f(s)ds$ and $G(x) = \int_0^x g(s)ds$.

Consider a $\mathcal{C}^\infty$ map $h : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that $h(0) = 0$. As usual, we denote by $\mu_0[h]$ its multiplicity at 0. Remember, for instance, that $\mu_0[h]$ is the number of complex h-preimages near 0 of a regular value of h near 0, see [3] and [4].

Theorem 1.1. *([9]) Consider $f, g \in \mathcal{L}(n, m)$. Set*

$$\mu(f, g) = \mu_{(0,0)} \left[\left(\frac{F(x) - F(y)}{x - y}, \frac{G(x) - G(y)}{x - y} \right) \right].$$

Then

(i) *if $\mu(f, g) = \infty$, the origin of (2) is a center,*
(ii) *if $\mu(f, g) < \infty$, $\phi(f, g) = \frac{\mu(f,g)}{2}$.*

A corollary of the above result is

Corollary 1.2.

$$H(n, m) = \frac{1}{2} \max_{\{(f,g)\in\mathcal{L}(n,m),\mu(f,g)<\infty\}} \mu(f, g).$$

The above results give a way to estimate the order of a weak focus for system (2) which is merely algebraic. In [9] this approach is used to improve previous results on Liénard systems.

The goals of this paper are: First, to give sharper bounds for $H(n, m)$ for general n and m (see Theorem 3.1); second, to take advantage of the properties of the multiplicity to implement a new algorithm to compute $H(n, m)$ for particular n and m, (as far as we know this algorithm allows us to enlarge the values of n and m for which $H(n, m)$ was known, see [13], [14], [15] and Table 1); and finally, to prove an extension of Theorem 1.1 for analytic degenerate (or not) Liénard equations, see Theorem 5.3.

Most of the computations of this paper have been carried out using MAPLE V.4.

This paper is organized as follows. In next section we introduce some more notation and we give some preliminary results. Section 3 deals with the theoretical study of $H(n, m)$, while in Section 4 we describe the algorithm that we have developed and the values $H(n, m)$ obtained by using it. The last section is devoted to study more general Liénard equations.

2. Preliminary Results.

A key point to study the order of a weak focus is the study of the multiplicity at zero of the map from $\mathbb{R}^2$ into $\mathbb{R}^2$

$$(\widetilde{P}(x, y), \widetilde{Q}(x, y)) := \left(\frac{F(x) - F(y)}{x - y}, \frac{G(x) - G(y)}{x - y}\right),$$

where F and G are the polynomials defined in Section 1. Observe that each component can be developed as

$$\begin{aligned} \widetilde{P}(x, y) &= \sum_{i=1}^{n} \widetilde{\alpha}_i \frac{x^{i+1} - y^{i+1}}{x - y}, \\ \widetilde{Q}(x, y) &= \sum_{i=1}^{m} \widetilde{\beta}_i \frac{x^{i+1} - y^{i+1}}{x - y}, \text{ with } \widetilde{\beta}_1 \neq 0, \end{aligned}$$

where the values $\widetilde{\alpha}_i$ and $\widetilde{\beta}_i$ are easily obtained from the expressions of $f(x)$ and $g(x)$.

To deal with the multiplicity will be useful to recall its properties. Next proposition lists some of them.

Proposition 2.1 (See [1] and[3]). *Let $f : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^n, 0)$ be a finite multiplicity map. Then:*

(i) *The multiplicity of f at zero does not depend on the choice of coordinates.*

(ii) *Let $f = (f_1, f_2, \ldots, f_n)$ and $f_i = f_i^{k_i}+$ higher order terms. Then $\mu_0[f] \geq \prod_{i=1}^n k_i$ and $\mu_0[f] = \prod_{i=1}^n k_i$ if and only if the system $f_i^{k_i} = 0$, $i = 1, \ldots, n$ has only the trivial solution in $\mathbb{C}^n$ (here $f_i^{k_i}$ is the homogeneous part of f_i of degree k_i).*

(iii) *If for some* $i \in \{1, \dots, n\}$, f_i *can be described as* $f_i = g_{i_1} \cdot g_{i_2}$ *and* $g_{i_1}(0) = g_{i_2}(0) = 0$, *then* $\mu_0[f] = \mu_0[g_1] + \mu_0[g_2]$ *where* $g_1 = (f_1, \dots, g_{i_1}, \dots, f_n)$ *and* $g_2 = (f_1, \dots, g_{i_2}, \dots, f_n)$.

(iv) *Let* $g : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^n, 0)$ *also be a finite multiplicity map. Then* $\mu_0[f \circ g] = \mu_0[f]\mu_0[g]$.

(v) *If* $g_i = f_i + \sum_{j<i} A^i_j f_j$, *then* $\mu_0[f] = \mu_0[g]$.

(vi) *If for some* $i \in \{1, \dots, n\}$, f_i *can be described as* $f_i = hg_i$ *with* $h(0) \neq 0$, *then* $\mu_0[f] = \mu_0[g]$ *where* $g = (f_1, \dots, g_i, \dots, f_n)$.

By using Proposition 2.1(i)-(vi) it turns out that we can compute $\mu_0[(\widetilde{P}, \widetilde{Q})]$ by taking the coordinates $u = x + y$, $v = x - y$ and by studying the map

$$(P, Q) : \mathbb{R}^2 \longrightarrow \mathbb{R}^2,$$

where

$$\begin{aligned} P(u, v) &= \sum_{i=1}^{n} \alpha_i R_i(u, v), \\ Q(u, v) &= u + \sum_{i=2}^{m} \beta_i R_i(u, v), \end{aligned} \tag{5}$$

and

$$R_i(u, v) = \frac{\left(\frac{u+v}{2}\right)^{i+1} - \left(\frac{u-v}{2}\right)^{i+1}}{v}, \tag{6}$$

and $\alpha_i = \widetilde{\alpha}_i$, $\beta_i = \frac{\widetilde{\beta}_i}{\widetilde{\beta}_1}$. Therefore

$$\mu_0[(\widetilde{P}, \widetilde{Q})] = \mu_0[(P, Q)]. \tag{7}$$

Next lemma gives some properties of the polynomials $R_i(u, v)$ that we will need in what follows. Note that they can easily be decomposed in product of complex linear factors by using the roots of unity.

Lemma 2.2. (i) $R_i(u, v)$ *is a homogeneous polynomial of degree* i *and satisfies*

$$\left\{ \begin{array}{lcl} R_i(-u, -v) & = & (-1)^i R_i(u, v), \\ R_i(-u, v) & = & (-1)^i R_i(u, v), \\ R_i(u, -v) & = & R_i(u, v). \end{array} \right.$$

(ii) *If* i *is odd* $R_i(u, v) = uS_{i-1}(u, v)$ *where* $S_j(u, v)$ *is a homogeneous polynomial of degree* j.

(iii) (a) R_s *divides* R_p *if and only if* $s + 1$ *divides* $p + 1$.
(b) $R_{\gcd(s+1,p+1)-1}$ *divides* R_s *and* R_p.

(iv) *Let* p, q *be integer numbers greater than zero,* $p > q$. *Then*

$$(p - q + 1)R_{2p} - R_{2q}S_{2p-2q} = u^2 T_{2p-2},$$

where $T_{2p-2}(u, v)$ *is a homogeneous polynomial of degree* $2p - 2$ *such that* $T_{2p-2}(0, v) \neq 0$, *for* $v \neq 0$.

(v) *For all k integer, $k \geq 1$,*

$$\sum_{i=0}^{k} \alpha^i \binom{k}{i} R_{nk-1+i} = (R_{n-1} + \alpha R_n) T_{(n+1)(k-1)},$$

where $T_{(n+1)(k-1)}(u,v)$ is a homogeneous polynomial of degree $(n+1)(k-1)$.

Proof. The proof of (i) and (ii) is straightforward; (iii) follows from the decomposition of $vR_i(u,v)$ in terms of the roots of unity.

Let us prove (iv). From its definition it is clear that

$$(p-q+1)R_{2p}(0,v) - R_{2q}(0,v)S_{2p-2q}(0,v) = 0.$$

Property (i) implies that $u=0$ is at least a double zero of the above expression. Therefore we have already obtained that there exists T_{2p-2} such that

$$(p-q+1)R_{2p} - R_{2q}S_{2p-2q} = u^2 T_{2p-2}.$$

Let us prove that $T_{2p-2}(0,v) \neq 0$. From their definition

$$\begin{aligned} R_{2p} &= \frac{v^{2p}}{2^{2p}} + \binom{2p+1}{2} \frac{v^{2p-2}u^2}{2^{2p}} + O(u^4), \\ S_{2r} &= \binom{2r+2}{1} \frac{v^{2r}}{2^{2r+1}} + \binom{2r+2}{3} \frac{v^{2r-2}u^2}{2^{2r+1}} + O(u^4), \end{aligned}$$

and direct computations putting $r = p - q$ give the desired result.

Let us prove (v). In this case it is easier to prove the equivalent expression in $(x,y)-$variables,

$$\sum_{i=0}^{k} \alpha^i \binom{k}{i} (x^{nk+i} - y^{nk+i}) = \left(x^n - y^n + \alpha\left(x^{n+1} - y^{n+1}\right)\right) T(x,y),$$

for some homogeneous polynomial T.

Observe that $\sum_{i=0}^{k} \binom{k}{i} (\alpha x)^i = (1+\alpha x)^k$.

Therefore the left-hand side of the above equation writes as

$$x^{nk}(1+\alpha x)^k - y^{nk}(1+\alpha y)^k,$$

which is equal to

$$\left(x^n - y^n + \alpha\left(x^{n+1} - y^{n+1}\right)\right) T(x,y),$$

because

$$a^k b^k - c^k d^k = (ab - cd)T,$$

for some polynomial $T(a,b,c,d)$ and a,b,c,d arbitrary real numbers. Hence the expression is proved. □

3. On the maximum order of weak focus.

In [9, Thm 2.3] it is proved that for Liénard systems $(f,g)\in\mathcal{L}(n,m)$

$$\left[\frac{1}{2}(n+m-1)\right]\le H(n,m)\le\left[\frac{1}{2}nm\right], \tag{8}$$

where [ ] denotes the integer part.

The first result of this section is an improvement of the above inequalities.

Given a positive natural number, for the sake of notation we denote by $e(n)$(resp. $o(n)$), the biggest even (resp. odd) integer number smaller than or equal to n. As usual $\delta_{ij}=0$ when $i\neq j$ and $\delta_{ii}=1$ for all i.

Theorem 3.1. *Let $H(n,m)$ be defined in (4). Then for $n,m\ge 4$*

$$\max\left(\frac{2\min(e(n),e(m))+\max(e(n),e(m))-2-4\delta_{e(n),e(m)}}{2},\frac{e(m)+o(n)-1}{2}\right)\le$$
$$H(n,m)\le\left[\frac{nm-\gcd(n+1,m+1)+1}{2}\right].$$

Before proving the theorem and to try to clarify our improvement we enunciate an immediate corollary corresponding to the case $m=n$.

Corollary 3.2. *Let $H(n,n)$ be defined in (4). Then for $n>1$*

$$\max\left(\left[\frac{3e(n)-6}{2}\right],n-1\right)\le H(n,n)\le\left[\frac{n^2-n}{2}\right].$$

Observe that specially our lower bound (for $n\ge 6$) is sharper than the one given in (8).

Proof of Theorem 3.1. By Theorem 1.1 and Corollary 1.2 it suffices to study multiplicities of maps. We begin by proving the upper bound. Arguing as in the proof of Christopher and Lloyd's paper [9], Bezout's Theorem implies that an upper bound for the multiplicity $\mu_0[(P,Q)]$ is nm. This result can be improved by taking into account the common zeros at infinity. Observe that the highest degree terms of P and Q are $\alpha_n R_n$ and $\beta_m R_m$, respectively. When $\gcd(n+1,m+1)\neq 1$, Lemma 2.2(iii) implies that $R_{\gcd(n+1,m+1)-1}$ divides both polynomials. Hence the polynomials P and Q have $\gcd(n+1,m+1)-1$ zeros at infinity in the projective plane. Again by Bezout's Theorem we deduce that $\mu_0[(P,Q)]$ can be at most $nm-\gcd(n+1,m+1)+1$.

Let us prove the lower bound $e(m)+o(n)-1$ for $n\ge 2$, $m\ge 3$. By using Lemma 2.2(ii) we write (5) as

$$\begin{aligned}P(u,v)&=uA(u,v)+B(u,v),\\ Q(u,v)&=uC(u,v)+D(u,v),\end{aligned}$$

where

$$A(u,v)=\sum_{i\geq 1,odd}^{o(n)}\alpha_i S_{i-1}(u,v), \qquad B(u,v)=\sum_{i,even}^{e(n)}\alpha_i R_i(u,v),$$
$$C(u,v)=1+\sum_{i\geq 3,odd}^{o(m)}\beta_i S_{i-1}(u,v), \qquad D(u,v)=\sum_{i,even}^{e(m)}\beta_i R_i(u,v).$$

Consider the case $B=0$, $D=\beta_{e(m)}R_{e(m)}(u,v)$ and $A=\alpha_{o(n)}S_{o(n)-1}(u,v)$. Then

$$\begin{aligned}\mu_0[(uA,uC+D)] &\overset{(iii)}{=} \mu_0[(u,uC+D)]+\mu_0[(A,uC+D)]=\\ &\overset{(v)}{=} \mu_0[(u,D)]+\mu_0[(A,uC+D)] \overset{(ii)}{\geq} e(m)+o(n)-1\end{aligned}$$

where we write over each equality or inequality the item of Proposition 2.1 that we have used. So we have proved $H(n,m)\geq [e(m)+o(n)-1]/2$.

Let us prove the other lower bound. Consider integer numbers $p>q\geq 2$, and define the polynomials

$$\begin{aligned}P(u,v)&=\alpha u S_{2p-2q}+\gamma R_{2p},\\ Q(u,v)&=u+\beta R_{2q},\end{aligned} \tag{9}$$

with $\gamma=\alpha\beta(p-q+1)$, and $\alpha\beta\neq 0$.

Then

$$\mu_0[(P,Q)]=\mu_0[\gamma R_{2p}-\alpha\beta S_{2p-2q}R_{2q},u+\beta R_{2q}],$$

where we have used Proposition 2.1(v) and we have changed P by $P-Q(\alpha S_{2p-2q})$. By using the value of γ we have by Lemma 2.2(iv) that

$$\gamma R_{2p}-\alpha\beta S_{2p-2q}R_{2q}=\alpha\beta u^2 T_{2p-2},$$

with $T_{2p-2}(0,v)\neq 0$ for $v\neq 0$. Therefore, using the same notation than in the previous proof,

$$\begin{aligned}\mu_0[(P,Q)] &\overset{(iii)}{=} \mu_0[(u^2T_{2p-2},u+\beta R_{2q})]\\ &\overset{(iii)}{=} 2\mu_0[(u,u+\beta R_{2q})]+\mu_0[(T_{2p-2},u+\beta R_{2q})]\\ &\overset{(iv)}{=} 2\mu_0[(u,\beta R_{2q})]+\mu_0[(T_{2p-2},u+\beta R_{2q})]\\ &\overset{(iii)}{=} 2(2q)+2p-2.\end{aligned}$$

Given $n,m\geq 4$, consider $e(n)$ and $e(m)$. Define

$$\begin{aligned}2p &= \max\{e(n),e(m)\},\\ 2q &= \min\{e(n),e(m)\}-2\delta_{e(n),e(m)}.\end{aligned}$$

Therefore if we consider the couple (P,Q) when $e(n)\geq e(m)$ or the couple $(Q,Q+P)$ when $e(n)<e(m)$, we can construct a couple $(f,g)\in\mathcal{L}(n,m)$ such that its multiplicity is

$$2\min\{e(n),e(m)\}+\max\{e(n),e(m)\}-2-4\delta_{e(n),e(m)},$$

as we wanted to prove. $\square$

A property of the function $H(n,m)$ which has already been observed in all previous papers and that our computations in next section also confirm is that $H(n,m)$ is equal to $H(m,n)$. The approach of [9] gives almost the complete answer to this problem. To be precise, let $\mathcal{L}_0(n,m)$ denote the subset of $\mathcal{L}(n,m)$ defined by $\mathcal{L}_0(n,m)=\{(f,g)\in\mathcal{L}(n,m), f'(0)>0\}$. Denote by $H_0(n,m)$ the maximum finite multiplicity inside $\mathcal{L}_0(n,m)$, then

Theorem 3.3. *([9])*

(i) *For* $n,m>1$, $H_0(n,m)=H_0(m,n)$.
(ii) *For* $n\geq m$, $H(n,m)=H_0(n,m)\leq H(m,n)$.
(iii) $H(1,m)=[\frac{m}{2}]$.

A key idea for proving the above theorem is that in general $\mu_0[(f,g)]=\mu_0[(g,f)]$ and that the only antisymmetric feature between f and g is that g is restricted to be $g'(0)>0$. By considering the additional assumption that $f'(0)>0$ the antisymmetry is broken.

From now on we will give some more properties of $H(n,m)$ addressed to study it without the assumption $f'(0)>0$.

Lemma 3.4. *The maximum multiplicity for a map of the form*

$$\begin{aligned} P &= R_{n-1}+\alpha R_n,\\ Q &= u+\sum_{i=2}^{m}\beta_i R_i, \end{aligned} \tag{10}$$

is attained for a map of the same form with $\beta_{(n+1)k-1}=0$ *for any* k *(*$k\neq 1$ *just for the case* $n=2$*) such that* $2\leq (n+1)k-1\leq m$.

Proof. Let us fix α and $\beta_2,\ldots,\beta_m$ such that the multiplicity of (P,Q) is finite and $\mu_0[(P,Q)]$ is the maximum inside this family.

Consider the biggest k satisfying $(n+1)k-1\leq m$ and call it K. By Lemma 2.2(v)

$$(R_{n-1}+\alpha R_n)T_{(n+1)(K-1)}=\sum_{i=0}^{K}\alpha^i\binom{K}{i}R_{nK-1+i},$$

and therefore by Proposition 2.1(v),

$$\begin{aligned} \mu_0[(P,Q)] &= \mu_0[(R_{n-1}+\alpha R_n, u+\sum_{i=2}^{m}\beta_i R_i-\frac{\beta_{(n+1)K-1}}{\alpha^K\binom{K}{K}}(R_{n-1}+\alpha R_n)T_{(n+1)(K-1)})]\\ &= \mu_0[(P,\overline{Q})], \end{aligned}$$

where $(P,\overline{Q})$ are in the form (10) with $\beta_{(n+1)K-1}=0$, and the other values of β_j with j less than this value, may be different. Then we can continue the process but taking the next value of K, in decreasing order, which satisfies $(n+1)K-1\leq m$ and cancelling it, and so on.

In the case $n=2$ notice that if $K=1$ then $T_{(n+1)(K-1)}=0$ and the previous argument does not apply. □

In [14] and [15] the following conjecture is made and it is proved until $m=12$.

Conjecture 3.5. *([14], [15]) $H(2,m) = H(m,2) = [\frac{2m+1}{3}]$.*

Observe that the above lemma supports the above conjecture. This is due to the fact that if we count the number of parameters which cannot be cancelled by using it for systems in $\mathcal{L}(2,m)$, we get that this number coincides with $[\frac{2m+1}{3}]$. We also have proved next result which also supports the conjecture.

Lemma 3.6. *Consider a system (2) with $n = 2$ and arbitrary m. The following statements hold:*

(i) *To find $H(2,m)$ it is not restrictive to study the multiplicity at $(0,0)$ of the map*

$$\begin{aligned} P(u,v) &= u + (3u^2 + v^2), \\ Q(u,v) &= u + \beta_2 R_2(u,v) + \sum_{i=3, i+1 \nmid 3}^{m} \beta_i R_i(u,v). \end{aligned} \tag{11}$$

(ii) *Consider $H(u) := Q(u,v)|_{v^2=-u-3u^2}$. Then either $H(u) \equiv 0$ and system (2) has a center at the origin or*

$$H(u) = K u^{\phi(f,g)} + O(u^{\phi(f,g)+1}), \quad K \neq 0,$$

where recall that $\phi(f,g)$ is the order of the weak focus at the origin for system (2).

(iii) *For $m \leq 50$, $H(2,m) = [\frac{2m+1}{3}]$.*

Proof. (i) It is easy to see that the maximum multiplicity has to be taken with $\alpha_1\alpha_2 \neq 0$. By making a scaling of the variables u and v, if necessary, and by applying Proposition 2.1 we can assume that $\alpha_1 = \alpha_2 = 1$. The fact that all $\beta_{3k-1}, k \geq 2$ can be taken as zero follows from Lemma 3.4.

(ii) From the above part $P(u,v) = u + 3u^2 + v^2$ and so

$$\begin{aligned} \phi(f,g) &= \mu_0[P,Q]/2 = \frac{1}{2}\left(\mu_0[v + \sqrt{-u-3u^2}, Q] + \mu_0[v - \sqrt{-u-3u^2}, Q]\right) \\ &= \mu_0[Q(u,v)|_{v^2=-u-3u^2}] = \mu_0[H(u)], \end{aligned}$$

where we have used that $Q(u,v)$ is a function of v^2. Hence the proof is done.

To prove (iii), observe that (ii) implies that the way of obtaining maximum multiplicity corresponds with the way of choosing β_i such that its associated function $H(u)$ has the maximum multiplicity at zero. We can compute the functions

$$\begin{aligned} R_2(u,v)|_{v^2=-u-3u^2} &= \frac{1}{2^3}(-u), \\ R_3(u,v)|_{v^2=-u-3u^2} &= \frac{1}{2^4}(-8u^2 - 16u^3), \\ R_4(u,v)|_{v^2=-u-3u^2} &= \frac{1}{2^5}(2u^2 - 8u^3 - 32u^4), \ldots \end{aligned}$$

and then the problem of obtaining the maximum multiplicity at zero is reduced to a problem of linear algebra. As an example we show all the computations for

$m=10$. The conditions for $H(u)$ to have at least multiplicity 7 at the origin write as

$$\begin{pmatrix} -8 & 2 & 0 & 0 & 0 & 0 \\ -16 & -8 & -2 & 0 & 0 & 0 \\ 0 & -32 & 24 & -16 & 0 & 0 \\ 0 & 0 & 128 & -32 & 20 & -2 \\ 0 & 0 & 128 & 128 & 0 & 80 \end{pmatrix} \begin{pmatrix} \gamma_3 \\ \gamma_4 \\ \gamma_6 \\ \gamma_7 \\ \gamma_9 \\ \gamma_{10} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

where $\gamma_i=\beta_i/2^{i+1}$. This system has its solutions of the form $\gamma_i=K_i\gamma_3$ for some nonzero K_i. For this values we obtain that

$$H(u)=Ku^7+O(u^8), \quad K\neq 0,$$

and therefore $H(2,10)=7$, as we wanted to prove.

Unfortunately we have not been able to make the above computations for an arbitrary m but we have proved that $H(2,m)=[(2m+1)/3]$ is true for $m\leq 50$.

To end this proof we want to comment that we have also obtained the values $H(2,m)$ for $m\leq 50$ by using the general algorithm developed in next section. $\square$

We also have proved next results. As far as we know, results (*ii*) and (*iii.b*) are new. Result (*i*) is a corollary of known results, see [9].

Theorem 3.7. (i) *$H(1,m)=H(m,1)$.*
(ii) *$H(2,m)=H(m,2)$.*
(iii) (a) *$H(3,m)=H(m,3)$, for $m\leq 8$.*
(b) *Assume that $H(2,m)>\frac{e(m)+2}{2}$. Then $H(3,m)=H(m,3)$.*

Remark 3.8. (i) *The lower bounds of Theorem 3.1 just imply that $H(2,m)\geq \frac{e(m)+2}{2}$.*
(ii) *For all $m\leq 50$ we have proved, in Lemma 3.6, that Conjecture 3.5 holds. Therefore, for $9\leq m\leq 50$, $H(2,m)=[\frac{2m+1}{3}]>\frac{e(m)+2}{2}$, which proves that $H(3,m)=H(m,3)$ for $m\leq 50$.*

Proof of Theorem 3.7. For cases n and m less than 5 it is well-known that $H(n,m)=H(m,n)$, see for instance [14], or next section.

So, from now on, we are just interested in the remainder cases.

Theorem 3.3(ii) implies that $H(n,m)\leq H(m,n)$ for $n=1,2,3$ and $m>4$.

Therefore to end our proof we need to prove the converse inequality,

$$H(n,m)\geq H(m,n), \tag{12}$$

when $n=1,2,3$ and $m>4$.

(*i*) Case $n=1$. A system $(f,g)\in\mathcal{L}(1,m)$ writes as

$$\begin{aligned} P &= \alpha_1 u, \\ Q &= u+\beta_2R_2+\ldots+\beta_mR_m. \end{aligned}$$

Therefore the maximum finite multiplicity should be attained by a case with $\alpha_1\neq 0$ which corresponds to a particular system in $\mathcal{L}(m,1)$ obtained by considering the couple $(Q, P/\alpha_1)$ instead of (P, Q). In other words

$$H(1, m) \leq H(m, 1),$$

as we wanted to prove.
(ii) Case $n = 2$. As before, if a system $(f, g) \in \mathcal{L}(2, m)$, it writes as

$$\begin{aligned} P &= \alpha_1 u + \alpha_2 R_2, \\ Q &= u + \beta_2 R_2 + \ldots + \beta_m R_m. \end{aligned}$$

If the system with (finite) maximum multiplicity had $\alpha_1 = 0$ this multiplicity would be

$$\mu_0[(P, Q)] = \mu_0[(\alpha_2 R_2, u + \beta_2 R_2 + \ldots + \beta_m R_m)],$$

which by Proposition 2.1(ii) is 2. This fact is in contradiction with the lower bounds given in Theorem 3.1. Therefore α_1 has to be different from zero and arguing as in the case $n = 1$ we obtain that

$$H(2, m) \leq H(m, 2),$$

and the proof is ended.
(iii) Case $n = 3$. The proof for cases $m \leq 8$ is done in next section. Let us consider $m > 8$. Let

$$\begin{aligned} P &= \alpha_1 u + \alpha_2 R_2 + \alpha_3 R_3, \\ Q &= u + \beta_2 R_2 + \ldots + \beta_m R_m, \end{aligned}$$

be a system inside $\mathcal{L}(3, m)$ with maximum (finite) multiplicity. If $\alpha_1 \neq 0$ arguing as in cases $(i), (ii)$ we are done. Assume that the maximum multiplicity is when $\alpha_1 = 0$. Then $\alpha_2 = 0$ (if not Proposition 2.1(ii) would imply that the maximum multiplicity is 2). Therefore the maximum multiplicity is

$$\mu_0[(\alpha_3 R_3, Q)] = \mu_0[(uS_2, Q)] = \mu_0[(u, Q)] + \mu_0[(S_2, Q)] \leq e(m) + 2,$$

which is in contradiction with our assumption. So we are done. □

Although we are convinced that the maximum multiplicity of a map in $\mathcal{L}(n, m)$, $n < m$, is always taken with $\alpha_1 \neq 0$ we have not been able to prove it. Note that this fact would imply that $H(n, m) = H(m, n)$.

Before ending this section we give some miscellaneous results with the aim to show the difficulty to find a general rule to obtain $H(n, m)$ for general n and m. See also Table 1 in next section.

Proposition 3.9. (i) $H(n, n) = \max\{H(n-1, n), H(n, n-1)\}$,
(ii) $H(4, m) \geq m + 1 - \left[\frac{m+1}{5}\right]$ *for* $6 \leq m \leq 20$,
(iii) $H(n, m) = \max\{H(m, n), H_0(n, m)\}$.

Proof. (i) Consider the system $(f, g) \in \mathcal{L}(n, n)$ with maximum(finite) multiplicity,

$$\begin{aligned} P &= \sum_{i=1}^{n} \alpha_i R_i, \\ Q &= u + \sum_{i=2}^{n} \beta_i R_i. \end{aligned}$$

If some α_n or β_n is zero we are done. Otherwise,

$$\mu_0[(P,Q)] = \mu_0[(P - \frac{\alpha_n}{\beta_n}Q, Q)].$$

Observe that this last map corresponds to some $(f,g) \in \mathcal{L}(n-1,n)$ and therefore

$$H(n-1,n) \geq H(n,n).$$

Since obviously $H(n,n) \geq H(n-1,n)$, we are done.
(ii) This lower bound for $n=4$ follows by studying the case $\alpha_1 = \alpha_2 = 0$, $\alpha_3 = 1$ and $\alpha_4 = \alpha$ in the expression of P. To obtain it we first apply Lemma 3.4 and later the algorithm developed in next section.
(iii) The value $H(n,m)$ can be obtained from a system with either $\alpha_1 = 0$ or $\alpha_1 \neq 0$. In the first case $H(n,m) = H_0(n,m)$, and in the second case we can interchange the functions P and Q to obtain the same multiplicity and so $H(n,m) = H(m,n)$. □

Remark 3.10. *The same idea used in Proposition 3.9.(ii) can be applied to give lower bounds for $H(2k,m)$.*

4. An algorithm for computing the multiplicity with applications.

In the first part of this section we describe an algorithm which takes advantage of the results of Theorem 1.1 and of the properties of the multiplicity (see Proposition 2.1) to compute the order of degeneracy of the origin of system (2). In the second part we use it to obtain the function $H(n,m)$ for several n and m. These results are summarized in Table 1. Afterwards, we detail the computation made to obtain $H(6,6)$ as an example of the way in which Table 1 is filled up. As a consequence of the algorithm we also characterize all the centers for system (2) when $n=m=6$.

4.1. **The Algorithm.** Let P, Q be polynomials in (u,v) of degree n and m respectively. We define the following functions:

(i) $\mathtt{small}(P) :=$ homogeneous part of less degree of P.
(ii) $\mathtt{subtract}(P,Q) := P - \frac{\mathtt{small}(P)}{u}Q$.
(iii) $\mathtt{coeffv}(P) := \mathtt{small}(P)|_{u=0}$.

Notice that $\mathtt{small}(P)$ is a polynomial and $\mathtt{subtract}(P,Q)$ is a polynomial if $\mathtt{coeffv}(P) = 0$.

To obtain the multiplicity of (P,Q) at the origin, i.e. $\mu_0[(P,Q)]$, we use the properties of Proposition 2.1 as follows: if $\mathtt{coeffv}(P) \neq 0$ then $\mu_0[(P,Q)] = \deg(\mathtt{small}(P))$; otherwise, we redefine the polynomial P as $P = \mathtt{subtract}(P,Q)$, and observe whether $\mathtt{coeffv}(P)$ is or not null. In the first case we can repeat the algorithm until we obtain a non null term. In the second case the algorithm finishes and the multiplicity is the degree of $\mathtt{small}(P)$. The algorithm is finite because when we obtain the degree of P greater than nm then the multiplicity has to be infinity, by Bezout's Theorem. See also Figure 1.

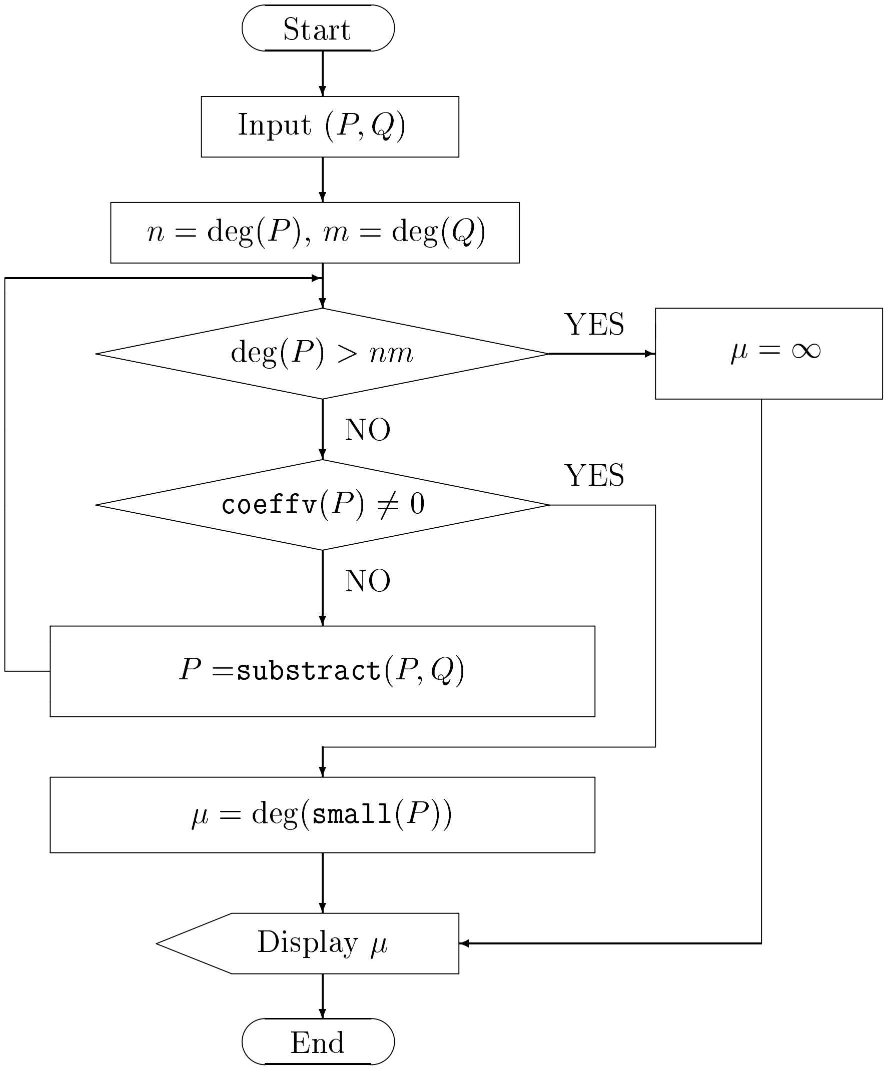

FIGURE 1. The Algorithm's Diagram for computing the multiplicity of (P, Q). The functions, `coeffv`, `substract` and `small` are defined in Section 4.1

Remark 4.1. *Observe that our algorithm does not give either the stability of the origin or its cyclicity. Although we are convinced that all this information is included in it, we do not consider these problems in this paper. In particular we think that in all the cases for which $H(n, m)$ is computed, this number coincides with the maximum cyclicity of the weak focus inside $\mathcal{L}(n, m)$.*

4.2. **Some values of $H(n, m)$.** The above algorithm allows to compute $H(n, m)$ for several values of n and m. The results obtained are shown in Table 1. In that table the numbers between parentheses give lower bounds for $H(n, m)$. As an example of the difficulties involved, we will give the detailed computation of

	1	2	3	4	5	6	7	8	9	10	11	12	$\cdots$	50	$\cdots$	n
1	0	1	1	2	2	3	3	4	4	5	5	6	$\cdots$	25	$\cdots$	$[\frac{n}{2}]$
2	1	1	2	3	3	4	5	5	6	7	7	8	$\cdots$	33		
3	1	2	2	4	4	6	6	7	8	9	9	11				
4	2	3	4	4	6	6	(8)	(8)								
5	2	3	4	6	6	7	(8)	(9)								
6	3	4	6	6	7	7	(9)									
7	3	5	6	(8)	(8)	(9)	(9)									
8	4	5	7	(8)												
9	4	6	8	(8)												
10	5	7	9	(9)												
$\vdots$	$\vdots$	$\vdots$		$\vdots$												
20	10	13		(17)												
$\vdots$	$\vdots$	$\vdots$														
50	25	33														
$\vdots$	$\vdots$															
m	$[\frac{m}{2}]$															

TABLE 1. Values of $H(n,m)$ defined in (4). The numbers between parentheses are just lower bounds.

$H(6,6)$. Also as a corollary of the computations involved we give all the centers for system (2) when $(f,g) \in \mathcal{L}(6,6)$.

As far as we know our main contributions are in the second row and column for n or m bigger than 12 and for the other values n and m for which $H(n,m)$ is bigger than 6. See [13], [14] and [15].

Proposition 4.2. *Consider the system (2) with $f(x)$ and $g(x)$ polynomials of degree 6,*

$$\begin{aligned} F(x) &= \int_0^x f(s)ds = a_2x^2 + a_3x^3 + \ldots + a_7x^7, \\ g(x) &= x + b_2x^2 + \ldots + b_6x^6. \end{aligned}$$

It is not restrictive to assume that when $b_2 \neq 0$ it is equal to $\frac{3}{2}$.

Then:

(a) *It has a center at the origin if and only if one of the following set of conditions hold:*
  (i) $a_3 = a_5 = a_7 = b_4 = b_6 = 0$ *and* $b_2 = 0$,
  (ii) $a_3 = 4a_4 - 2a_2b_3 = 5a_5 - a_2b_4 = 6a_6 - a_2b_5 = 7a_7 - a_2b_6 = 0$ *and* $b_2 = 0$,
  (iii) $a_2 - a_3 = 4a_4 - 2a_3b_3 = 5a_5 - 2a_3b_4 = 6a_6 - 2a_3b_5 = 7a_7 - 2a_3b_6 = 0$ *and* $b_2 = \frac{3}{2}$,
  (iv) $5b_3 - 2b_4 = 5b_5 - 3b_4 = b_6 = a_2 - a_3 = 2a_4 - a_5 = 2a_6 - a_5 = a_7 = 0$ *and* $b_2 = \frac{3}{2}$.

(b) *The maximum order of degeneracy of the focus is 7, i.e. $H(6,6) = 7$.*

Proof. To prove it we will use Theorem 1.1 and the notations introduced in (5). Furthermore note that it is not restrictive to assume that $\beta_2 = \frac{2}{3}b_2$ is either 0 or 1, and that the proof of the theorem is reduced to the computation of $\mu_0[P,Q]$.

(a) In this part of the proof we also follow the same notations than in the algorithm developed in Subsection 4.1. Therefore if j is the degree of $\texttt{small}(P)$ in each step of the algorithm and we put $c_j := \texttt{coeffv}(P)$, then $c_{2j} = 0$ for $j = 1, \ldots, J$ give necessary conditions to have a weak focus of order at least $J+1$.

In the first case, $\beta_2 = 0$, we obtain,

$$\begin{aligned}
c_2 &= \frac{1}{2^2}\alpha_2,\\
c_4 &= -\frac{1}{2^4}(\alpha_1\beta_4 - \alpha_4),\\
c_6 &= -\frac{1}{2^6}(-2\alpha_4\beta_3 + 2\alpha_3\beta_4 + \alpha_1\beta_6 - \alpha_6),\\
c_8 &= -\frac{1}{2^8}(-2\alpha_6\beta_3 + 3\alpha_5\beta_4 - 3\alpha_4\beta_5 + 2\alpha_3\beta_6),\\
c_{10} &= -\frac{3}{2^{10}}(-\beta_5\alpha_6 + \alpha_5\beta_6),\\
c_{12} &= -\frac{5}{2^{13}}\beta_4(-\alpha_6\beta_4 + \alpha_4\beta_6),\\
c_{14} &= -\frac{17}{2^{15}3}\beta_6(-\alpha_6\beta_4 + \alpha_4\beta_6).
\end{aligned}$$

The system $\{c_2 = c_4 = \ldots = c_{12} = 0\}$ has two solutions,

(i) $\alpha_2 = \alpha_4 = \alpha_6 = \beta_2 = \beta_4 = \beta_6 = 0$, and
(ii) $\alpha_2 = \alpha_3 - \alpha_1\beta_3 = \alpha_4 - \alpha_1\beta_4 = \alpha_5 - \alpha_1\beta_5 = \alpha_6 - \alpha_1\beta_6 = \beta_2 = 0$.

The first family, (i), has a center at the origin because the polynomials P and Q, defined in (5) have the common factor u and therefore the multiplicity at the origin of (P,Q) is infinity. The second family, (ii), satisfies $\alpha_1 P = Q$, and for the same reason the system (2) has a center at the origin. From the definition of $\alpha_i = a_{i+1}$ and $\beta_i = \frac{2}{i+1}b_i$ we obtain the families (i-ii) of the statement.

In the second case, $\beta_2 = 1$, we obtain:

$$\begin{aligned}
c_2 &= -\frac{1}{2^2}(-\alpha_2 + \alpha_1),\\
c_4 &= \frac{1}{2^4}(\alpha_4 - 2\alpha_3 - \alpha_2\beta_4 + 2\alpha_2\beta_3),\\
c_6 &= -\frac{1}{2^7}(-4\alpha_4\beta_3 + 3\alpha_2\beta_4 + 4\alpha_3\beta_4 - 6\alpha_2\beta_5 + 2\alpha_2\beta_6 - 3\alpha_4 + 6\alpha_5 - 2\alpha_6),\\
c_8 &= -\frac{1}{2^{10}}(-8\alpha_6\beta_3 + 11\alpha_2\beta_4 + 12\alpha_5\beta_4 - 22\alpha_2\beta_5 - 12\alpha_4\beta_5 + 22\alpha_2\beta_6 +\\
&\quad 8\alpha_3\beta_6 - 11\alpha_4 + 22\alpha_5 - 22\alpha_6),
\end{aligned}$$

$$
\begin{aligned}
c_{10} = & -\frac{1}{2^{13}}(-104\alpha_5\beta_3 + 169\alpha_2\beta_4 + 156\alpha_5\beta_4 - 52\alpha_6\beta_4 - 338\alpha_2\beta_5 + 104\alpha_3\beta_5 - \\
& 156\alpha_4\beta_5 - 24\alpha_6\beta_5 + 286\alpha_2\beta_6 + 52\alpha_4\beta_6 + 24\alpha_5\beta_6 - 169\alpha_4 + 338\alpha_5 - \\
& 286\alpha_6), \\
c_{12} = & -\frac{1}{2^{16}13^2}(2873\alpha_4 - 5746\alpha_5 + 5746\alpha_6 - 2873\alpha_2\beta_4 + 5746\alpha_2\beta_5 - \\
& 5746\alpha_2\beta_6 + 45968\alpha_5\beta_4 + 67652\alpha_4\beta_5 - 34476\alpha_6\beta_4 + 96408\alpha_6\beta_5 - \\
& 53404\alpha_4\beta_6 + 76752\alpha_5\beta_6 - 28730\alpha_4\beta_4 + 28730\alpha_2\beta_4^2 - 112320\alpha_5\beta_5 + \\
& 112320\alpha_2\beta_5^2 - 65000\alpha_6\beta_6 + 28080\alpha_4\beta_5^2 + 65000\alpha_2\beta_6^2 - 6760\alpha_6\beta_4^2 + \\
& 87880\alpha_2\beta_4\beta_6 - 113620\alpha_2\beta_4\beta_5 - 173160\alpha_2\beta_5\beta_6 - 28080\alpha_5\beta_5\beta_4 + \\
& 6760\alpha_4\beta_4\beta_6 - 28080\alpha_4\beta_5\beta_6 - 9360\alpha_6\beta_4\beta_6 + 28080\alpha_5\beta_4\beta_6 - \\
& 4320\alpha_6\beta_5\beta_6 + 9360\alpha_4\beta_6^2 + 4320\alpha_5\beta_6^2), \\
c_{14} = & \frac{1}{2^{19}13^2 75}(-600457\alpha_4 + 1200914\alpha_5 - 1200914\alpha_6 + 600457\alpha_2\beta_4 - \\
& 1200914\alpha_2\beta_5 + 1200914\alpha_2\beta_6 + 2401828\alpha_5\beta_4 - 3691168\alpha_4\beta_5 - \\
& 4803656\alpha_6\beta_4 + 4106128\alpha_6\beta_5 + 8913736\alpha_4\beta_6 - 14904968\alpha_5\beta_6 + \\
& 2578680\alpha_5\beta_5 - 2578680\alpha_2\beta_5^2 + 7587840\alpha_6\beta_6 + 4268160\alpha_4\beta_5^2 - \\
& 7587840\alpha_2\beta_6^2 - 3211000\alpha_6\beta_4^2 - 4110080\alpha_2\beta_4\beta_6 + 1289340\alpha_2\beta_4\beta_5 + \\
& 10798840\alpha_2\beta_5\beta_6 - 4268160\alpha_5\beta_5\beta_4 + 3211000\alpha_4\beta_4\beta_6 - 4268160\alpha_4\beta_5\beta_6 - \\
& 273520\alpha_6\beta_4\beta_6 + 7291440\alpha_5\beta_4\beta_6 - 656640\alpha_6\beta_5\beta_6 - 3023280\alpha_6\beta_4\beta_5 - \\
& 2183480\alpha_4\beta_5\beta_4 - 773600\alpha_5\beta_5\beta_6 + 273520\alpha_4\beta_6^2 + 656640\alpha_5\beta_6^2 + \\
& 44773600\alpha_6\beta_5^2 + 2183480\alpha_5\beta_4^2), \\
c_{16} = & -\frac{1}{2^{18}13^2 289}(-163761\alpha_4 + 327522\alpha_5 - 327522\alpha_6 + 163761\alpha_2\beta_4 - \\
& 327522\alpha_2\beta_5 + 327522\alpha_2\beta_6 + 655044\alpha_5\beta_4 - 1744808\alpha_4\beta_5 - \\
& 1310088\alpha_6\beta_4 - 264784\alpha_6\beta_5 + 5821543\alpha_4\beta_6 - 10937654\alpha_5\beta_6 + \\
& 2179528\alpha_5\beta_5 - 2179528\alpha_2\beta_5^2 + 8286850\alpha_6\beta_6 + 1529424\alpha_4\beta_5^2 - \\
& 8286850\alpha_2\beta_6^2 - 1605500\alpha_6\beta_4^2 - 4511455\alpha_2\beta_4\beta_6 + 1089764\alpha_2\beta_4\beta_5 + \\
& 11202438\alpha_2\beta_5\beta_6 - 1529424\alpha_5\beta_5\beta_4 + 1605500\alpha_4\beta_4\beta_6 - 988\alpha_4\beta_5\beta_6 + \\
& 1547260\alpha_6\beta_4\beta_6 + 4569500\alpha_5\beta_4\beta_6 + 1881000\alpha_6\beta_5\beta_6 - 4568512\alpha_6\beta_4\beta_5 - \\
& 2195856\alpha_5\beta_5\beta_6 - 1547260\alpha_4\beta_6^2 - 1881000\alpha_5\beta_6^2 + 2195856\alpha_6\beta_5^2).
\end{aligned}
$$

To obtain the possible centers we can solve the system $\{c_2 = \ldots = c_{16} = 0\}$, and we obtain the next two families

(iii) $\alpha_1 - \alpha_2 = \alpha_3 - \alpha_2\beta_3 = \alpha_4 - \alpha_2\beta_4 = \alpha_5 - \alpha_2\beta_5 = \alpha_6 - \alpha_2\beta_6$,
(iv) $\beta_3 - 1/2\beta_4 = \beta_5 - 1/2\beta_4 = \beta_6 = 0 = \alpha_1 - \alpha_2 = \alpha_3 - 1/2\alpha_4 = \alpha_5 - 1/2\alpha_4 = \alpha_6 = 0$.

For the above two families the system (2) has a center at the origin because in both cases $\mu_0[P, Q] = \infty$. In the first one this is due to the fact that $\alpha_2 P = Q$. In the second case the reason is that $(4u+3u^2+v^2)|P$ and $(4u+3u^2+v^2)|Q$. Hence the proof of the first part of the theorem is done.

(b) It is easy to see that $c_{14}=0$, or $c_{16}=0$, if the parameters are in the families (i-ii), or (iii-iv), respectively. In other words we have that the condition $c_{14}=0$ (resp. $c_{16}=0$) is redundant to characterize the centers of system (2) when $\beta_2=0$ (resp. $\beta_2=1$). This fact forces an upper bound for $H(6,6)$, i.e. $H(6,6)\leq \frac{14}{2}=7$. To obtain a lower bound, we study the particular case

$$\alpha_1=0, \alpha_2=0, \alpha_3=1, \alpha_4=2, \alpha_5=-\frac{988}{189}, \alpha_6=-\frac{715}{63},$$

$$\beta_1=1, \beta_2=1, \beta_3=-\frac{11}{6}, \beta_4=0, \beta_5=-5203441, \beta_6=-25289882,$$

which satisfies $c_2=\ldots=c_{12}=0$ and $c_{14}=-\frac{2737867}{146313216}$. This proves that $H(6,6)=7$. $\square$

5. Degenerate Liénard equation

In this section we extend the results of Theorem 1.1 to analytic degenerate (or not) Liénard equations of the form

$$\dot{x}=\frac{dx}{dt}=\varphi(y)-F(x),\ \dot{y}=\frac{dy}{dt}=-g(x), \tag{13}$$

where φ, F and g are analytic functions satisfying

$$\begin{aligned} \varphi(y) &= y^{2m-1}+O(y^{2m}),\\ F(x) &= a_kx^k+O(x^{k+1}),\\ G(x) &= \int_0^x g(s)\,ds=\frac{x^{2l}}{2l}+O(x^{2l+1}), \end{aligned}$$

with $m,k,l\in\mathbb{N}$ being non zero. As in the non degenerate polynomial case ($m=l=1$) we can define

$$\mu(f,g)=\mu_{(0,0)}\left[\left(\frac{F(x)-F(y)}{x-y},\frac{G(x)-G(y)}{x-y}\right)\right].$$

Our aim is to relate $\mu(f,g)$ to some qualitative property of equation (13). First we need two technical lemmas.

Lemma 5.1. *Consider $G(x)$ given in equation (13). Define in a neighbourhood of 0 the analytic function*

$$u=\phi(x)=x\sqrt[2l]{\frac{2lG(x)}{x^{2l}}}=x+O(x^2).$$

Let $\phi^{-1}(u)$ denote its inverse. Therefore

$$G(x)-G(y)=\left(\prod_{\{\sigma\in\mathbb{C}\,:\,\sigma^{2l}=1\}}\left(\phi^{-1}\left(\sigma\phi(x)\right)-y\right)\right)\widetilde{G}(x,y),$$

where $\widetilde{G}$ is an analytic function which does not vanish in a $\mathbb{C}^2$-neighbourhood of 0.

Proof. By Weirstrass' preparation Theorem and the results of Chapter XIII.32 of [2] we have that

$$G(x) - G(y) = \left(y^{2l} + A_1(x)y^{2l-1} + A_2(x)y^{2l-2} + \cdots + A_{2l}(x)\right) \widetilde{G}(x,y),$$

for some analytic functions $A_j(x)$ and $\widetilde{G}$ and with $\widetilde{G}(0,0) \neq 0$. On the other hand observe that by (13), $u^{2l} = 2lG\left(\phi^{-1}(u)\right)$. By substituting in this last expression $u = \sigma\phi(x)$ where σ is a $2l$ root of unity we obtain, for all σ, that

$$\phi^{2l}(x) = 2lG\left(\phi^{-1}\left(\sigma\phi(x)\right)\right) = 2lG(x).$$

Therefore we have found the $2l$ roots of $y^{2l} + A_1(x)y^{2l-1} + A_2(x)y^{2l-2} + \cdots + A_{2l}(x)$ when we consider it as a polynomial in y and the proof follows. ☐

Lemma 5.2. *(see* [10]*)*

(i) *Consider the change of variables and time*

$$u = \phi(x), \quad y = y, \quad \frac{dt}{ds} = \frac{u^{2l-1}}{g(x)},$$

where ϕ is defined in the previous the lemma. Then system (13) writes in a neighbourhood of $(0,0)$ *as*

$$u' = \varphi(y) - F\left(\phi^{-1}(u)\right), \;\; y' = -u^{2l-1}. \tag{14}$$

(ii) *Set $F\left(\phi^{-1}(u)\right) = \sum_{i\geq k} f_i u^i$ in system (14) and assume that $k > l(2m-1)/m$. Then it has a center at the origin if and only if $F\left(\phi^{-1}(u)\right)$ is an even function.*

From the above lemma it seems natural to define *the order of degeneracy of the origin for equation (13)* as $S := \inf\{i \,:\, f_{2i+1} \neq 0\}$, where $F\left(\phi^{-1}(u)\right) = \sum_{i\geq k} f_i u^i$. When this order of degeneracy does not exist, we say that it is infinity and the point is a center. Observe that when $l = m = 1$ the above definition coincides with the definition of order of a weak focus. Furthermore, it can be proved that if the first odd term in $F\left(\phi^{-1}(u)\right)$ is $f_{2S+1}u^{2S+1}$ then the sign of f_{2S+1} determines the stability of the origin. So F_{2S+1} can be thought as a kind of Lyapunov constant for degenerate critical points of Liénard equations.

Our main result is given in the next theorem. Observe that it extends Theorem 1.1, to analytic degenerate (or not) Liénard equations.

Theorem 5.3. *Consider the analytic Liénard equation (13). Assume that associated to it, we have $F\left(\phi^{-1}(u)\right) = \sum_{i\geq k} f_i u^i$ with $k > l(2m-1)/m$, where ϕ^{-1} is defined in Lemma 5.1. Let S be the order of degeneracy of the origin of (13). Then*

(i) *If $\mu(f,g) = \infty$ then the origin of (13) is a center.*

(ii) *If* $\mu(f,g)<\infty$ *then*

$$S\leq \frac{\mu(f,g)}{2}-(k-1)(l-1).$$

Furthermore the above inequality is an equality if and only if either $l=1$ *or* $l>1$ *and* k *and* $2l$ *are coprime.*

Proof. (i) The proof of first part follows from Lemma 5.1 and next assertion which is proved in Theorem 2.6 of [10]: "Under the hypotheses of our theorem the origin of (13) is a center if and only if for x small enough the system $F(x)=F(y), G(x)=G(y)$ has a unique solution $y=z(x)$ satisfying $z(0)=0$ and $z'(0)<0$". This result is a generalization of a well-known characterization of the centers for non degenerate Liénard equations, see [6] and [8].

(ii) Let us compute $\mu(f,g)$. From Lemma 5.1 the equation $(G(x)-G(y))/(x-y)=0$ has the same solutions in a neighbourhood of $(0,0)$ that

$$\prod_{\{\sigma\in\mathbb{C}\,:\,\sigma^{2l}=1\}\setminus\{1\}}\left(\phi^{-1}\left(\sigma\phi(x)\right)-y\right).$$

Therefore

$$\begin{aligned}\mu(f,g) &= \mu_0\left[\frac{F(x)-F(y)}{x-y},\prod_{\{\sigma\in\mathbb{C}\,:\,\sigma^{2l}=1\}\setminus\{1\}}\left(\phi^{-1}\left(\sigma\phi(x)\right)-y\right)\right]\\ &= \mu_0\left[\frac{F(x)-F(y)}{x-y},\phi^{-1}\left(-\phi(x)\right)-y\right]+\\ &\qquad\sum_{\{\sigma\in\mathbb{C}\,:\,\sigma^{2l}=1\}\setminus\{1,-1\}}\mu_0\left[\frac{F(x)-F(y)}{x-y},\phi^{-1}\left(\sigma\phi(x)\right)-y\right].\end{aligned}\tag{15}$$

Let us compute the two terms of the last expression. The first one coincides with the multiplicity at zero of the map

$$\frac{F(x)-F\left(\phi^{-1}\left(-\phi(x)\right)\right)}{x-\phi^{-1}\left(-\phi(x)\right)}.$$

This multiplicity can be calculated by making the change of variables $x=\phi^{-1}(u)=u+O(u^2)$. With this new variables the above expression writes as

$$\frac{F\left(\phi^{-1}(u)\right)-F\left(\phi^{-1}(-u)\right)}{2u+O(u^2)}=\frac{2f_{2S+1}u^{2S+1}+O(u^{2S+2})}{2u+O(u^2)}=f_{2S+1}u^{2S}+O(u^{2S+1}).$$

Therefore the first term in (16) is $2S$. Note that if $l=1$ the second term in (16) does not exist and we are done. So from now one we will assume that $l>1$.

Arguing as in the previous situation, to get the second term in (16) we have to compute the multiplicity at zero of

$$\frac{F(x)-F\left(\phi^{-1}\left(\sigma\phi(x)\right)\right)}{x-\phi^{-1}\left(\sigma\phi(x)\right)}=\frac{F(x)-F\left(\sigma x+O(x^2)\right)}{(1-\sigma)x+O(x^2)}=a_k\frac{1-\sigma^k}{1-\sigma}x^{k-1}+O(x^k).$$

Observe that this multiplicity is greater or equal than $k-1$ and that it is equal to $k-1$ if and only if $\sigma^k \neq 1$. Recall that σ is a complex number satisfying $\sigma^{2l}=1$ and that $\sigma \neq 1, -1$.

In other words we have proved that

$$\mu(f,g) \geq 2S + (2l-2)(k-1),$$

and that this last inequality is an equality if and only if k and $2l$ are coprime, as we wanted to prove. □

Remark 5.4. *Observe that the proofs of the above theorem and Lemma 5.1 suggest another algorithm, different from the one developed in the previous section for $l=m=1$, to compute the multiplicity $\mu(f,g)$. It consists of the following steps:*

(i) *Define $\phi(x)$ as in Lemma 5.1*

$$u = \phi(x) = x\sqrt[2l]{\frac{2lG(x)}{x^{2l}}} = x + O(x^2).$$

(ii) *Define another similar function associated to F*

$$u = \psi(x) = x\sqrt[k]{\frac{F(x)}{a_k x^k}} = x + O(x^2).$$

(iii) *The multiplicity is*

$$\mu(f,g) = \sum_{(\{\sigma\in\mathbb{C}\,:\sigma^{2l}=1\}\setminus\{1\})\cup(\{\omega\in\mathbb{C}\,:\omega^k=1\}\setminus\{1\})} \mu_0\left[\phi^{-1}\left(\sigma\phi(x)\right) + \psi^{-1}\left(\omega\psi(x)\right)\right].$$

Although the above algorithm reduces the computation of the multiplicity to a problem in just one variable we think that the algorithm developed in previous section for $l=m=1$ is more efficient than this new approach.

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