A NEW APPROACH TO THE COMPUTATION OF THE LYAPUNOV CONSTANTS

ARMENGOL GASULL AND JOAN TORREGROSA

Dedicated to Professor Mauricio Peixoto in occasion of his 80th birthday

Abstract. The problem of distinguishing whether a critical point of an analytic planar vector field with pure imaginary eigenvalues is a center or a focus was already solved by Lyapunov by searching an appropriate Lyapunov function. During the last years different methods have been developed to improve his approach. In this paper we give a new one. It is based on the study of the higher order Melnikov functions of a perturbation of the Hamiltonian system associated to $H(x, y) = (x^2 + y^2)/2$. From our point of view, our method has the following advantages: it is easy to be implemented; it allows to express the Lyapunov constants as words; it permits to prove –in an easy way– the algebraic properties of the Lyapunov constants when they are regarded as polynomials in the coefficients of the systems; it can also be applied to special types of degenerate monodromic critical points and, finally, it allows to relate the cyclicity of the critical point with the number of limit cycles which appear from the level curves of the linear center by using small perturbations.

1. Introduction and main results

The center-focus problem (that is to distinguish whether a critical point of an analytic planar vector field with pure imaginary eigenvalues is a center or a focus) was already solved by Lyapunov. He gave a method for obtaining a set of polynomial conditions which have to be satisfied by the coefficients of the system in order to have a critical point of center type. These expressions –which equated to zero give the center conditions– are usually called *Lyapunov constants*.

Different algorithms to compute these constants have been developed during these last years. Without trying to be exhaustive, we quote some papers on this subject, collected according the approaches that they use: computation of a Lyapunov function, following the first method introduced by Lyapunov ([Shi81], [Cha94], [PLC96], [LP99]); use of normal forms ([HW78], [LL90]); computation of the power expansion of a solution of the system in polar coordinates ([ALGM73], [GGM97]); use of the algebraic structure of the Lyapunov constants ([LL90],

Date: January, 2001.

1991 *Mathematics Subject Classification.* 34C25,34C07.

Key words and phrases. Center-focus problem, Lyapunov constants, Melnikov functions.

Partially supported by the DGICYT grant number PB96-1153 and CONACIT grant number 1999SGR 00349.

[Żoł94b], [CGMM97], [GGM99]); method of Lyapunov-Schmidt ([FLLL89]); method of averaging ([SV85]), ...

The aim of this paper is to give a different approach based on the study of the higher order Melnikov functions of a perturbation of the Hamiltonian system associated to $H(x,y) = (x^2 + y^2)/2$. Our study starts from previous results of Françoise [Fra96]. As far as we know, this new approach has been used to obtain center conditions just for systems with homogeneous nonlinearities, see [FP95]. In this paper we give a general compact expression of the Lyapunov constants, see Theorem 1.1.

From our point of view our approach has several advantages:

- It is easy to be implemented in a computer algebra system. In fact our method is used in the program P4 (Polynomial Planar Phase Portraits). This program studies a given polynomial planar vector field of any degree and has been developed by Dumortier, Llibre, Herssens and Artés. See the web page `http://mat.uab.es/artes/p4.htm` or the paper [DH99] for more details. We have also implemented it in Maple, see Section 3. See also Section 5 for some examples where it is applied to illustrate its efficiency.
- It allows to express the Lyapunov constant as *words* in an special alphabet, see Theorem 3.3. Similar approaches to the computation and knowledge of the structure of the Lyapunov constants, specially for Abel equations, are developed in [Dev90, Dev91].
- From the expression of Lyapunov constants as words, it is not difficult to obtain properties of the Lyapunov constants when they are regarded as polynomials in the coefficients of system. In Section 4 we give a new, unified and short proof of all the known properties about these polynomials, see again [Żoł94b], [CGMM97], [Joy98], [GGM99], [LP99].
- The obtention of several Lyapunov constants allows to study which is the highest order of degeneracy of a weak focus in a specific family, as well as the *cyclicity* of the critical point (number of limit cycles which appear from degenerate Hopf bifurcation), see Section 6.
- It permits to relate the cyclicity of the origin with the number of limit cycles which bifurcate from the level curves of the linear center by using small perturbations and higher order Melnikov functions, see [GT01b].
- It can be extended to solve the center focus-problem for some special type of degenerate monodromic critical points, see [GT01a].

In order to state our main theorem we introduce some notation. Consider the general expression of a planar analytic system with a weak focus at the origin,

$$\left\{\begin{array}{rcl} \dot{x} & = & -y + P(x,y) = -y + \sum\limits_{k=2}^{\infty} P_k(x,y), \\ \dot{y} & = & x + Q(x,y) = x + \sum\limits_{k=2}^{\infty} Q_k(x,y), \end{array}\right. \tag{1}$$

where P_k and Q_k are homogeneous polynomials of degree k.

System (1) can be written, in polar coordinates (r, θ), as

$$\frac{dr}{d\theta} = \sum_{k=2}^{\infty} S_k(\theta) r^k, \tag{2}$$

or, in complex coordinates $(z, \overline{z})$, as

$$\dot{z} = iz + \sum_{k=2}^{\infty} R_k(z, \overline{z}), \tag{3}$$

where R_k are homogeneous polynomials of degree k in $z, \overline{z}$, and $S_k(\theta)$ are homogeneous trigonometric polynomials.

Denote by $r(\theta, r_0)$ the solution of (2) such that $r = r_0$ when $\theta = 0$. For r small enough, we can write

$$r(\theta, r_0) = r_0 + \sum_{k=2}^{\infty} u_k(\theta) r_0^k,$$

with $u_k(0) = 0$ for $k \geq 2$. The Poincaré return map is defined as

$$\Pi(r_0) = r(2\pi, r_0) = r_0 + \sum_{k=2}^{\infty} u_k(2\pi) r_0^k.$$

Furthermore, when $u_2(2\pi) = \cdots = u_{n-1}(2\pi) = 0$, the value of $V_n = u_n(2\pi)$ is the *n-th Lyapunov constant* of system (1), (or (2), or (3)). Recall the well-known property that the first n such that $V_n \neq 0$ is always odd, see [ALGM73, p. 243], or also Proposition 4.1.(i).

The principal result of this paper, proved in the next section, is:

Theorem 1.1. *Consider the differential equation (1), or the equivalent expression*

$$dH(x, y) + \omega_1(x, y) + \omega_2(x, y) + \cdots = 0,$$

where $H(x, y) = \frac{1}{2}(x^2 + y^2)$ and $\omega_l(x, y) = -Q_{l+1}(x, y)dx + P_{l+1}(x, y)dy$, for all $l \in \mathbb{N}$. If $V_2 = V_3 = \cdots = V_{n-1} = 0$ then its n-th Lyapunov constant is

$$V_n = \frac{1}{2^{\frac{n+1}{2}}} \frac{1}{\rho^{\frac{n+1}{2}}} \int_{H=\rho} \sum_{l=1}^{n-1} \omega_l h_{n-1-l},$$

where $h_0 \equiv 1$ and h_m, for all $m = 1, \ldots, n-2$, are polynomials in two variables defined recurrently by

$$d\left(\sum_{l=1}^{m} \omega_l h_{m-l}\right) = -d\left(h_m dH\right).$$

Notice that the key point to make the above theorem useful is to be able to compute all the polynomials h_m which appear in its statement. A method to obtain them was already developed in [Fra96]. It is recalled –with minor modifications– in Lemma 2.3.

2. Proof of the main Theorem

In order to prove Theorem 1.1 we introduce some preliminary results. They provide a method to compute the functions h_m which appear in Theorem 1.1 and will also be useful to state our approach to the Lyapunov constants through words.

Lemma 2.1. *Let $\omega = A(z,\overline{z})dz + B(z,\overline{z})d\overline{z}$ be a 1-form, with A and B polynomials in z and $\overline{z}$. Then*

$$\int_{H=\rho} \omega = -2\pi i \sum_k \operatorname{coef}\left(-\frac{\partial}{\partial \overline{z}}A + \frac{\partial}{\partial z}B, z^k\overline{z}^k\right)\left(\frac{(2\rho)^{k+1}}{k+1}\right),$$

where $\operatorname{coef}(f, z^k\overline{z}^l)$ *denotes the coefficient of the monomial $z^k\overline{z}^l$ of f for any k and l.*

Proof. From Stokes' Theorem

$$\begin{aligned}\int_{H=\rho}\omega &= \int_{H=\rho}(A(z,\overline{z})dz + B(z,\overline{z})d\overline{z}) = \int_{H\le\rho} d\left(A(z,\overline{z})dz + B(z,\overline{z})d\overline{z}\right)\\ &= \int_{H\le\rho}\left(\frac{\partial A}{\partial \overline{z}}d\overline{z}\wedge dz + \frac{\partial B}{\partial z}dz\wedge d\overline{z}\right) = \int_{H\le\rho}\left(-\frac{\partial A}{\partial \overline{z}} + \frac{\partial B}{\partial z}\right)dz\wedge d\overline{z}.\end{aligned}$$

Since the functions A and B are polynomials, we just need to compute the integral $\int_{H\le\rho} z^k\overline{z}^l dz\wedge d\overline{z}$, for any k,l. The change of variables $z = Re^{i\theta}$ gives the following equalities:

$$\begin{aligned}\int_{H\le\rho} z^k\overline{z}^l dz\wedge d\overline{z} &= \int_0^{\sqrt{2\rho}}\int_0^{2\pi} R^{k+l}e^{i(k-l)\theta}\left(e^{i\theta}dR + Rie^{i\theta}d\theta\right)\wedge\left(e^{-i\theta}dR - iRe^{-i\theta}d\theta\right)\\ &= \int_0^{\sqrt{2\rho}}\int_0^{2\pi} R^{k+l}e^{i(k-l)\theta}(-2iR)dR\wedge d\theta\\ &= -2i\int_0^{\sqrt{2\rho}} R^{k+l+1}dR\int_0^{2\pi} e^{i(k-l)\theta}d\theta = \begin{cases}0 & \text{if } k\ne l,\\ -2\pi i\frac{(2\rho)^{k+1}}{k+1} & \text{if } k = l.\end{cases}\end{aligned}$$

□

The above lemma suggests the next definition:

Definition 2.2. *Let $\mathcal{P}$ be the set of all polynomials in $z,\overline{z}$ variables and let $\mathcal{P}_0$ be the subset of $\mathcal{P}$ formed by the polynomials vanishing at zero. Let $\mathcal{P}_1\subset\mathcal{P}_0$ be the subset of all polynomials without monomials of the form $z^k\overline{z}^k$ for all $k>0$, and let $\mathcal{P}_2\subset\mathcal{P}_0$ be the subset of all polynomials without monomials of the form $z^{k+1}\overline{z}^k$ for all $k\ge 0$. In these subsets of $\mathcal{P}_0$ we define the following linear operators:*

(i)
$$\begin{array}{rccc}\mathcal{G}: & \mathcal{P}_1 & \longrightarrow & \mathcal{P}_1\\ & R = \sum\limits_{k\ne l} r_{kl}z^k\overline{z}^l & \longmapsto & \sum\limits_{k\ne l}\frac{2}{k-l}r_{kl}z^k\overline{z}^l,\end{array}$$

(ii)
$$\begin{array}{rccc}\mathcal{F}: & \mathcal{P}_2 & \longrightarrow & \mathcal{P}_1\\ & R & \longmapsto & -\operatorname{Im}\left(\mathcal{G}\left(\frac{\partial R(z,\overline{z})}{\partial z}\right)\right).\end{array}$$

Lemma 2.3 ([Fra96])**.** *Consider the function $H(x,y)=\frac{1}{2}(x^2+y^2)$. Let ω be a polynomial 1-form such that $\int_{H=\rho}\omega\equiv 0$. Then, there exists a polynomial h such that $d\omega=d(hdH)$. Furthermore, if the expression of ω in complex coordinates is*

$$\omega=A(z,\overline{z})dz+B(z,\overline{z})d\overline{z},$$

then

$$h(z,\overline{z})=\mathcal{G}\left(-\frac{\partial A(z,\overline{z})}{\partial\overline{z}}+\frac{\partial B(z,\overline{z})}{\partial z}\right),$$

where $\mathcal{G}$ is given in Definition 2.2(i).

Proof of Lemma 2.3. Following Lemma 2.1, $\int_{H=\rho}\omega\equiv 0$ if and only if the polynomial $-\frac{\partial}{\partial\overline{z}}A+\frac{\partial}{\partial z}B$ has no monomials of the form $(z\overline{z})^k$. Hence the polynomial $h(z,\overline{z})$ given above is well defined. Write

$$-\frac{\partial}{\partial\overline{z}}A+\frac{\partial}{\partial z}B=\sum_{k\neq l}d_{kl}z^k\overline{z}^l,$$

and consider h given as in the statement of the lemma. Then

$$d\left(h(z,\overline{z})dH\right)=d\left(\left(\sum_{k\neq l}\frac{2}{k-l}d_{kl}z^k\overline{z}^l\right)\frac{1}{2}\left(\overline{z}dz+zd\overline{z}\right)\right)=$$

$$\sum_{k\neq l}\frac{d_{kl}}{k-l}d\left(z^k\overline{z}^{l+1}dz+z^{k+1}\overline{z}^ld\overline{z}\right)=\sum_{k\neq l}d_{kl}z^kz^ldz\wedge d\overline{z}=d\omega,$$

as we wanted to see. □

Next result gives the first non zero term of the return map associated to a perturbation of the Hamiltonian system $dH=0$, where $H(x,y)=\frac{1}{2}(x^2+y^2)$. It is an extension of a result of Françoise, see [Fra96]. This extension is already described in [Pog94], [Ili98], [Rou98], [IP99] or [GT01b]. We include it here for the sake of completeness.

Theorem 2.4. *Consider the analytic differential equation*

$$\omega_\varepsilon=dH+\sum_{i=1}^{\infty}\varepsilon^i\omega_i=0, \tag{4}$$

where $\omega_i=\omega_i(x,y)$ are polynomial 1-forms vanishing at the origin, and $H(x,y)=\frac{1}{2}(x^2+y^2)$. Let

$$L:(\rho,\varepsilon)\longrightarrow L(\rho,\varepsilon)=\rho+\varepsilon L_1(\rho)+\cdots+\varepsilon^mL_m(\rho)+O(\varepsilon^{m+1}),$$

be the first return map associated to the flow of (4) and to a transversal section Σ (we choose $H=\rho$ to parameterize Σ). If $L_1(\rho)\equiv L_2(\rho)\equiv\cdots L_{m-1}(\rho)\equiv 0$, then there exist polynomials $h_0\equiv 1,h_1,\ldots,h_{m-1}$ such that

$$d\left(\sum_{i=1}^{m}\omega_ih_{m-i}\right)=-d(h_mdH),$$

and the m-th ε-derivative of L at zero is $m!L_m$, where

$$L_m(\rho) = -\int_{H=\rho} \sum_{i=1}^{m} \omega_i h_{m-i}.$$

Proof. Let γ_ε be the solution curve of $\omega_\varepsilon = 0$ which define the return map $L(\rho,\varepsilon)$. Note that γ_0 is given by the level sets $H=\rho$. Let us prove the theorem by induction on m.

In order to prove the formula for $m=1$, observe that the 1-form ω_ε is zero over γ_ε. Therefore

$$\int_{\gamma_\varepsilon} \omega_\varepsilon = 0.$$

If we substitute the expression of ω_ε in the above equality, we obtain that

$$\begin{aligned}
0 =& \int_{\gamma_\varepsilon} \big(dH + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \cdots\big) \\
=& H(L(\rho,\varepsilon)) - H(\rho) + \varepsilon \int_{\gamma_\varepsilon} \omega_1 + \varepsilon^2 \int_{\gamma_\varepsilon} \omega_2 + \cdots \\
=& \big(\rho + L_1(\rho)\varepsilon + L_2(\rho)\varepsilon^2 + \cdots\big) - \rho + \varepsilon \int_{\gamma_0} (\omega_1 + O(\varepsilon)) \\
&+ \varepsilon^2 \int_{\gamma_0} (\omega_2 + O(\varepsilon)) + \cdots ,
\end{aligned}$$

where, by equating the terms in ε, we obtain the next expression for $L_1(\rho)$,

$$L_1(\rho) = -\int_{H=\rho} \omega_1,$$

and therefore the result is proved for $m=1$.

Now, suppose that $L_k(\rho) \equiv 0$ for $k=1,\ldots,m-1$. From Lemma 2.3 we have that there exist polynomials h_k and S_k satisfying the next relations

$$-\sum_{i=1}^{k} \omega_i h_{k-i} = h_k dH + dS_k, \text{ for } k=1,\ldots,m-1.$$

Consider the 1-form $(1+\varepsilon h_1 + \varepsilon^2 h_2 + \cdots + \varepsilon^{m-1}h_{m-1})\omega_\varepsilon$, which is also zero over γ_ε. By integrating over γ_ε and equating the terms in ε^m we obtain that

$$L_m = -\int_{H=\rho} \sum_{i=1}^{m} \omega_i h_{m-i},$$

as we wanted to prove. ∎

Now, we can prove our main result.

Proof of Theorem 1.1. From system (1), by the scaling

$$(x,y) = (\varepsilon X, \varepsilon Y), \tag{5}$$

we obtain the next one

$$\left\{ \begin{array}{rcl} \dot{X} & = & \frac{1}{\varepsilon}(-\varepsilon Y + P(\varepsilon X, \varepsilon Y)) = -Y + \sum\limits_{k=2}^{\infty} \varepsilon^{k-1} P_k(x,y), \\ \dot{Y} & = & \frac{1}{\varepsilon}(\varepsilon X + Q(\varepsilon X, \varepsilon Y)) = X + \sum\limits_{k=2}^{\infty} \varepsilon^{k-1} Q_k(x,y). \end{array} \right. \tag{6}$$

The return map for (6), $L(\rho, \varepsilon)$, is well defined for every point $(\rho, 0)$, with ρ small enough, in (ρ, θ)-coordinates (action-angle). This point, with the change (5), moves to the point $(\varepsilon\sqrt{2\rho}, 0)$ in cartesian coordinates, and the first return of this last point, for system (1), is

$$\left(\varepsilon\sqrt{2\rho} + V_m(\varepsilon\sqrt{2\rho})^m + \cdots, 0\right).$$

With the inverse change of (5), we obtain the point

$$\left(\sqrt{2\rho} + V_m \varepsilon^{m-1}(\sqrt{2\rho})^m + \cdots, 0\right),$$

which level curve, H, is

$$\begin{aligned} & \frac{1}{2}\left(\sqrt{2\rho} + V_m \varepsilon^{m-1}(\sqrt{2\rho})^m + \cdots\right)^2 \\ & = \frac{1}{2}\left(2\rho + 2\varepsilon^{m-1}(\sqrt{2\rho})^{m+1} + \cdots\right) \\ & = \rho + \varepsilon^{m-1} 2^{(m+1)/2} \rho^{(m+1)/2} V_m + \cdots. \end{aligned}$$

This value is the first return of $(\rho, 0)$ for system (6), and so, the value of $L(\rho, \varepsilon)$. From this last equation, we obtain the equality

$$L_{m-1}(\rho) = 2^{(m+1)/2} \rho^{(m+1)/2} V_m.$$

By using Theorem 2.4, the proof is complete. □

3. The algorithm

The method induced by Theorem 1.1 to compute the Lyapunov constants of system (1) can also be read in terms of words, as it is shown in this section. With this aim we introduce some more notation. Remember that system (1) in complex coordinates can be written as

$$\dot{z} = iz + \sum_{k=2}^{\infty} R_k(z, \bar{z}).$$

Definition 3.1. *Let $R \in \mathcal{P}$ be a polynomial of degree n. For all integer number $k \geq 2$, and associated to system (3) we define the operators $\mathcal{F}_k$ and $\mathcal{H}_k$ as*

$$\begin{array}{rccc} \mathcal{F}_k : & \mathcal{P} & \longrightarrow & \mathcal{P} \\ & R & \longmapsto & \mathcal{F}(R_k R), \end{array}$$

$$\begin{array}{rccc} \mathcal{H}_k: & \mathcal{P} & \longrightarrow & \mathbb{R} \\ & R & \longmapsto & -\frac{1}{(2\rho)^{\frac{n+1+k}{2}}}\displaystyle\int_{H=\rho} \operatorname{Im}\left(R_k R d\overline{z}\right), \end{array}$$

where R_k are given in (3) and $\mathcal{F}$ is introduced by Definition 2.2. Notice that both families of operators are well defined just on some subsets of $\mathcal{P}$.

For sake of simplicity we introduce some notation. Let $m_0, m_1, \ldots, m_s$ be $s+1$ natural numbers. Then

$$\mathcal{H}_{m_0} \prod_{m} \mathcal{F}_{m_i} := \mathcal{H}_{m_0}\mathcal{F}_{m_1}\mathcal{F}_{m_2}\cdots\mathcal{F}_{m_s} := \mathcal{H}_{m_0}\left(\mathcal{F}_{m_1}\left(\mathcal{F}_{m_2}\left(\cdots\left(\mathcal{F}_{m_s}\left(1\right)\right)\right)\right)\right),$$

where $m=(m_1,\ldots,m_s)$; in the case $m=(1)$, by convention, $\prod_{(1)} \mathcal{F}_{m_i} = 1$.

In the next lemma we give some properties of these operators.

Lemma 3.2. *(i) Let $\omega = \operatorname{Im}(f(z,\overline{z})d\overline{z})$ a 1-form such that $\int_{H=\rho}\omega = 0$, then the function $h(z,\overline{z})$ given in Lemma 2.3 is $\mathcal{F}(f)$.*
(ii) The image of the operators $\mathcal{F}$ and $\mathcal{F}_k$ are real polynomials.
(iii) If R is a polynomial of degree n then the degree of $\mathcal{F}_k(R)$ is $n+k-1$.

Proof. (i) Consider $\omega = \operatorname{Im}(f d\overline{z}) = \frac{f d\overline{z} - \overline{f} dz}{2i}$. The function h given in Lemma 2.3 is

$$\begin{aligned} h &= \mathcal{G}\left(\frac{1}{2i}\left(\frac{\partial \overline{f}}{\partial \overline{z}} + \frac{\partial f}{\partial z}\right)\right) = \frac{1}{2i}\left(\mathcal{G}\left(\frac{\partial \overline{f}}{\partial \overline{z}}\right) + \mathcal{G}\left(\frac{\partial f}{\partial z}\right)\right) \\ &= \frac{1}{2i}\left(-\overline{\mathcal{G}\left(\frac{\partial f}{\partial z}\right)} + \mathcal{G}\left(\frac{\partial f}{\partial z}\right)\right) = \operatorname{Im}\left(\mathcal{G}\left(\frac{\partial f}{\partial z}\right)\right) = \mathcal{F}(f), \end{aligned}$$

as we wanted to see. The proof of (ii) and (iii) is straightforward. □

With this new notation we can rewrite Theorem 1.1 as:

Theorem 3.3. *Assume that for system*

$$\dot{z} = iz + \sum_{k=2} R_k(z,\overline{z}),$$

the first $n-2$ Lyapunov constants vanish ($V_2 = V_3 = \cdots = V_{n-1} = 0$). Then its n-th Lyapunov constant is

$$V_n = \sum_{k=2}^{n} \mathcal{H}_k \left(\sum_{m\in S_{n-k}} \prod_{m} \mathcal{F}_{m_i} \right),$$

where $\mathcal{H}_k$ and $\mathcal{F}_m$ are the operators introduced in Definition 3.1 and S_l is the set

$$S_l = \bigcup_{s\in\mathbb{N}^+} \left\{ m = (m_1,\ldots,m_s) \in \left(\mathbb{N}^+ \setminus \{1\}\right)^s \text{ such that } \sum_{j=1}^{s}(m_j - 1) = l \right\},$$

when $l \neq 0$ and $S_0 = \{(1)\}$.

Proof. The expression of system (3), after using the scaling $z \to \varepsilon z$ (as in the proof of Theorem 2.4), is

$$dH + \sum_{k=2}^{\infty} \varepsilon^{k-1} \operatorname{Im}(R_k(z,\overline{z})d\overline{z}) = dH + \sum_{k=1}^{\infty} \varepsilon^k \omega_k = 0.$$

We prove the theorem by induction on n.

Case $n = 1$. From Theorem 2.4 and Definition 3.1 we have that

$$L_1(\rho) = -\int_{H=\rho} \omega_1 = -\int_{H=\rho} \operatorname{Im}(R_2 d\overline{z}) = (2\rho)^{3/2}\mathcal{H}_2(1) = (2\rho)^{3/2}\mathcal{H}_2.$$

On the other hand Theorem 1.1 shows that

$$V_2 = \frac{1}{(2\rho)^{3/2}} L_1(\rho) = \mathcal{H}_2.$$

To clarify the proof we also include the case $n = 2$. If $L_1(\rho) \equiv V_2 = 0$, and by using Lemma 2.3 there exists a polynomial h_1 such that $d\left(-\omega_1\right) = d\left(h_1 dH\right)$. From Lemma 3.2.(i) we also have that $h_1 = \mathcal{F}(R_2) = \mathcal{F}_2$ and it has degree 1 in z and $\overline{z}$.

Consider now the 1-form $\omega_2 + \omega_1 h_1$, with the same arguments as above and using also Lemma 3.2.(ii), we can write:

$$\begin{aligned}
L_2(\rho) &= -\int_{H=\rho} \omega_2 + \omega_1 h_1 \\
&= -\int_{H=\rho} \left(\operatorname{Im}(R_3 d\overline{z}) + \operatorname{Im}(R_2 d\overline{z})\mathcal{F}_2\right) \\
&= -\int_{H=\rho} \left(\operatorname{Im}(R_3 d\overline{z}) + \operatorname{Im}(R_2 \mathcal{F}_2 d\overline{z})\right) \\
&= (2\rho)^{4/2}\mathcal{H}_3(1) + (2\rho)^{4/2}\mathcal{H}_2(\mathcal{F}_2),
\end{aligned}$$

and

$$V_3 = \frac{1}{(2\rho)^{4/2}} L_2(\rho) = \mathcal{H}_3 + \mathcal{H}_2\mathcal{F}_2.$$

Note also that when $V_2 = V_3 = 0$, from Lemma 3.2, the polynomial h_2 such that $-d\left(\omega_2 + \omega_1 h_1\right) = d\left(h_2 dH\right)$ is

$$h_2 = \mathcal{F}_3 + \mathcal{F}_2\mathcal{F}_2,$$

and it has degree 2.

In the general case, we have that $V_2 = V_3 = \cdots = V_n = 0$, and we have to compute V_{n+1}. By the induction hypothesis we have that for all $l = 1, \ldots, n-1$, the polynomials h_l have degree l and satisfy the following properties:

(i) $-d\left(\sum\limits_{k=0}^{l-1} \omega_{l-k} h_k\right) = d\left(h_l dH\right)$ and,

(ii) $h_l = \sum\limits_{m \in S_l} \prod\limits_{m} \mathcal{F}_{m_i}$.

From Theorem 2.4 and Definition 3.1 we obtain

$$\begin{aligned}
L_n(\rho) &= -\int_{H=\rho}\left(\sum_{k=0}^{n-1}\omega_{n-k}h_k\right) = -\int_{H=\rho}\left(\sum_{k=0}^{n-1}\operatorname{Im}\left(R_{n-k+1}d\overline{z}\right)\sum_{m\in S_k}\prod_m \mathcal{F}_{m_i}\right)\\
&= -\int_{H=\rho}\left(\sum_{k=0}^{n-1}\operatorname{Im}\left(R_{n-k+1}\left(\sum_{m\in S_k}\prod_m \mathcal{F}_{m_i}\right)d\overline{z}\right)\right)\\
&= \sum_{k=0}^{n-1}-\int_{H=\rho}\left(\operatorname{Im}\left(R_{n-k+1}\left(\sum_{m\in S_k}\prod_m \mathcal{F}_{m_i}\right)d\overline{z}\right)\right)\\
&= \sum_{k=0}^{n-1}(2\rho)^{\frac{n-k+1+k+1}{2}}\mathcal{H}_{n-k+1}\left(\sum_{m\in S_k}\prod_m \mathcal{F}_{m_i}\right)\\
&= (2\rho)^{\frac{n+2}{2}}\sum_{k=2}^{n+1}\mathcal{H}_k\left(\sum_{m\in S_{n-k+1}}\prod_m \mathcal{F}_{m_i}\right),
\end{aligned}$$

and by using Theorem 1.1, V_{n+1} can be computed as

$$V_{n+1} = \frac{1}{(2\rho)^{(n+2)/2}}L_n(\rho) = \sum_{k=2}^{n+1}\mathcal{H}_k\left(\sum_{m\in S_{n-k+1}}\prod_m \mathcal{F}_{m_i}\right),$$

which is the expression of the statement of the theorem.

To conclude the proof, we have to compute h_n when $V_{n+1} = 0$. Following similar arguments than above, we obtain

$$\begin{aligned}
h_n &= \sum_{k=0}^{n-1}\mathcal{F}_{n-k+1}\left(\sum_{m\in S_k}\prod_m \mathcal{F}_{m_i}\right)\\
&= \sum_{k=2}^{n+1}\mathcal{F}_k\left(\sum_{m\in S_{n-k+1}}\prod_m \mathcal{F}_{m_i}\right) = \sum_{m\in S_n}\prod_m \mathcal{F}_{m_i},
\end{aligned}\tag{7}$$

and this fact ends the proof. □

As a corollary of this result, we can obtain a compact expression of all Lyapunov constants for the homogeneous family of degree k.

Corollary 3.4. *Let $R_k(z,\overline{z})$ be a homogeneous polynomial of degree k. The non identically zero Lyapunov constants of the system*

$$\dot{z} = iz + R_k(z,\overline{z}),$$

are

$$V_{k+j(k-1)} = \mathcal{H}_k\Big(\prod_{m=1}^{j}\mathcal{F}_k\Big) := \mathcal{H}_k(\mathcal{F}_k^j),$$

where j is any natural number (resp. any odd natural number) when k is odd, (resp. even).

Proof. From Theorem 3.3, in the expression of V_n for this family, they only appear the operators $\mathcal{H}_k$ and $\mathcal{F}_k$. Therefore the sum $\sum\limits_{m\in S_{n-k}} \prod\limits_{m} \mathcal{F}_{m_i}$ is reduced to the term $\mathcal{F}_k^j$. From the definition of the sets S_l, the non identically zero constants appear just for the values n satisfying $j(k-1) = n-k$. Furthermore it is well known that $n = k + j(k-1)$ has to be an odd number (see also Proposition 4.1.(i) in the next section). These two facts prove the result. □

We end this section with some comments about the practical implementation of Theorem 3.3.

The first Lyapunov constants expressed as words are:

$$\begin{aligned} V_3 &= \mathcal{H}_3 + \mathcal{H}_2(\mathcal{F}_2), \\ V_5 &= \mathcal{H}_5 + \mathcal{H}_4(\mathcal{F}_2) + \mathcal{H}_3(\mathcal{F}_3 + \mathcal{F}_2\mathcal{F}_2) + \mathcal{H}_2(\mathcal{F}_4 + \mathcal{F}_3\mathcal{F}_2 + \mathcal{F}_2\mathcal{F}_3 + \mathcal{F}_2\mathcal{F}_2\mathcal{F}_2). \end{aligned}$$

Notice that the expression of $\mathcal{F}_2$ obtained in the computation of V_3 is used in several places to compute V_5.

Now we illustrate how to obtain the expression of V_7. First, we need to get S_k for $k = 0, \ldots, 5$.

$$\begin{aligned} S_0 &= \{(1)\}, \\ S_1 &= \{(2)\}, \\ S_2 &= \{(2,2),(3)\}, \\ S_3 &= \{(2,2,2),(3,2),(2,3),(4)\}, \\ S_4 &= \{(2,2,2,2),(3,2,2),(2,3,2),(2,2,3),(3,3),(4,2),(2,4),(5)\}, \\ S_5 &= \{(2,2,2,2,2),(3,2,2,2),(2,3,2,2),(2,2,3,2),(2,2,2,3),(3,3,2),(3,2,3), \\ &\quad (2,3,3),(4,2,2),(2,4,2),(2,2,4),(4,3),(3,4),(5,2),(2,5),(6)\}. \end{aligned}$$

The first four sets are the ones needed to obtain V_3 and V_5, the other two are new. From their expressions we have to compute the factor $\prod\limits_{m} \mathcal{F}_{m_i}$ for every $m \in S_k$. Notice that the terms,

$$\mathcal{F}_2, \mathcal{F}_2\mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_3\mathcal{F}_2, \mathcal{F}_2\mathcal{F}_3, \mathcal{F}_2\mathcal{F}_2\mathcal{F}_2 \text{ and } \mathcal{F}_4,$$

which appear in the expression of V_7 have been already computed to get the previous Lyapunov constants.

Note also that the number of elements of each S_k is related to the well-known partition function $p(l)$, which associates to each natural number l the number of ways to obtain l as sum of positive integers. This function grows fast with l, see [NZ80]. In the implementation of our method we use the following idea to generate the S_k:

Define $\mathbf{S} := \bigcup_{k\in\mathbb{N}} \mathbb{N}^k$, and the following operators:

$$\begin{array}{rccc} \text{addl}: & \mathbb{N}\times\mathbf{S} & \longrightarrow & \mathbf{S} \\ & (a,(u_1,\ldots,u_n)) & \longmapsto & (a,u_1,\ldots,u_n), \end{array}$$

$$\begin{array}{rccc} \text{addr}: & \mathbf{S}\times\mathbb{N} & \longrightarrow & \mathbf{S} \\ & ((u_1,\ldots,u_n),a) & \longmapsto & (u_1,\ldots,u_n,a), \end{array}$$

and

$$\begin{array}{rccc} \text{add}: & \mathbb{N}\times\mathcal{P}(\mathbf{S}) & \longrightarrow & \mathcal{P}(\mathbf{S}) \\ & (a,S) & \longmapsto & \left(\bigcup_{u\in S}\text{addl}(a,u)\right)\cup\left(\bigcup_{u\in S}\text{addr}(u,a)\right), \end{array}$$

where, as usual, $\mathcal{P}(\mathbf{S})$ denotes the set of parts of $\mathbf{S}$.

By using this method we have that

$$\begin{aligned} S_2 &= \{(3)\}\bigcup\{\text{add}(2,S_1)\}, \\ S_3 &= \{(4)\}\bigcup\{\text{add}(2,S_2)\}\bigcup\{\text{add}(3,S_1)\}, \text{ and, in general} \\ S_n &= \{(n)\}\bigcup_{k=1}^{n-1}\{\text{add}(n-k+1,S_k)\}. \end{aligned}$$

4. The algebraic properties

From Theorem 3.3 we can give a simple and unified proof of the algebraic properties of the Lyapunov constants when they are regarded as polynomials in the coefficients of system (1) (or equivalently of system (3)). Different proofs of some of these results can be found in [Żoł94b], [CGMM97], [Joy98] and [GGM99]. To state these properties we recall the following definition: It is said that a function f, from $\mathbb{C}^n$ to $\mathbb{C}$, is a quasi-homogeneous function with weight $\mathbf{a}=(a_1,\ldots,a_n)\in\mathbb{Z}^n$ and quasi-degree $d\in\mathbb{Z}$ if it satisfies

$$f(\lambda^{a_1},\lambda^{a_2},\ldots,\lambda^{a_n}x_n)=\lambda^d f(x_1,x_2,\ldots,x_n),$$

for each $\lambda\in\mathbb{R}^+$. In particular, notice that any monomial $x_1^{r_1}x_2^{r_2}\cdots x_n^{r_n}$ is a quasi-homogeneous function with arbitrary weight $\mathbf{a}=(a_1,\ldots,a_n)$ and quasi-degree $d=a_1r_1+a_2r_2+\cdots+a_nr_n$.

In the next result, to emphasize the dependence of the Lyapunov constant V_n with respect the coefficients of (3), where $R_m(z,\overline{z})=\sum\limits_{k+l=m} r_{k,l}z^k\overline{z}^l$, we write $V_n = V_n(\{r_{k,l},\overline{r}_{k,l}\})$.

Proposition 4.1. *Let $V_n(\{r_{k,l},\overline{r}_{k,l}\})$ be a Lyapunov constant of system (3). Then, the following statements hold:*

(i) $V_n=0$ for all even n.

(ii) *V_{2m+1} is a quasi-homogeneous polynomial of quasi-degree 0 when we consider the variables $r_{k,l}$ (resp. $\overline{r}_{k,l}$) with weight $-k+l+1$ (resp. $k-l-1$), i.e. it satisfies*
$$V_{2m+1}(\{\lambda^{-k+l+1}r_{k,l},\lambda^{k-l-1}\overline{r}_{k,l}\}) = V_{2m+1}(\{r_{k,l},\overline{r}_{k,l}\}),$$
for all positive real number λ.

(iii) *V_{2m+1} is a quasi-homogeneous polynomial of quasi-degree $2m$ when we consider both variables $r_{k,l}$ and $\overline{r}_{k,l}$ with weight $k+l-1$; i.e. it satisfies*
$$V_{2m+1}(\{\lambda^{k+l-1}r_{k,l},\lambda^{k+l-1}\overline{r}_{k,l}\}) = \lambda^{2m}V_{2m+1}(\{r_{k,l},\overline{r}_{k,l}\}),$$
for all positive real number λ.

(iv) *Let q be the coefficient of a monomial of V_{2m+1}. Then, if the monomial has odd (resp. even) degree then $q/\pi\in\mathbb{Q}$ (resp. $iq/\pi\in\mathbb{Q}$), the conjugated of this monomial also appears in V_{2m+1} and its coefficient is q (resp. $-q$). In other words, V_{2m+1} can be written as $\mathrm{Re}\left(V^o_{2m+1}\right)+\mathrm{Im}\left(V^e_{2m+1}\right)$, where V^e_{2m+1}/π and V^o_{2m+1}/π are polynomials with rational coefficients with all their monomials of degree even and odd, respectively.*

To clarify the meaning of the above properties next remark gives the expressions of the first two Lyapunov constants in the variables $r_{k,l}$. These expressions are obtained by using Theorem 1.1.

Remark 4.2. *The expressions of the first two Lyapunov constants for system (3) are:*
$$V_3 = 2\pi\left(\mathrm{Re}(r_{2,1})+\mathrm{Im}(-r_{2,0}r_{1,1})\right),$$
$$V_5 = \frac{2}{3}\pi(\mathrm{Re}(V_{5,1})+\mathrm{Im}(V_{5,2})+\mathrm{Re}(V_{5,3})+\mathrm{Im}(V_{5,4})),$$
where
$$\begin{aligned}
V_{5,1} &= 3r_{32},\\
V_{5,2} &= -4r_{02}r_{40}-6r_{31}r_{11}-3r_{30}r_{12}-r_{02}\overline{r}_{13}-3\overline{r}_{22}r_{11}-3r_{31}\overline{r}_{20},\\
V_{5,3} &= -6r_{30}r_{11}^2-3r_{30}r_{11}\overline{r}_{20}-2r_{30}r_{02}r_{20}+5r_{30}r_{02}\overline{r}_{11}\\
&\quad +3r_{12}r_{20}\overline{r}_{11}+2r_{12}\overline{r}_{02}\overline{r}_{20}+3r_{12}\overline{r}_{02}r_{11}-30r_{21}\overline{r}_{20}r_{20}\\
&\quad -24r_{21}r_{20}r_{11}-21r_{21}\overline{r}_{20}\overline{r}_{11}-15r_{21}\overline{r}_{11}r_{11}+r_{03}\overline{r}_{02}\overline{r}_{11}+2r_{03}\overline{r}_{02}r_{20},\\
V_{5,4} &= 24r_{11}^2r_{20}^2-2\overline{r}_{02}r_{11}^3+30\overline{r}_{20}r_{11}r_{20}^2+15r_{11}^2\overline{r}_{11}r_{20}\\
&\quad +3r_{02}\overline{r}_{11}^2r_{20}-2\overline{r}_{11}r_{02}r_{20}^2-4r_{11}\overline{r}_{02}r_{02}r_{20}.
\end{aligned}$$

Before proving Proposition 4.1, we need to check similar properties for the polynomials $h_N=\sum\limits_{m\in S_N}\prod\limits_{m}\mathcal{F}_{m_i}$, which appear in the proof of Theorem 3.3, see (7). We use the same notation than in the above proposition to make explicit the dependence of any function with respect the coefficients of system (3).

Lemma 4.3. *Consider the polynomial $h_N=\sum\limits_{m\in S_N}\prod\limits_{m}\mathcal{F}_{m_i}=\sum\limits_{k,l}f_{k,l}(\{r_{a,b},\overline{r}_{a,b}\})z^k\overline{z}^l$ given in (7). It satisfies the following properties:*

(i) *h_N is a homogeneous polynomial of degree N in $z,\overline{z}$.*

(ii) $f_{k,l}$ *is a quasi-homogeneous polynomial of quasi-degree* $-k+l$ *when we assume that the variables* $r_{a,b}$ *(resp.* $\overline{r}_{a,b}$*) have weight* $-a+b+1$ *(resp.* $a-b-1$*).*

(iii) $f_{k,l}$ *is a quasi-homogeneous polynomial of quasi-degree* $k+l$ *when we assume that both variables* $r_{a,b}$ *and* $\overline{r}_{a,b}$ *have weight* $a+b-1$*.*

(iv) h_N *can be written as* $\mathrm{Im}\,(h_N^o)+\mathrm{Re}\,(h_N^e)$ *, where* h_N^o *(resp.* h_N^e*) is a polynomial in* z *and* $\overline{z}$*, such that each one of its monomials has a coefficient* C*, which is again a polynomial in the variables* $r_{a,b}$ *and* $\overline{r}_{a,b}$*. Furthermore the polynomial* iC *(resp.* C*) has rational coefficients and all its monomials have odd (resp. even) degree.*

Proof. Note that, by linearity, it suffices to prove the four properties for each element $\prod\limits_m \mathcal{F}_{m_i}$ of the sum $\sum_{m\in S_n}\prod\limits_m \mathcal{F}_{m_i}$. Remember that $R_m=\sum\limits_{a+b=m} r_{a,b}z^a\overline{z}^b$. We will prove our result by induction on s, the number of elements of the product. Remember that $m=(m_1,m_2,\ldots,m_s)$.

Case $s=1$. Consider $m=(m_1)$. We have

$$\begin{aligned}
\mathcal{F}_{m_1}&=\mathcal{F}_{m_1}(1)=\mathcal{F}(R_{m_1})=\mathcal{F}(\sum_{a+b=m_1} r_{a,b}z^a\overline{z}^b)=\sum_{a+b=m_1}\mathcal{F}(r_{a,b}z^a\overline{z}^b)\\
&=-\sum_{a+b=m_1}\mathrm{Im}\left(\mathcal{G}\left(ar_{a,b}z^{a-1}\overline{z}^b\right)\right)=\sum_{a+b=m_1}\mathrm{Im}\left(-\frac{2a}{a-b-1}r_{a,b}z^{a-1}\overline{z}^b\right)\\
&=\sum_{a+b=m_1}\frac{1}{i}\frac{a}{a-b-1}\left(-r_{a,b}z^{a-1}\overline{z}^b+\overline{r}_{a,b}\overline{z}^{a-1}z^b\right).
\end{aligned}$$

From the above expression, $\mathcal{F}(R_{m_1})$ is a polynomial of degree m_1-1 in the variables z and $\overline{z}$; the monomials $r_{a,b}z^{a-1}\overline{z}^b$ and $\overline{r}_{a,b}\overline{z}^{a-1}z^b$ have coefficients which are quasi-homogeneous polynomials of weight $-a+b+1=-(a-1)+b$ and $a-b-1=-b+(a-1)$ respectively; each coefficient has quasi-degree $a+b-1$; and (iv) is also trivially satisfied.

Assume by the induction hypothesis that, $\prod\limits_{i=1}^{n}\mathcal{F}_{m_i}=\sum\limits_{k,l}f_{k,l}z^k\overline{z}^l$ satisfies all the statements of the enunciate. Then we have that

$$\begin{aligned}
\prod_{j=1}^{n+1}\mathcal{F}_{m_j}&=\mathcal{F}_{m_{n+1}}\prod_{j=1}^{n}\mathcal{F}_{m_j}=\mathcal{F}(R_{m_{n+1}}\sum_{c+d=\sum_{j=1}^n(m_j-1)} f_{c,d}z^c\overline{z}^d)\\
&=\mathcal{F}(\sum_{a+b=m_{n+1}} r_{a,b}z^a\overline{z}^b\sum_{c+d=\sum_{j=1}^n(m_j-1)} f_{c,d}z^c\overline{z}^d)\\
&=\sum_{\substack{a+b=m_{n+1}\\ c+d=\sum_{j=1}^n(m_j-1)}}\mathcal{F}(r_{a,b}f_{c,d}z^{a+c}\overline{z}^{b+d})\\
&=-\sum_{\substack{a+b=m_{n+1}\\ c+d=\sum_{j=1}^n(m_j-1)}}\mathrm{Im}\left(\mathcal{G}((a+c)r_{a,b}f_{c,d}z^{a+c-1}\overline{z}^{b+d})\right)
\end{aligned}$$

$$= - \sum_{\substack{a+b=m_{n+1} \\ c+d=\sum_{j=1}^{n}(m_j-1)}} \operatorname{Im}\left(\frac{2(a+c)}{a+c-1-b-d} r_{a,b} f_{c,d} z^{a+c-1} \overline{z}^{b+d}\right)$$

$$= \sum_{\substack{a+b=m_{n+1} \\ c+d=\sum_{j=1}^{n}(m_j-1)}} \frac{(a+c)}{a+c-1-b-d} \frac{1}{i} \left(\overline{r}_{a,b} \overline{f}_{c,d} \overline{z}^{a+c-1} z^{b+d} - r_{a,b} f_{c,d} z^{a+c-1} \overline{z}^{b+d}\right).$$

Note that $a+c-1-b-d$ never can be zero. Therefore:

(i) $\prod_{j=1}^{n+1} \mathcal{F}_{m_j}$ is a homogeneous polynomial of degree $m_{n+1} + \sum_{j=1}^{n}(m_j - 1) - 1 = \sum_{j=1}^{n+1}(m_j - 1)$, in variables $z, \overline{z}$.

(ii,iii) The coefficients of every monomial in $z, \overline{z}$, $r_{a,b} f_{c,d}$ and $\overline{r}_{a,b} \overline{f}_{c,d}$, are quasi-homogeneous polynomials with weight $-a+b+1-c+d = -(a+c-1)+(b+d)$ and $a-b-1+c-d = (a+c-1)-(b+d)$, respectively, and they have quasi-degree $a+c-1+b+d = (a+b-1)+(c+d)$.

(iv) The operator $\mathcal{F}_{m_{n+1}}$ changes the parity of the degree in $r_{k,l}$ and $\overline{r}_{k,l}$ of each monomial of the product. From this fact, the above formulas and using of one of the following expressions (according to the parity of the degree)

$$\begin{aligned} \operatorname{Im}(A \operatorname{Im}(B)) &= \operatorname{Re}\left(-\frac{AB - A\overline{B}}{2}\right), \\ \operatorname{Im}(A \operatorname{Re}(B)) &= \operatorname{Im}\left(\frac{AB + A\overline{B}}{2}\right), \end{aligned}$$

the proof of (iv) follows. □

Now, we can prove the Proposition 4.1.

Proof of Proposition 4.1. From Theorem 3.3, the n-th Lyapunov constant is

$$V_n = \sum_{k=2}^{n} \mathcal{H}_k \left(\sum_{m \in S_{n-k}} \prod_{m} \mathcal{F}_{m_i} \right).$$

From the above lemma, we only need to check that all the properties are satisfied after applying the operator $\mathcal{H}_k$. From the linearity of this operator it suffices to study how it acts on a monomial of the form $R = f_{c,d} z^c \overline{z}^d$, satisfying that $c + d = n - k$.

By the definition of $\mathcal{H}_k$, we have

$$\begin{aligned} \mathcal{H}_k(R) &= -\frac{1}{(2\rho)^{\frac{c+d+1+k}{2}}} \int_{H=\rho} \operatorname{Im}(R_k R d\overline{z}) \\ &= -\frac{1}{(2\rho)^{\frac{n+1}{2}}} \int_{H=\rho} \operatorname{Im}\left(\sum_{a+b=k} r_{a,b} z^a \overline{z}^b f_{c,d} z^c \overline{z}^d d\overline{z} \right) \end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{(2\rho)^{\frac{n+1}{2}}} \int_{H=\rho} \operatorname{Im}\left(\sum_{a+b=k} r_{a,b} f_{c,d} z^{a+c}\overline{z}^{b+d} d\overline{z}\right) \\
&= -\frac{1}{(2\rho)^{\frac{n+1}{2}}} \sum_{a+b=k} \int_{H=\rho} \frac{1}{2i}\left(r_{a,b} f_{c,d} z^{a+c}\overline{z}^{b+d} d\overline{z} - \overline{r}_{a,b}\overline{f}_{c,d}\overline{z}^{a+c} z^{b+d} dz\right) \\
&= -\frac{1}{(2\rho)^{\frac{n+1}{2}}} \sum_{a+b=k} \frac{1}{2i}\left(r_{a,b} f_{c,d} \int_{H=\rho} z^{a+c}\overline{z}^{b+d} d\overline{z} - \overline{r}_{a,b}\overline{f}_{c,d} \int_{H=\rho} \overline{z}^{a+c} z^{b+d} dz\right) \\
&= \frac{\pi}{(2\rho)^{\frac{n+1}{2}}} \left(\sum_{\substack{a+b=k \\ a+c-b-d-1=0}} r_{a,b} f_{c,d} (2\rho)^{\frac{a+c+b+d+1}{2}} + \sum_{\substack{a+b=k \\ b+d-a-c+1=0}} \overline{r}_{a,b}\overline{f}_{c,d} (2\rho)^{\frac{b+d+a+c+1}{2}}\right) \\
&= \pi \sum_{\substack{a+b=k \\ a+c-b-d-1=0}} \left(r_{a,b} f_{c,d} + \overline{r}_{a,b}\overline{f}_{c,d}\right) = 2\pi \sum_{\substack{a+b=k \\ a+c-b-d-1=0}} \operatorname{Re}\left(r_{a,b} f_{c,d}\right).
\end{aligned}
$$

Let us prove statement (i). From the relations $c+d=n-k$, $a+b=k$ and $a+c-b-d=1$, it follows that $a+c=\frac{n+1}{2}$ and $b+d=\frac{n-1}{2}$. Since n is an even number, and a,b,c,d are integers, the sum of the above expression has no factors. Therefore (i) follows.

(ii-iii) From Lemma 4.3, when we associate to each of the coefficients $\mathcal{H}_k(R)$ the weights given in statements (ii) or (iii), it is quasi-homogeneous polynomial of quasi-degree $(-a+b+1)+(-c+d)=0$, or $(a+b-1)+(c+d)=(k-1)+(n-k)=n-1$, respectively, as we wanted to prove.

(iv) The proof of this statement is similar to the proof of (iv) of Lemma 4.3 using here the following equalities

$$
\begin{aligned}
\operatorname{Re}(A\operatorname{Im}(B)) &= \operatorname{Im}\left(-\frac{AB - A\overline{B}}{2}\right), \\
\operatorname{Re}(A\operatorname{Re}(B)) &= \operatorname{Re}\left(\frac{AB + A\overline{B}}{2}\right).
\end{aligned}
$$

□

5. Aplications

This section is devoted to apply our method for obtaining Lyapunov constants to solve the center-focus problem to several families of planar systems.

5.1. **Some cubic systems.**

In complex coordinates, a cubic system writes as

$$
\dot{z} = iz + r_{20}z^2 + r_{11}z\overline{z} + r_{02}\overline{z}^2 + r_{30}z^3 + r_{21}z^2\overline{z} + r_{12}z\overline{z}^2 + r_{03}\overline{z}^3, \tag{8}
$$

where $r_{i,j} \in \mathbb{C}$.

As a first test for our method we apply it to some families of cubic systems studied in [LPR96].

If $r_{11} = 0$ and $\overline{r}_{i,j} = r_{i,j}$, then the first Lyapunov constants[1] obtained by using the implementation in Maple of our method are

$$
\begin{aligned}
V_3 &= 2\pi r_{21}\\
V_5 &= -\frac{4}{3}\pi r_{02} r_{20}(r_{30} - r_{12} - r_{03}),\\
V_7 &= -\frac{1}{4}\pi r_{03}(3r_{30} + r_{12})(r_{30} - 3r_{12}),\\
V_9 &= \frac{1}{30}\pi(60r_{02}^2 r_{30}^3 - 160r_{02}^2 r_{30}^2 r_{12} + 60r_{02}^2 r_{30} r_{12}^2 - 270r_{02}^2 r_{30} r_{03} r_{12}\\
&\quad +105r_{02}^2 r_{03}^2 r_{30} + 40r_{02}^2 r_{12}^3 - 90r_{02}^2 r_{12}^2 r_{03} + 35r_{02}^2 r_{03}^2 r_{12}\\
&\quad +72r_{20} r_{02} r_{03}^2 r_{12} + 54r_{03}^3 r_{02} r_{20} + 20r_{30}^2 r_{12} r_{20}^2 - 40r_{12}^2 r_{30} r_{20}^2\\
&\quad -30r_{12} r_{20}^2 r_{03} r_{30} - 27r_{03}^2 r_{20}^2 r_{30} + 20r_{12}^3 r_{20}^2 - 42r_{03} r_{20}^2 r_{12}^2\\
&\quad +43r_{12} r_{03}^2 r_{20}^2),\\
V_{11} &= \frac{1}{720}\pi(1260r_{02}^4 r_{30}^2 r_{12} - 840r_{02}^4 r_{30} r_{12}^2 + 630r_{02}^4 r_{30} r_{03} r_{12}\\
&\quad -945r_{02}^4 r_{03}^2 r_{30} - 420r_{02}^4 r_{12}^3 + 210r_{02}^4 r_{12}^2 r_{03} - 315r_{02}^4 r_{03}^2 r_{12}\\
&\quad -3240r_{12} r_{02}^3 r_{03}^2 r_{20} - 2430r_{03}^3 r_{02}^3 r_{20} + 3780r_{03}^3 r_{02}^2 r_{20}^2\\
&\quad -1116r_{03}^3 r_{20}^3 r_{02} + 20r_{30}^2 r_{20}^4 r_{12} - 2880r_{30} r_{03} r_{12}^3 - 40r_{30} r_{20}^4 r_{12}^2\\
&\quad +720r_{30} r_{03}^3 r_{12} - 338r_{30} r_{20}^4 r_{12} r_{03} + 15r_{30} r_{20}^4 r_{03}^2 - 960r_{12}^4 r_{03}\\
&\quad +20r_{12}^3 r_{20}^4 + 240r_{12}^2 r_{03}^3 - 6r_{12}^2 r_{20}^4 r_{03} + 97r_{12} r_{20}^4 r_{03}^2\\
&\quad +168r_{03}^3 r_{20}^4),\\
V_{13} &= \frac{1}{756000}\pi(-92637900r_{20} r_{03}^4 r_{02} r_{12} + 71856r_{20}^6 r_{12} r_{30} r_{03}\\
&\quad -5799456r_{30} r_{20}^2 r_{03}^2 r_{12}^2 - 3061800r_{02}^4 r_{03}^3 r_{20}^2\\
&\quad +7560000r_{02}^2 r_{30}^2 r_{12}^3 - 38094525r_{20}^2 r_{03}^4 r_{12} + 18192r_{20}^6 r_{12}^2 r_{03}\\
&\quad +30080r_{20}^6 r_{12}^2 r_{30} + 2184000r_{20}^2 r_{30} r_{12}^4 + 23726925r_{20}^2 r_{03}^4 r_{30}\\
&\quad -50400r_{20}^2 r_{30}^4 r_{12} - 15040r_{20}^6 r_{30}^2 r_{12} - 65580300r_{03}^5 r_{02} r_{20}\\
&\quad -35144r_{20}^6 r_{03}^2 r_{12} - 5040000r_{02}^2 r_{30} r_{12}^4 + 67388526r_{30} r_{12} r_{03}^3 r_{20}^2\\
&\quad -504000r_{20}^2 r_{30}^3 r_{12}^2 - 66720000r_{03} r_{12}^4 r_{20}^2 + 27094368r_{20}^2 r_{03}^2 r_{12}^3\\
&\quad -48249078r_{12}^2 r_{03}^3 r_{20}^2 + 187332750r_{02}^2 r_{30} r_{03}^3 r_{12}\\
&\quad +2332800r_{02}^5 r_{03}^2 r_{12} r_{20} + 10440r_{20}^6 r_{03}^2 r_{30} - 235200r_{20}^2 r_{30}^2 r_{12}^3\\
&\quad -95043375r_{02}^2 r_{03}^4 r_{30} + 1749600r_{02}^5 r_{03}^3 r_{20} + 422712r_{20}^5 r_{03}^3 r_{02}\\
&\quad -31681125r_{02}^2 r_{03}^4 r_{12} + 62444250r_{02}^2 r_{12}^2 r_{03}^3 - 15040r_{20}^6 r_{12}^3\\
&\quad -2520000r_{02}^2 r_{12}^5 - 1394400r_{20}^2 r_{12}^5 - 30816r_{20}^6 r_{03}^3).
\end{aligned}
$$

[1]From now on, the explicit expressions of the Lyapunov constants for other systems studied will not be given.

Using again a computer algebra system we obtain that the solutions of the system $\{V_3 = V_5 = V_7 = V_9 = V_{11} = V_{13} = 0\}$ are given by the systems which satisfy one of the next conditions:

- $r_{21} = r_{12} = r_{02} = r_{03} = 0$,
- $r_{21} = r_{02} = r_{20} = r_{03} = 0$,
- $r_{21} = r_{30} = r_{12} = r_{03} = 0$,
- $r_{21} = 3r_{30} + r_{12} = r_{20} = 0$,
- $r_{21} = r_{30} - 3r_{12} = r_{03} - 2r_{12} = r_{02} = r_{20} = 0$,
- $r_{21} = r_{30} - 3r_{12} = r_{03} + 2r_{12} = r_{02} = r_{20} = 0$.

The computations has been run on a PC-Pentium 450Mhz in 1700 seconds. This result coincides with the one obtained in [LPR96]. In that paper it is also proved that all these families have a center at the origin.

Another cubic system that we have studied –also considered in [LPR96]– is the one corresponding to the case $r_{20} = r_{02} = r_{12} = 0$. For this system we can compute the first seven Lyapunov constants, up to V_{15}, in 560 seconds. Our computer need 4000 seconds to solve the system $\{V_3 = V_5 = \ldots = V_{15} = 0\}$, obtaining the families:

- $r_{21} - \overline{r}_{21} = r_{11} = \overline{r}_{03}\overline{r}_{30}^2 + r_{30}^2 r_{03} = 0$,
- $r_{21} - \overline{r}_{21} = \overline{r}_{03} r_{11}^4 + r_{03}\overline{r}_{11}^4 = r_{30} r_{11}^2 + \overline{r}_{30}\overline{r}_{11}^2 = 0$.

See again [LPR96] for a proof that all the systems corresponding to these cases have a center at the origin.

5.2. **BiLiénard systems.** The system

$$\begin{array}{rcl} \dot{x} & = & -y + P(x), \\ \dot{y} & = & x + Q(y), \end{array} \tag{9}$$

where $P(x)$ and $Q(x)$ are analytic functions starting with second order terms at the origin will be called BiLiénard system. It is very easy to find some centers inside this family:

Lemma 5.1. *Consider the BiLiénard system (9). The next cases have a center at the origin:*

(i) $P \equiv 0$, $Q(x) = Q(-x)$,
(ii) $Q \equiv 0$, $P(x) = P(-x)$,
(iii) $P(x) = -Q(x)$,
(iv) $P(x) = Q(-x)$.

Proof. The first two cases are the classical Liénard families with a center. The last two cases are centers because they are invariant by the change of variables $(x, y, t) \to (y, x, -t)$. $\square$

A more difficult problem is to know if there are more centers inside this family. Our method allows to investigate this problem.

Consider $Q(x) = Q_n(x)$ and $P(x) = P_n(x)$, being Q_n and P_n polynomials of degree at most n and having neither constant nor linear terms. We have proved that when $n = 2, 3, 4$ the cases given in Lemma 5.1 are the only centers inside (9). We describe our computations in the sequel. The cases $n = 2, 3$ are easy to handle. For the case $n = 4$ we have computed the first seven Lyapunov and proved that the following ideals are different

$$< V_3, V_5, V_7, V_9, V_{11}, V_{13} > \neq < V_3, V_5, V_7, V_9, V_{11}, V_{13}, V_{15} > .$$

Anyway $V_{15}^2 \in < V_3, V_5, V_7, V_9, V_{11}, V_{13} >$. So, in principle, it seems that to solve the center-focus problem we just have to consider the first six Lyapunov constants. By scaling the variables we can assume that $a_3 \in \{0, \pm 1\}$, where $P_4(x) = a_2x^2 + a_3x^3 + a_4x^4$. Then by using a computer algebra system we obtain that the only values of the coefficients of the equation for which the six Lyapunov constants vanish are the ones given in Lemma 5.1.

For $n = 5$, we can obtain also the first seven Lyapunov constants but our computers (using Maple or Mathematica) have not been able to solve the associated system.

For these polynomial BiLiénard families, the next proposition gives a lower bound of the number of Lyapunov constants which are needed to solve the center-focus problem.

Proposition 5.2. *The origin of the system*

$$\begin{aligned} \dot{x} &= -y + ax^n, \\ \dot{y} &= x + by^n, \end{aligned}$$

with $n \leq 100$ *and even, is a weak focus of order* $4n-3$ *if* $ab(a^2-b^2) \neq 0$. *Otherwise, the origin is a center.*

Proof. Since n is even, by Corollary 3.4 the first significative Lyapunov constant is $V_{2n-1} = \mathcal{H}_n(\mathcal{F}_n(1))$, but for our system, it is zero. When $ab(a-b)(a+b) \neq 0$ we get that the first non zero Lyapunov constant is $V_{4n-3} = \mathcal{H}_n(\mathcal{F}_n(\mathcal{F}_n(\mathcal{F}_n(1))))$. Making computations we have that, in general,

$$V_{4n-3} = K_n \pi ab(a-b)(a+b).$$

We have obtained all the constants K_n for $n \leq 100$ by using our algorithm and they are not zero. In particular $K_2 = -1$, $K_4 = -125/4$, $K_6 = 84057/128$, $K_8 = -9309451/768, \ldots, K_{18} = -453038231859206353983/27917287424, \ldots$ □

5.3. **Homogeneous Kukles systems.** The center-focus problem for systems with homogeneous nonlinearities of degree n have been lengthly studied. The cases $n = 2$ ([Bau54]) and $n = 3$ ([Sib65]) are completely solved. On the other hand for $n \geq 4$, just partial results are known. For this reason, one of the easiest open center-focus problems that is currently studied is the center focus-problem for the so called homogenous Kukles system, see [VI99] or [Gin00]. This system writes as

$$\begin{aligned} \dot{x} &= -y, \\ \dot{y} &= x + Q_n(x, y), \end{aligned}$$

where $Q_n(x, y)$ is a homogeneous polynomial of degree n.

In complex coordinates $z = x + iy$, it can be written as

$$\dot{z} = iz + R_n(z, \overline{z}) = iz + \sum_{k+l=n} r_{k,l} z^k \overline{z}^l,$$

where $\overline{r}_{k,l} = -r_{l,k}$, that is, $\overline{R(z, \overline{z})} = -R(z, \overline{z})$.

As in the case of BiLiénard systems it is very easy to find some cases of reversible centers inside this family. We give them in the next lemma.

Lemma 5.3. *The Kukles system*

$$\dot{z} = iz + R_n(z, \overline{z}),$$

where R_n is a homogeneous polynomial satisfying $\overline{R_n(z, \overline{z})} = -R_n(z, \overline{z})$, has a center at the origin in the next cases:

(i) $r_{i,n-i} = r_{n-i,i}$, *for* $i = 0, \ldots, n$,

(ii) n *even and* $r_{i,n-i} = -r_{n-i,i}$, *for* $i = 0, \ldots, n$.

It is known that the centers given in the above lemma are the only ones when $n = 2, 3$. In [VI99] the authors propose the problem of studying if the same is true for any n. In [Gin00] the author solves the cases $n = 4, 5$ obtaining again that the only centers are the ones given in the above lemma. Here we also study the cases $n = 4, 5$ in a more compact form.

From our method we can obtain the first six (resp. nine) Lyapunov constants when $n = 4$ (resp. $n = 5$) in 1800 seconds (resp. 250 seconds) with a PC-Pentium(500Mhz).

For $n = 4$, by scaling the variables, we can assume that $r_{2,2} \in \{0, 1\}$. By solving the system $\{V_4 = V_{10} = V_{16} = V_{22} = V_{28} = V_{34} = 0\}$ in both cases we get the known centers.

For $n = 5$, we have the following properties for the Lyapunov constants:

$\{V_{29}^2, V_{33}^2, V_{37}\} \subset < V_5, V_9, V_{13}, V_{17}, V_{21}, V_{25} >$, and

$\{V_{33}, V_{37}\} \subset < V_5, V_9, V_{13}, V_{17}, V_{21}, V_{25}, V_{29} >$.

That is, to solve the center problem it seems that it suffices to consider first six constants. After a new scaling ($r_{2,3} \in \{0, 1\}$) we can solve the associated systems. Again we just obtain the well-known centers.

6. Final Comments

There is a different question very related with the center-focus problem: the study of the cyclicity of a weak focus. Remember the following well known fact, see [Rou98]: Assume that the Lyapunov constants $V_3(\alpha), V_5(\alpha), \ldots V_{2n+1}(\alpha)$ associated to a parametric family (with parameter $\alpha \in \mathbb{R}^l$) can take arbitrary values. Then the cyclicity of the critical point inside this family is at least $n - 1$.

The hypothesis on the $\{V_{2k+1}\}$ is clearly not satisfied if there exists some $m \in \mathbb{N}$ such that

$$V_{2k+1}^m \in < V_3, V_5, \ldots, V_{2k-1} > .$$

Our method provides expressions of the Lyapunov constants for several values of k and then the use of computer algebra systems and some tests based on the Gröbner basis theory (see [BW93]) allow us to check this kind of relations. For instance we have the following criterium:

Criterium 6.1. *Given the polynomials* $V_3(\alpha), V_5(\alpha), \ldots V_{2k+1}(\alpha)$, *there exists an* $m \in \mathbb{N}^+$ *such that*

$$V_{2k+1}^m \in < V_3, V_5, \ldots, V_{2k-1} >$$

if and only if the Gröbner basis of the polynomials $V_3(\alpha), V_5(\alpha), \ldots V_{2k-1}(\alpha)$ *and* $V_{2k+1}(\alpha)T - 1$, *with respect the variables* α *and* T *is* $< 1 >$.

Finally, we want to comment that although we think that the method that we develop in this paper is a good one for obtaining the Lyapunov constants of polynomial families of planar differential equations, two more difficulties have to be overcome to solve the center-focus problem:

- Obtaining more efficient algorithms to solve non linear algebraic equations.
- Once some solutions are found, develop methods to check that the differential systems corresponding to these solutions have effectively a center at the critical point.

This paper does not contribute to solve the above problems. The second one is treated in most papers devoted to solve the center-focus problem for specific families. Apart from the already quoted papers we also want to mention the approaches to characterize centers used in [Żoł94a], [CL99] and [ALP00]. More details about the first problem are given in [Wan99].

References

[ALGM73] A. A. Andronov, E. A. Leontovich, I. I. Gordon, and A. G. Maĭer, *Theory of bifurcations of dynamic systems on a plane*, Halsted Press [A division of John Wiley & Sons], New York-Toronto, Ont., 1973, Translated from the Russian.

[ALP00] W. A. Albarakati, N. G. Lloyd, and J. M. Pearson, *Transformation to Liénard form*, Electron. J. Differential Equations **2000** (2000), no. 76, 1–11.

[Bau54] N. N. Bautin, *On the number of limit cycles which appear with the variation of coefficients from an equilibrium position of focus or center type*, American Math. Soc. Translation **1954** (1954), no. 100, 19.

[BW93] T. Becker and V. Weispfenning, *Gröbner bases*, Springer-Verlag, New York, 1993, A computational approach to commutative algebra, In cooperation with Heinz Kredel.

[CGMM97] A. Cima, A. Gasull, V. Mañosa, and F. Mañosas, *Algebraic properties of the Lyapunov and period constants*, Rocky Mountain J. Math. **27** (1997), no. 2, 471 – 501.

[Cha94] J. Chavarriga, *Integrable systems in the plane with center type linear part*, Appl. Math. (Warsaw) **22** (1994), no. 2, 285–309.

[CL99] C. Christopher and J. Llibre, *Algebraic aspects of integrability for polynomial systems*, Qual. Theory Dyn. Syst. **1** (1999), no. 1, 71–95.

[Dev90] J. Devlin, *Word problems related to periodic solutions of a nonautonomous system*, Math. Proc. Cambridge Philos. Soc. **108** (1990), no. 1, 127 – 151.

[Dev91] J. Devlin, *Word problems related to derivatives of the displacement map*, Math. Proc. Cambridge Philos. Soc. **110** (1991), no. 3, 569 – 579.

[DH99] F. Dumortier and C. Herssens, *Tracing phase portraits of planar polynomial vector fields with detailed analysis of the singularities*, Qual. Theory Dyn. Syst. **1** (1999), no. 1, 97 – 131.

[FLLL89] W. W. Farr, C. Li, I. S. Labouriau, and W. F. Langford, *Degenerate Hopf bifurcation formulas and Hilbert's 16th problem*, SIAM J. Math. Anal. **20** (1989), no. 1, 13–30.

[FP95] J. P. Françoise and R. Pons, *Computer algebra methods and the stability of differential systems*, Random Comput. Dynam. **3** (1995), no. 4, 265 – 287.

[Fra96] J. P. Françoise, *Successive derivatives of a first return map, application to the study of quadratic vector fields*, Ergodic Theory Dynam. Systems **16** (1996), no. 1, 87 – 96.

[GGM97] A. Gasull, A. Guillamon, and V. Mañosa, *An explicit expression of the first Lyapunov and period constants with applications*, J. Math. Anal. Appl. **211** (1997), no. 1, 190 – 212.

[GGM99] A. Gasull, A. Guillamon, and V. Mañosa, *An analytic–numerical method for computation of the Lyapunov and period constants derived from their algebraic structure*, SIAM J. Numer. Anal. **36** (1999), no. 4, 1030 – 1043 (electronic).

[Gin00] J. Giné, *Conditions for the existence of a center for the Kukles homogeneous systems*, Preprint, 2000.

[GT01a] A. Gasull and J. Torregrosa, *A new algorithm for the computation of the Lyapunov constants for some degenerated critical points*, To appear in the Proceedings of WCNA2000, in Nonlinear Analysis, 2001.

[GT01b] A. Gasull and J. Torregrosa, *A relation between small amplitude and big limit cycles*, To appear in Rocky Mountain Journal of Mathematics, 2001.

[HW78] B. Hassard and Y. H. Wan, *Bifurcation formulae derived from center manifold theory*, J. Math. Anal. Appl. **63** (1978), no. 1, 297–312.

[Ili98] I. D. Iliev, *On second order bifurcations of limit cycles*, J. London Math. Soc. (2) **58** (1998), no. 2, 353–366.

[IP99] I. D. Iliev and L. M. Perko, *Higher order bifurcations of limit cycles*, J. Differential Equations **154** (1999), no. 2, 339–363.

[Joy98] P. Joyal, *Invariance of Poincaré–Lyapunov polynomials under the group of rotations*, Electron. J. Differential Equations **1998** (1998), No. 23, p. 8 (electronic).

[LL90] Y. R. Liu and J. B. Li, *Theory of values of singular point in complex autonomous differential systems*, Sci. China Ser. A **33** (1990), no. 1, 10 – 23.

[LP99] N. G. Lloyd and J. M. Pearson, *Bifurcation of limit cycles and integrability of planar dynamical systems in complex form*, J. Phys. A **32** (1999), no. 10, 1973–1984.

[LPR96] N. G. Lloyd, J. M. Pearson, and V. A. Romanovsky, *Computing integrability conditions for a cubic differential system*, Comput. Math. Appl. **32** (1996), no. 10, 99 – 107.

[NZ80] I. Niven and H. S. Zuckerman, *An introduction to the theory of numbers*, fourth ed., John Wiley & Sons, New York-Chichester-Brisbane, 1980.

[PLC96] J. M. Pearson, N. G. Lloyd, and C. J. Christopher, *Algorithmic derivation of centre conditions*, SIAM Rev. **38** (1996), no. 4, 619–636.

[Pog94] J. C. Poggiale, *Applications des variétés invariantes à la modélisation de l'hétérogénéité en dynamique des populations*, Ph.D. thesis, Université de Bourgogne, 1994.

[Rou98] R. Roussarie, *Bifurcation of planar vector fields and Hilbert's sixteenth problem*, Birkhäuser Verlag, Basel, 1998.

[Shi81] S. L. Shi, *A method of constructing cycles without contact around a weak focus*, J. Differential Equations **41** (1981), no. 3, 301–312.

[Sib65] K. S. Sibirskiĭ, *On the number of limit cycles in the neighborhood of a singular point*, Differencial'nye Uravnenija **1** (1965), 53–66.

[SV85] J. A. Sanders and F. Verhulst, *Averaging methods in nonlinear dynamical systems*, Springer-Verlag, New York, 1985.

[VI99] E. P. Volokitin and V. V. Ivanov, *Isochronicity and commutability of polynomial vector fields*, Sibirsk. Mat. Zh. **40** (1999), no. 1, 30–48, i.

[Wan99] D. Wang, *Polynomial systems from certain differential equations*, J. Symbolic Comput. **28** (1999), no. 1-2, 303–315, Polynomial elimination—algorithms and applications.

[Żoł94a] H. Żoładek, *The classification of reversible cubic systems with center*, Topol. Methods Nonlinear Anal. **4** (1994), no. 1, 79–136.

[Żoł94b] H. Żoładek, *Quadratic systems with center and their perturbations*, J. Differential Equations **109** (1994), no. 2, 223–273.

Dept. de Matemàtiques, Universitat Autònoma de Barcelona, Edifici Cc 08193 Bellaterra, Barcelona.Spain

E-mail address: `gasull@mat.uab.es, torre@mat.uab.es`