A NEW ALGORITHM FOR THE COMPUTATION OF THE LYAPUNOV CONSTANTS FOR SOME DEGENERATED CRITICAL POINTS

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Abstract. The center problem for degenerated monodromic critical points is far to be solved in general. In this paper we give a procedure to solve it for a particular perturbation of critical points which dominant part near the critical point is $x^{2n-1}\frac{\partial}{\partial x} + y^{2m-1}\frac{\partial}{\partial y}$. For these critical points the problem is solved by writing its associated differential equation in the generalized polar coordinates introduced by Lyapunov and by developing a new method of computation of the so called generalized Lyapunov contants. This method is based into the transformation of the differential equation into the perturbation of a Hamiltonian system. Finally, the method is applied to solve the center and the stability problem for a particular family of differential equations.

1. Introduction and Main Results

Remember that the so called stability problem and the center problem can be solved for smooth non degenerated critical points via the Lyapunov constants. On the other hand the same problem but for arbitrary monodromic critical points is far to be solved in general. This paper will deal with a new method for solving the above mentioned problems for a special kind of planar degenerated critical points. This method is based into a different way of computing the so called generalized Lyapunov constants, see [10]. Before state our results we need to introduce some well known definitions.

Given m, n, $s \in \mathbb{N}$, it is said that *a function* $f : \mathbb{R}^2 \to \mathbb{R}$ *is* (m,n)*-quasi-homogeneous of degree* s if $f(\lambda^m x, \lambda^n y) = \lambda^s f(x,y)$ for all $\lambda \in \mathbb{R}$. A *vector field* $X = (P,Q) : \mathbb{R}^2 \to \mathbb{R}^2$ *is called* (m,n)*-quasi-homogeneous of degree* k if P and Q are (m,n)-quasi-homogeneous functions of degrees $m+k-1$ and $n+k-1$ respectively, see [2, Chap. 7].

Observe that quasi-homogeneous vector fields are a natural generalization of homogeneous vector fields because while the second one can be integrated by using polar coordinates, the first one can be integrated by using the so called (m,n)-polar coordinates. These generalized polar coordinates were introduced by Lyapunov in his study of the stability of degenerate critical points, see [10].

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This paper is devoted to give a method which allows to compute all the generalized Lyapunov constants for a degenerated critical point of an smooth vector field X which first jet (in the decomposition induced by the (m,n)-quasi-homogeneous degree) is given by

$$x^{2n-1}\frac{\partial}{\partial x}+y^{2m-1}\frac{\partial}{\partial y}.$$

By sake of simplicity we will continue just considering the case $m=1$, although all the results of this paper could be obtained for a general m.

More concretely we deal with systems of the form

$$\left\{\begin{array}{rcl}\dot{x} & = & -y+\sum\limits_{k=n+1}^{\infty}P_k(x,y),\\ \dot{y} & = & x^{2n-1}+\sum\limits_{k=2n}^{\infty}Q_k(x,y),\end{array}\right. \tag{1}$$

where the vector fields (P_k,Q_k) are $(1,n)$-quasi-homogeneous of degree k.

Following [10], let us to introduce here these $(1,n)$-polar coordinates. Let $\operatorname{Cs}\theta$ and $\operatorname{Sn}\theta$ be the only solutions of the Cauchy problem

$$\left\{\begin{array}{rcl}\dot{\operatorname{Cs}}\theta & = & -\operatorname{Sn}\theta,\\ \dot{\operatorname{Sn}}\theta & = & \operatorname{Cs}^{2n-1}\theta,\end{array}\right.$$

with initial condition $\operatorname{Cs}(0)=1$ and $\operatorname{Sn}(0)=0$.

Among other properties we recall that $\operatorname{Cs}\theta$ is an even function, that $\operatorname{Sn}\theta$ is an odd function, both are T-periodic, where

$$T=2\sqrt{\frac{\pi}{n}}\frac{\Gamma\left(\frac{1}{2n}\right)}{\Gamma\left(\frac{n+1}{2n}\right)},$$

being Γ the gamma function, and satisfy

$$\operatorname{Cs}^{2n}\theta+n\operatorname{Sn}^2\theta=1.$$

The $(1,n)$-*polar coordinates* are given by $(x,y)=(r\operatorname{Cs}\theta,r^n\operatorname{Sn}\theta)$. With these coordinates, system (1) writes as

$$\left\{\begin{array}{rcl}\dot{r} & = & \frac{x^{2n-1}\dot{x}+y\dot{y}}{r^{2n-1}}=r^{n+1}\sum\limits_{k=0}^{\infty}r^kR_{3n+k}(\theta),\\ \dot{\theta} & = & \frac{-ny\dot{x}+x\dot{y}}{r^{n+1}}=r^{n-1}\left(1+\sum\limits_{k=1}^{\infty}r^k\Theta_{2n+k}(\theta)\right),\end{array}\right. \tag{2}$$

where $R_k(\theta)$ i $\Theta_k(\theta)$ are $(1,n)$-quasi-homogeneous polynomials with the variables $\operatorname{Sn}\theta$ and $\operatorname{Cs}\theta$ of $(1,n)$-quasi-degree k.

In order to define the generalized Lyapunov constants, we write system (2) in the form

$$\frac{dr}{d\theta} = \frac{r^2 \sum_{k=0}^{\infty} r^k R_{3n+k}(\theta)}{1 + \sum_{k=1}^{\infty} r^k \Theta_{2n+k}(\theta)}.$$

We search for a solution $r(\theta, r_0)$ of the form

$$r = r_0 + u_2(\theta) r_0^2 + u_3(\theta) r_0^3 + \cdots,$$

and satisfying $r(0) = r_0$. By evaluating this solution at T, we get the return map near the origin for system (2) as

$$r(T, r_0) = r_0 + u_m(T) r_0^m + \cdots,$$

where $u_m(T)$ is the first non zero term in the Taylor expansion of $r(T, r_0)$ at 0. If such an m does not exists then the origin of (2) is a center. This value $u_m(T)$ is denoted by $V_m = u_m(T)$ and called *m-th generalized Lyapunov constant.* It is clear that its sign gives the stability of the origin of (2). It is also proved in [10] that $m \equiv n \pmod 2$.

Our main result is the following theorem.

Theorem A. *Consider the differential equation (1). In the $(1,n)$-polar coordinates $x = r \operatorname{Cs}\theta$, $y = r^n \operatorname{Sn}\theta$, it can be written as (2), or equivalently in the form*

$$dH + \omega_1 + \omega_2 + \omega_3 \cdots = 0,$$

where $H = \frac{1}{2n}(x^{2n} + ny^2) = \frac{1}{2n} r^{2n}$, and $\omega_k(r,\theta)$ are the 1-forms

$$\omega_k(r,\theta) = r^{2n+k-1}\Theta_{2n+k}(\theta) dr - r^{2n+k} R_{3n+k-1}(\theta) d\theta.$$

Then its K-th Lyapunov constant can be computed as

$$V_K = -\frac{1}{(\sqrt[2n]{2\rho n})^{K+1}} \int_{H=\rho} \sum_{l=1}^{K-1} \omega_l h_{K-1-l},$$

where $h_0 = 1$ and for $m = 1, \ldots, K-1$, h_m are defined by the recurrent expression

$$d\left(\sum_{l=1}^{m} \omega_l h_{m-l}\right) = -d(h_m dH).$$

The above theorem generalizes a similar result obtained for non degenerated centers ($n = 1$) in [7]. Its proof is based on a generalization of results of Françoise, where a perturbation of special kind of Hamiltonian systems is studied, see [4]. An advantage of this new method in relation with other methods (see for instance [1] and [5]) is that it can be used (at least theoretically) to compute V_K for any K.

Here we give a corollary of the above result which computes the first two non zero constants, in terms of the coefficients of the differential equation, for the first degenerated case $n = 2$.

Corollary B. *Consider the following differential equation*

$$\left\{ \begin{array}{rcl} \dot{x} & = & -y + \sum\limits_{k=3}^{\infty} P_k(x,y), \\ \dot{y} & = & x^3 + \sum\limits_{k=4}^{\infty} Q_k(x,y), \end{array} \right. \tag{3}$$

where P_k, Q_k are $(1,2)$-quasi-homogeneous vector fields of degree k. Set

$$\begin{array}{rcl} P_3(x,y) & = & p_{3,0}x^3 + p_{1,1}xy, \\ Q_3(x,y) & = & q_{4,0}x^4 + q_{2,1}x^2y + q_{0,2}y^2, \\ P_4(x,y) & = & p_{4,0}x^4 + p_{2,1}x^2y + p_{0,2}y^2, \\ Q_4(x,y) & = & q_{5,0}x^5 + q_{3,1}x^3y + q_{1,2}xy^2, \\ P_5(x,y) & = & p_{5,0}x^5 + p_{3,1}x^3y + p_{1,2}xy^2, \\ Q_5(x,y) & = & q_{6,0}x^6 + q_{4,1}x^4y + q_{2,2}x^2y^2 + q_{0,3}y^3. \end{array}$$

Then the first Lyapunov constants of system (3) are

$$\begin{array}{rcl} V_2 & = & \dfrac{4}{5}\dfrac{\Gamma(3/4)^2}{\pi^{1/2}}(3p_{3,0} + q_{2,1}), \\ V_3 & = & 0, \\ V_4 & = & \dfrac{2}{63}\dfrac{\pi^{3/2}}{\Gamma(3/4)^2}(3q_{2,1}p_{2,1} + 12q_{0,2}^2q_{2,1} + 3p_{1,2} + 3q_{4,1} + 15p_{5,0} + 9q_{0,3} \\ & & +6p_{0,2}q_{0,2} + 3p_{0,2}p_{1,1} - 3q_{0,2}q_{3,1} - 6q_{0,2}p_{4,0} + 3p_{1,1}p_{4,0} + 45p_{3,0}q_{0,2}p_{1,1} \\ & & +15p_{3,0}p_{1,1}^2 + 15p_{3,0}q_{1,2} + 30p_{3,0}q_{0,2}^2 + 4p_{1,1}^2q_{2,1} - 12q_{4,0}p_{4,0} + 3q_{2,1}q_{1,2} \\ & & +14p_{1,1}q_{2,1}q_{0,2} - 3q_{4,0}q_{3,1} + 15p_{3,0}p_{2,1} + 2q_{4,0}q_{2,1}q_{0,2} + q_{4,0}q_{2,1}p_{1,1}), \\ V_5 & = & 0. \end{array}$$

When a particular system of the form (3) is studied, more Lyapunov constants can be computed. Our last result is an application of the above two results to solve completely the center problem and the computation of the generalized Lyapunov constants for the system

$$\begin{array}{rcl} \dot{x} & = & -y + p_{3,0}x^3 + p_{1,1}xy, \\ \dot{y} & = & x^3 + q_{4,0}x^4 + q_{2,1}x^2y + q_{0,2}y^2. \end{array} \tag{4}$$

This problem has been already studied in [3] and [6]. In this last paper the center problem is solved by using the so called Cherkas' method, which consists in making a change of variables that transforms (4) into a Liénard differential equation. Here we solve the problem by directly computing its generalized Lyapunov constants. Our result is:

Theorem C. *Consider system (4). Then the following hold:*

(i) Its first generalized Lyapunov constants are:

$$\begin{aligned} V_2 &= \frac{4}{5}\frac{\Gamma(3/4)^2}{\pi^{1/2}}(3p_{3,0}+q_{2,1}), \\ V_4 &= -\frac{2}{63}\frac{\pi^{3/2}}{\Gamma(3/4)^2 q_{2,1}}(2q_{0,2}+p_{1,1})(p_{1,1}-q_{0,2}-q_{4,0}), \\ V_6 &= \frac{8}{2025}\frac{\Gamma(3/4)^2}{\pi^{1/2}}q_{2,1}(3q_{4,0}+2q_{0,2})(6q_{4,0}^2+3q_{4,0}q_{0,2}+2q_{2,1}^2)(2q_{0,2}+p_{1,1}). \end{aligned}$$

(ii) It has a center at the origin if and only if $V_2=V_4=V_6=0$. Furthermore this situation happens if and only if

(a) $p_{3,0}=q_{2,1}=0$.

(b) $p_{1,1}+2q_{0,2}=q_{2,1}+3p_{3,0}=0$.

(c) $2p_{1,1}+q_{4,0}=q_{2,1}+3p_{3,0}=2q_{0,2}+3q_{4,0}=0$.

(d) $p_{1,1}-q_{2,1}-q_{4,0}=6p_{3,0}^2+q_{0,2}q_{4,0}+2q_{4,0}^2=q_{2,1}+3p_{3,0}=0$.

2. Generalized trigonometrical functions

In order to be able to apply Theorem A we need an easy way of computing $\int_{H=\rho}\omega$, where $H(r,\theta)=r^{2n}/(2n)$ and $\omega=\omega(r,\theta)$ is a 1-form. We prove the following result.

Lemma 2.1. *Let ω be the 1-form*

$$\omega=\alpha(r,\theta)dr+\beta(r,\theta)d\theta,$$

where α and β are T-periodic analytic functions in the variable θ and $H=\frac{1}{2n}r^{2n}$. Then the following hold:

(i) $\int_{H=\rho}\omega=\int_0^T\beta(\sqrt[2n]{2\rho n},\theta)d\theta$.

(ii) The function $h=-\frac{1}{r^{2n-1}}\int_0^\theta\left(\frac{\partial\alpha}{\partial\psi}-\frac{\partial\beta}{\partial r}\right)d\psi$ satisfies $d(\omega)=d(hdH)$; moreover, it turns out that if $\int_{H=\rho}\omega=0$, then h is also T-periodic.

Proof. The first part follows from the fact that H just depends on r, and hence integrate on $H=\rho$ is equivalent to replace r by $\sqrt[2n]{2\rho n}$, and make integration with respect to θ on the interval $[0,T]$.

The second part can be verified just by replacing the value of h inside the equality $d\omega=d(hdH)$. □

In our situation the functions α and β given above are $(1,n)$-quasi-homogeneous polynomials evaluated on the generalizated trigonometrical functions. Therefore we need to get primitives of these functions. Next result of [10], solves partially the problem.

Lemma 2.2. *Let* Sn *and* Cs *denote the $(1,n)$-trigonometrical functions, let T denote their period and let p and q be fixed natural numbers. The following holds:*

(i) $\int \operatorname{Sn}\theta \operatorname{Cs}^q \theta d\theta = -\frac{\operatorname{Cs}^{q+1}\theta}{q+1} + c.$

(ii) $\int \operatorname{Sn}^p\theta \operatorname{Cs}^{2n-1}\theta d\theta = \frac{\operatorname{Sn}^{p+1}\theta}{p+1} + c.$

(iii) $\int \operatorname{Sn}^p\theta \operatorname{Cs}^q\theta d\theta = -\frac{\operatorname{Sn}^{p-1}\theta \operatorname{Cs}^{q+1}\theta}{(p-1)n+q+1} + \frac{p-1}{(p-1)n+q+1}\int \operatorname{Sn}^{p-2}\theta \operatorname{Cs}^q\theta d\theta.$

(iv) $\int \operatorname{Sn}^p\theta \operatorname{Cs}^q\theta d\theta = \frac{n\operatorname{Sn}^{p+1}\theta \operatorname{Cs}^{q-2n+1}\theta}{(p-1)n+q+1} + \frac{q-2n+1}{(p-1)n+q+1}\int \operatorname{Sn}^{p}\theta \operatorname{Cs}^{q-2n}\theta d\theta.$

(v) *There exist a polynomial in two variables $E(x,y)$ and a real constant A such that*
$$\int \operatorname{Sn}^p\theta \operatorname{Cs}^q\theta d\theta = E(\operatorname{Sn}\theta, \operatorname{Cs}\theta) + A\int \operatorname{Cs}^r\theta d\theta,$$
with $r \equiv q \,(mod 2n)$.

Furthermore, the constant A is non zero if and only if p is even and $q \not\equiv 0\,(mod 2n)$.

(vi) $\int_0^T \operatorname{Sn}^p\theta \operatorname{Cs}^q\theta d\theta = 0$, *when either p or q are odd.*

(vii)
$$\int_0^T \operatorname{Sn}^p\theta \operatorname{Cs}^q\theta d\theta = \frac{2}{n^{\frac{p+1}{2}}}\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2n}\right)}{\Gamma\left(\frac{p+1}{2}+\frac{q+1}{2n}\right)},$$
when p and q are even.

As it is discussed in [10] the above result allows to get an explicit expression of the primitives of $\operatorname{Sn}^p\theta \operatorname{Cs}^q\theta$ in terms of the functions $\operatorname{Cs}\theta$ and $\operatorname{Sn}\theta$ just when p is odd. If p is even the lemma gives the primitives also in terms of the functions $\int \operatorname{Cs}^r\theta d\theta$ with $r < 2n$. This fact induces to define the following functions
$$\psi_r(\theta) = \int_0^\theta \operatorname{Cs}^r\phi d\phi - \frac{\int_0^T \operatorname{Cs}^r\phi d\phi}{T}\theta,$$
for any $r = 1, 2, \ldots, 2n-1$, which are also T-periodic. By using these new functions all primitives of $\operatorname{Sn}^p\theta \operatorname{Cs}^q\theta$ as well as the primitives of $\psi_r^m(\theta)\operatorname{Sn}^p\theta \operatorname{Cs}^q\theta$ can be computed as it is shown in the next lemma.

Lemma 2.3. *With the same notations than in Lemma 2.2, and being m a natural number, the following statements hold:*

(i) *There exist a polynomial in two variables $E(x,y)$ and real constants A and B such that*
$$\int \operatorname{Sn}^p\theta \operatorname{Cs}^q\theta d\theta = E(\operatorname{Sn}\theta, \operatorname{Cs}\theta) + A\psi_r(\theta) + B\theta,$$
where $0 < r < 2n$ and $r \equiv q\,(mod 2n)$. Furthermore B is non zero if and only if p and q are even numbers.

(ii) *There exist two polynomials in two variables $E(x,y)$ and $F(x,y)$ and a real contant B (the same E and B than in the above paragraph) such that*
$$\int \psi_r^m(\theta)\operatorname{Sn}^p\theta \operatorname{Cs}^q\theta d\theta = \quad \psi_r^{m+1}(\theta) + \psi_r^m(\theta)E(\operatorname{Sn}\theta, \operatorname{Cs}\theta) +$$
$$\int \psi_r^{m-1}(\theta)F(\operatorname{Sn}\theta, \operatorname{Cs}\theta)d\theta + BG(\theta),$$

where $G(\theta)$ is some unknown function.

Remark 2.4. *Observe that the above lemma allows to compute, in a recurrent way, all primitives of the functions $\psi_r^m(\theta)\operatorname{Sn}^p\theta\operatorname{Cs}^q\theta$ in terms of $\operatorname{Sn}\theta, \operatorname{Cs}\theta$ and ψ_r varying r, when the constant B appearing in the primitive of $\operatorname{Sn}^p\theta\operatorname{Cs}^q\theta$ is zero. This is the situation that happens when we use the above lemma to compute Lyapunov constants for system (1). This described situation is similar to the situation occurring when one compute the usual Lyapunov constants: It suffices to know how to compute primitives of functions of the form $\sin^p\theta\cos^q\theta$ without taking into account the cases in which some θ appears in the integration process.*

Proof of Lemma 2.3. (i) This part follows straightforward from the definition of ψ_r.

(ii) This equality follows by applying integration by parts, taking $u(\theta) = \psi_r^m(\theta)$ and $v'(\theta) = \operatorname{Sn}^p\theta\operatorname{Cs}^q\theta$, and using part (i). □

3. Proof of the main results

In order to prove Theorem A we need to extend some results of [4]. In that paper the author computes the first non zero derivative of the return map associated to a first order perturbation in ε of some Hamiltonian systems, including $H(x,y) = \frac{1}{2}(x^2+y^2) = r^2/2$. Here we study the case of arbitrary analytic perturbations of Hamiltonian systems of the form $\frac{1}{2n}(x^{2n}+ny^2)$. The proof of our generalization can be done following the same ideas than in [8] or [9]. We state it in the following theorem.

Theorem 3.1. *Let (r,θ) be some type of polar coordinates defined on the cylinder $C := \mathbb{R}^+ \times \mathbb{R}/[0,T)$, with $T > 0$. Consider the solution of the planar analytic differential equation defined on C*

$$\omega_\varepsilon = dH + \sum_{i=1}^{\infty} \varepsilon^i \omega_i = 0,$$

where $\omega_i(r,\theta)$ are analytic 1-forms, T-periodics in θ, and $H(r,\theta) \equiv H(r)$. Let

$$L : (\rho,\varepsilon) \to L(\rho,\varepsilon) = \rho + \varepsilon L_1(\rho) + \cdots + \varepsilon^k L_k(\rho) + O(\varepsilon^{k+1}),$$

be the first return map given by its flow and associated to the transversal section Σ, (we choose $H = \rho$ as a parametrization of Σ). If we assume that $L_1(\rho) \equiv \cdots \equiv L_{m-1}(\rho) \equiv 0$, then there exist functions $h_0 \equiv 1, h_1(r,\theta), \ldots, h_{m-1}(r,\theta)$, T-periodics in θ and such that

(i)

$$d\left(\sum_{i=1}^{m} \omega_i h_{m-i}\right) = -d(h_m dH).$$

(ii) The m-th derivative at $\varepsilon = 0$, with respect to ε, of $L(\rho, \varepsilon)$ is given by $m!L_m(\rho)$, where

$$L_m(\rho) = -\int_{H=\rho} \sum_{i=1}^{m} \omega_i h_{m-i}.$$

Proof of Theorem A. Assume that for system (1), which in polar coordinates writes as (2), the first $K-1$ generalized Lyapunov constants are zero, i.e. $V_1 = V_2 = \cdots = V_{K-1} = 0$. Then a solution $r(\theta, r_0)$, starting at $r(0, r_0) = r_0$ satisfies that

$$r(T, r_0) = r_0 + V_K r_0^K + O(r_0^{K+1}). \tag{5}$$

To compute this V_K we make the change of scale $\tilde{r} = \frac{1}{\varepsilon} r$ in the differential equation (2). It is transformed into

$$\frac{d\tilde{r}}{d\theta} = \frac{d\tilde{r}}{dr}\frac{dr}{d\theta} = \varepsilon \frac{\tilde{r}^2 \sum\limits_{k=0}^{\infty} (\varepsilon \tilde{r})^k R_{3n+k}(\theta)}{1 + \sum\limits_{k=1}^{\infty} (\varepsilon \tilde{r})^k \Theta_{2n+k}(\theta)},$$

which can also be written as

$$d\tilde{r} + \sum_{k=1}^{\infty} \varepsilon^k \left(\tilde{r}^k \Theta_{2n+k}(\theta) d\tilde{r} - \tilde{r}^{k+1} R_{3n+k-1}(\theta) d\theta \right) = 0.$$

Finally, the above expression, for $\tilde{r} > 0$, has the same solutions than

$$dH + \varepsilon \omega_1 + \varepsilon^2 \omega_2 + \cdots = 0, \tag{6}$$

where remember that $H = \frac{1}{2n}(x^{2n} + ny^2) = \frac{1}{2n}\tilde{r}^{2n}$ and $\omega_k = \tilde{r}^{2n+k-1}\Theta_{2n+k}(\theta) d\tilde{r} - \tilde{r}^{2n+k} R_{3n+k-1}(\theta) d\theta$. To compute the solution of (6) as a power expansion in ε we can use Theorem 3.1. To do this we parametrize the transversal section $\Sigma = \{(\rho, \theta), \quad \theta = 0\}$ by the energy level ρ. In other words we parametrice it by ρ where $r = \varepsilon \tilde{r} = \varepsilon \sqrt[2n]{2\rho n}$. In these coordinates the return map associated to Σ writes as

$$L(\rho, \varepsilon) = \rho + \varepsilon^M L_M(\rho) + O(\rho^{M+1}), \tag{7}$$

for some integer M.

On the other hand, a expression of L can be obtained from (5). By using both expressions and Theorem 3.1 the result will follow. We make these computations in the sequel.

A point in Σ with energy ρ has radius $r = \varepsilon \sqrt[2n]{2\rho n}$ in quasi-degenerated polar coordinates. Hence by using (5) it is transformed by the return map into a point with r-coordinate $\varepsilon \sqrt[2n]{2\rho n} + u_K(T) \left(\varepsilon \sqrt[2n]{2\rho n} \right)^K + \cdots$. This point, after the scaling, writes as $\sqrt[2n]{2\rho n} + \varepsilon^{K-1} \left(\sqrt[2n]{2\rho n} \right)^K V_k + \cdots$ and hence it is on the energy level

$$\frac{1}{2n} \left(\sqrt[2n]{2\rho n} + \varepsilon^{K-1} V_K \left(\sqrt[2n]{2\rho n} \right)^K + \cdots \right)^{2n} = \rho + (2\rho n)^{\frac{K+1}{2n}} V_K \varepsilon^{K-1} + O(\varepsilon^K).$$

By equating the above expression and (7) we get that $L_1 \equiv L_2 \equiv \cdots \equiv L_{M-2} \equiv 0$, that $M-1 = K$ and that

$$V_K = \frac{1}{(\sqrt[2n]{2\rho n})^{K+1}} L_{K-1}(\rho).$$

By using Theorem 3.1 we have the expression for L_{K-1}, which gives the proof of the theorem. □

Proof of Corollary B. We want to apply Theorem A to system (3). We start with the more general system (1). Its equivalent expression, in generalized polar coordinates, is

$$dH + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \cdots = 0,$$

where

$$\omega_k(r,\theta) = r^{2n+k-1}\Theta_{2n+k}(\theta)dr - r^{2n+k}R_{3n+k-1}(\theta)d\theta.$$

Hence

$$L_1(\rho) = -\int_{H=\rho}\omega_1 = \int_{H=\rho} r^{2n+1}R_{3n}(\theta)d\theta = (\sqrt[2n]{2\rho n})^{2n+1}\int_0^T R_{3n}(\theta)d\theta.$$

Observe that $R_{3n}(\theta) = r_{3n,0}\operatorname{Cs}^{3n}\theta + r_{2n,1}\operatorname{Cs}^{2n}\theta\operatorname{Sn}\theta + r_{n,2}\operatorname{Cs}^{n}\theta\operatorname{Sn}^2\theta + r_{0,3}\operatorname{Sn}^3\theta$, and $\Theta_{2n+1}(\theta) = t_{2n+1,0}\operatorname{Cs}^{2n+1}\theta + t_{n+1,1}\operatorname{Cs}^{n+1}\theta\operatorname{Sn}\theta + t_{1,2}\operatorname{Cs}\theta\operatorname{Sn}^2\theta$, for some constants $r_{i,j}$ and $t_{i,j}$. Hence by using Lema 2.2 and some properties of the Γ function we get

$$\begin{aligned}
\int_0^T R_{3n}(\theta)d\theta &= r_{3n,0}\int_0^T \operatorname{Cs}^{3n}\theta d\theta + r_{n,2}\int_0^T \operatorname{Cs}^{n}\theta\operatorname{Sn}^2\theta d\theta \\
&= \begin{cases} \frac{4\sqrt{n}}{(2n+1)}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n+1}{2n}\right)}{\Gamma\left(\frac{1}{2n}\right)}((n+1)r_{3n,0} + r_{n,2}) & \text{when } n \text{ is even,} \\ 0 & \text{when } n \text{ is odd.} \end{cases}
\end{aligned}$$

When $L_1(\rho) \equiv 0$, it is possible to compute h_1 in order to get an expression for $L_2(\rho) = -\int_{H=\rho}(\omega_2 + h_1\omega_1)$. Lemma 2.1 gives that

$$\begin{aligned}
h_1 &= -\frac{1}{r^{2n-1}}\int_0^\theta \left(\frac{\partial}{\partial\phi}\left(r^{2n}\Theta_{2n+1}(\phi)\right) + \frac{\partial}{\partial r}\left(r^{2n+1}R_{3n}(\phi)\right)\right)d\phi \\
&= -r\left(\Theta_{2n+1}(\theta) + (2n+1)\int_0^\theta R_{3n}(\phi)d\phi\right),
\end{aligned}$$

and, after substituting the expressions of Θ_{2n+1} and R_{3n}, we obtain that

$$\begin{aligned} h_1 &= -r\Big(t_{2n+1,0}\operatorname{Cs}^{2n+1}\theta + t_{n+1,1}\operatorname{Cs}^{n+1}\theta\operatorname{Sn}\theta + t_{1,2}\operatorname{Cs}\theta\operatorname{Sn}^2\theta \\ &\quad +(2n+1)\big(r_{3n,0}\int_0^\theta \operatorname{Cs}^{3n}\phi d\phi + r_{2n,1}\int_0^\theta \operatorname{Cs}^{2n}\phi\operatorname{Sn}\phi d\phi \\ &\quad +r_{n,2}\int_0^\theta \operatorname{Cs}^{n}\phi\operatorname{Sn}^2\phi d\phi + r_{0,3}\int_0^\theta \operatorname{Sn}^3\phi d\phi\big)\Big). \end{aligned}$$

It is clear that the above primitives depend on the parity of n. By using Lemma 2.2 We get the following general expression

$$\begin{aligned} h_1 &= -r\big((t_{2n+1,0}-r_{2n,1})\operatorname{Cs}^{2n+1}\theta + (t_{n+1}+nr_{3n,0}-r_{n,2})\operatorname{Cs}^{n+1}\theta\operatorname{Sn}\theta \\ &\quad +(t_{1,2}-r_{0,3})\operatorname{Cs}\theta\operatorname{Sn}^2\theta + \frac{1-(-1)^n}{2}((n+1)r_{3n,0}+r_{n,2})\psi_n(\theta)\big). \end{aligned}$$

Observe that in the expression of h_1 the function ψ_n just appears when n is odd. For this reason L_2 is zero when n is even. The computation in the odd case would need the use of Lemmas 2.2 and 2.3. At this point we continue just for the case $n = 2$. Straightforward computations by using Lemmas 2.2 and 2.3 gives the desired result. $\square$

Proof of Theorem C. (i) By making the same computations than in the above corollary, but for system (4) we obtain that the first h_l are given by

$$\begin{aligned} h_1 &= r(2q_{0,2}+p_{1,1})\operatorname{Cs}(\theta), \\ h_2 &= \frac{r^2}{3}(2q_{0,2}+p_{1,1})(3\operatorname{Cs}(\theta)^2p_{1,1}+\operatorname{Sn}(\theta)q_{2,1}+3\operatorname{Cs}(\theta)^2q_{0,2}), \\ h_3 &= \frac{r^3}{9}(2q_{0,2}+p_{1,1})\operatorname{Cs}(\theta)(\operatorname{Cs}(\theta)^2(9p_{1,1}^2+15q_{0,2}p_{1,1}+6q_{0,2}^2+q_{2,1}^2) \\ &\quad +\operatorname{Sn}(\theta)(12q_{0,2}q_{2,1}+3q_{4,0}q_{2,1})), \\ h_4 &= \frac{r^4}{36}(2q_{0,2}+p_{1,1})((2q_{2,1}^3+6q_{4,0}^2q_{2,1}+3q_{4,0}q_{2,1}q_{0,2})\psi_1(\theta) \\ &\quad +(36p_{1,1}^3+78p_{1,1}^2q_{0,2}+54p_{1,1}q_{0,2}^2+6q_{4,0}q_{2,1}^2+14q_{0,2}q_{2,1}^2+12q_{0,2}^3)\operatorname{Cs}(\theta)^4 \\ &\quad +(18q_{4,0}^2q_{2,1}+120q_{2,1}q_{0,2}^2+81q_{4,0}q_{2,1}q_{0,2}+2q_{2,1}^3)\operatorname{Sn}(\theta)\operatorname{Cs}(\theta)^2). \end{aligned}$$

From these expressions and Theorem A it is easy to get the expressions of V_2, V_4 and V_6 that appear in the statement of the theorem. Note that we have needed to use Lemmas 2.2 and 2.3. Observe that the computation of V_6 already needs part (ii) of this last lemma because it involves the integration of the one form $h_4\omega_1$ and h_4 has a term with the function $\psi_1(\theta)$.

(ii) It is easy to see that the only solutions of the system $V_2 = V_4 = V_6 = 0$ are given by the four families of the statement of the theorem. The families (a), (b) and (c) have a center at the origin because: (a) is invariant by the change of variables

$y_1 = -y$ and $t_1 = -t$; (b) is Hamiltonian and (c) has an integrating factor. To prove that in case (d) the origin is a center is more complicated. We remit to [6] for a proof. Here we just say that in this case system (4) can be transformed into a Liénard differential equation and then the application of a generalization of Cherkas' criterion for Liénard equations allows to finish the problem. □

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