A RELATION BETWEEN SMALL AMPLITUDE AND BIG LIMIT CYCLES.

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Abstract. There are two well known methods for generating limit cycles for planar systems with a non degenerate critical point of focus type: the degenerate Hopf bifurcation, and the Poincaré-Melnikov method; that is, the study of small perturbations of Hamiltonian systems. The first one gives the so called small amplitude limit cycles, while the second one gives limit cycles which tend to some concrete periodic orbits of the Hamiltonian system when the perturbation goes to zero (big limit cycles for short). The goal of this paper is to relate both methods. In fact, in all the families of differential equations that we have studied both methods generate the same number of limit cycles. The families studied include Liénard systems and systems with homogeneous nonlinearities.

1. Introduction and Main Results

One of the most interesting and difficult problems in the theory of planar differential equations is the control of the number of limit cycles that a differential equation or a family of differential equations can have. Two well-known methods used for generating limit cycles and hence for giving lower bounds for this number for a given family are: Degenerate Hopf bifurcation and the Poincaré-Melnikov method; that is, the study of perturbations of Hamiltonian systems.

Although the above two methods are usually considered as independent there has been several attempts to relate both for concrete families of differential equations. See the results of [3] on quadratic systems and the results of [4] on Liénard systems.

The main goal of this paper is to relate both approaches when we study the number of limit cycles surrounding a nondegenerate critical point. To be more precise we need to introduce some notation.

Let $\mathcal{F}$ be a family of systems of the form

$$\begin{aligned} \dot{x} &= \alpha x - y + p(x,y,\lambda), \\ \dot{y} &= x + \alpha y + q(x,y,\lambda), \end{aligned} \tag{1}$$

where $\lambda \in \mathbb{R}^m$ and the lowest order terms of the analytic functions $p(x,y,\lambda)$ and $q(x,y,\lambda)$ are second order.

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Remember that when $\alpha = 0$ it is said that $\mathcal{F}$ has a *weak focus* at the origin. We say that a weak focus at the origin of $(1)_{\lambda_0}$ has cyclicity $\mathbf{c}(\lambda_0)$ inside $\mathcal{F}$ if:

(i) it is possible to find numbers $\varepsilon_0 > 0$ and $\delta_0 > 0$ such that every system of the form $(1)_\lambda$ with $||\lambda - \lambda_0|| < \varepsilon_0$ cannot have more than $\mathbf{c}(\lambda_0)$ limit cycles within the δ_0-neighborhood of the origin in $\mathbb{R}^2$, and

(ii) for any choice of positive numbers $\varepsilon < \varepsilon_0$ and $\delta < \delta_0$ there exists $\lambda \in \mathbb{R}^m$ satisfying $||\lambda - \lambda_0|| < \varepsilon$ and such that $(1)_\lambda$ has $\mathbf{c}(\lambda_0)$ limit cycles within the $\delta-$neighborhood of the origin in $\mathbb{R}^2$.

Finally, we define $\mathbf{C}(\mathcal{F}) = \sup_{\lambda \in \mathbb{R}^m} \{\mathbf{c}(\lambda)\}$.

In the sequel we describe the usual approach for the computation of $\mathbf{C}(\mathcal{F})$. N.N. Bautin proved that the return map associated to the OX^+-axis can be written as

$$\Pi(x, \alpha, \lambda) = x + \sum_{n=1}^{\infty} \mathbf{V}_n(\alpha, \lambda) x^n,$$

where each function $\mathbf{V}_n$ is an entire function in (α, λ), the coefficients of equation (1). Moreover, if $\alpha = 0$, the function $V_n := V_n(\lambda) := \mathbf{V}_n(0, \lambda)$ is a polynomial of degree $n - 1$, and $V_1 = V_2 = 0$. S. Yakovenko [15], defined the Bautin ideal, I, to be the ideal generated by these coefficients; that is,

$$I = \langle V_3, V_4, \ldots, V_n, \ldots \rangle \in \mathbb{R}[\lambda].$$

Since the family $\mathcal{F}$ has finitely many coefficients, $\lambda \in \mathbb{R}^m$, from Hilbert Basis Theorem, I is finitely generated and hence there exists a minimum $\mathbf{b} \in \mathbb{N}$ such that $I = \langle V_3, V_4, \ldots, V_{\mathbf{b}} \rangle$.

In general, it is difficult to find explicit expressions for the V_n. Usually, instead of these polynomials, people search for corresponding polynomials v_n such that

$v_3 = V_3$, and
$v_n - V_n \in \langle V_3, V_4, \ldots, V_{n-1} \rangle = \langle v_3, v_4, \ldots, v_{n-1} \rangle$, for $n \geq 4$.

The method that we develop to obtain an expression for v_n (see Theorem 2.8) implies that for each $l \geq 2$, $v_{2l} = 0$, or in other words that

$$V_{2l} \in \langle v_3, v_5, \ldots, v_{2l-1} \rangle.$$

We call the polynomials v_n the Lyapunov constants of (1).

We consider the set $\{v_3, v_5, \ldots, v_{2L+1}\}$ where $2L + 1 = \mathbf{b}$ and we eliminate from this set the polynomial v_{2l+1} if $v_{2l+1} \in \langle v_3, v_5, \ldots, v_{2l-1} \rangle$. In this way, we obtain $I = \langle v_{2l_1+1}, v_{2l_2+1}, \ldots, v_{2l_{\mathbf{B}}+1} \rangle$. It is easy to see that $\mathbf{B}$ does not depend on the choice of v_n, and we call it the *Bautin number* of $\mathcal{F}$, $\mathbf{B}(\mathcal{F}) = \mathbf{B}$. In this situation,

$$\Pi(x, 0, \lambda) = x + \sum_{j=1}^{\mathbf{B}(\mathcal{F})} v_{2l_j+1} x^{2l_j+1}(1 + O(x)). \tag{2}$$

For fixed $\lambda = \lambda_0$, the return map in a neighborhood of the origin is either $\Pi(x, 0, \lambda_0) \equiv x$ or $\Pi(x, 0, \lambda_0) = x + v_{2K+1}(\lambda_0) x^{2K+1}(1 + O(x))$, with $v_{2K+1}(\lambda_0) \neq 0$. This K is called the *order* of the origin as a weak focus of $(1)_{\alpha=0,\lambda=\lambda_0}$. It is clear that the maximum order of the origin inside our family is smaller or equal than $l_{\mathbf{B}(\mathcal{F})}$ (this value is not always attained, see Proposition 5.2).

Note that from expression (2) and the works of R. Roussarie [13], or C. Zuppa [16], it is easy to see that the cyclicity of the origin for our family, $\mathbf{C}(\mathcal{F})$, is also bounded above by $\mathbf{B}(\mathcal{F})-1$ ($\mathbf{B}(\mathcal{F})$ varying also α). If $v_{2l_1+1},\ldots,v_{2l_{\mathbf{B}(\mathcal{F})}+1}$ can take arbitrary values, then

$$\mathbf{C}(\mathcal{F})=\mathbf{B}(\mathcal{F})-1. \tag{3}$$

This is the situation for the families $\mathcal{F}=\mathcal{H}_2,\mathcal{H}_3$, and $\mathcal{L}_n$, defined as follows:

(i) $\mathcal{H}_n$, the family of vector fields with homogeneous nonlinearities, whose members are differential equations

$$\begin{aligned}\dot{x} &= -y+P_n(x,y),\\ \dot{y} &= \quad x+Q_n(x,y)\end{aligned}$$

where P_n and Q_n are homogeneous polynomials of degree $n>1$, and

(ii) $\mathcal{L}_n$, the family of Liénard systems given by

$$\begin{aligned}\dot{x} &= -y+p_n(x),\\ \dot{y} &= \quad x\end{aligned} \tag{4}$$

where $p_n(x)$ is a polynomial of degree n without constant and linear terms.

In fact, $\mathbf{C}(\mathcal{H}_2)=2$, $\mathbf{C}(\mathcal{H}_3)=4$, and $\mathbf{C}(\mathcal{L}_n)=\left[\frac{n-3}{2}\right]$, where [ ] denotes the integer part function. These values are calculated in [1], [14], and [2, 16], respectively. We want to stress that (3) is not always true as shown in Proposition 5.2 where there is a family with $\mathbf{C}(\mathcal{F})=2$ and $\mathbf{B}(\mathcal{F})=4$.

From now on, we consider families $\mathcal{F}$ of the form (1), with $\alpha=0$, for which p and q satisfy

$$\begin{aligned}p(x,y,a\lambda+b\mu) &= ap(x,y,\lambda)+bp(x,y,\mu),\\ q(x,y,a\lambda+b\mu) &= aq(x,y,\lambda)+bq(x,y,\mu),\end{aligned}$$

for all $\lambda,\mu\in\mathbb{R}^m$ and $a,b\in\mathbb{R}$. Notice that this is true for $\mathcal{H}_n$ and $\mathcal{L}_n$.

We define the *k-th order Melnikov number* of $\mathcal{F}$, $\mathbf{M}^k(\mathcal{F})$, as the maximum number of limit cycles for system

$$\begin{aligned}\dot{x} &= -H_y+p(x,y,\lambda_k(\varepsilon)),\\ \dot{y} &= \quad H_x+q(x,y,\lambda_k(\varepsilon)),\end{aligned}$$

which bifurcate from the closed orbits of $H=\frac{1}{2}(x^2+y^2)=h$, when ε is small enough and $\lambda_k(\varepsilon)=\lambda_1\varepsilon+\lambda_2\varepsilon^2+\cdots+\lambda_k\varepsilon^k$, varying $\lambda_i\in\mathbb{R}^m$ for $i=1,\ldots,k,$.

Note that the above differential equation is equivalent to

$$\begin{aligned}\dot{x} &= -H_y+\varepsilon p(x,y,\lambda_1)+\varepsilon^2p(x,y,\lambda_2)+\cdots+\varepsilon^kp(x,y,\lambda_k),\\ \dot{y} &= \quad H_x+\varepsilon q(x,y,\lambda_1)+\varepsilon^2q(x,y,\lambda_2)+\cdots+\varepsilon^kq(x,y,\lambda_k).\end{aligned} \tag{5}$$

From Poincaré's work it is well known that the first Melnikov number, $\mathbf{M}^1(\mathcal{F})$, coincides with the maximum number of positive simple zeros of

$$L_1(\rho):=\int_{H=\rho}(p(x,y,\lambda_1)dy-q(x,y,\lambda_1)dx).$$

In Theorem 2.2 we give a generalization of J.P. Françoise's results [5] (see also [13, Chap. 4]) which allows us to compute the first nonzero term, $L_k(\rho)$, of the ε-expansion of the return map associated with system (5) and the OX^+-axis so that

$$L(\rho,\varepsilon) = \rho + \varepsilon^k L_k(\rho) + O(\varepsilon^{k+1}).$$

Furthermore, we prove that $L_k(\rho)$ is a polynomial in ρ.

For each $k \in \mathbb{N}$, define $\widetilde{\mathbf{M}}^k(\mathcal{F})$ to be one less than the number of nonzero $\rho-$monomials that appear in $L_k(\rho)$. In general, $\mathbf{M}^k(\mathcal{F}) \le \widetilde{\mathbf{M}}^k(\mathcal{F})$. As far as we know there are few results about the computation of $\mathbf{M}^k(\mathcal{F})$. In [3] all $\mathbf{M}^k(\mathcal{H}_2)$ are computed and it is proved that $\mathbf{M}^k(\mathcal{H}_2) = 2$ for $k \ge 6$. In [11] it is proved that $\mathbf{M}^1(\mathcal{L}_n) = \left[\frac{n-3}{2}\right]$.

If

$$\mathbf{M}(\mathcal{F}) := \sup_{k\in\mathbb{N}} \mathbf{M}^k(\mathcal{F}) \in \mathbb{N}\cup\{\infty\}$$

and $\mathcal{F}$ is a family with a bounded number of limit cycles, then

$$\mathbf{M}(\mathcal{F}) = \mathbf{M}^{k_0}(\mathcal{F})$$

for some k_0 and all $k \ge k_0$ since $\mathbf{M}^k(\mathcal{F})$ does not decrease with k. We also define

$$\widetilde{\mathbf{M}}(\mathcal{F}) = \sup_{k\in\mathbb{N}} \widetilde{\mathbf{M}}^k(\mathcal{F}).$$

In general $\mathbf{M}(\mathcal{F}) \le \widetilde{\mathbf{M}}(\mathcal{F})$. When we can guarantee that the coefficients of $L_k(\rho)$ are such that it has at least as many zeros as one less than coefficients then $\mathbf{M}(\mathcal{F}) = \widetilde{\mathbf{M}}(\mathcal{F})$.

Remark 1.1. *A more general perturbation of $\dot{x} = -H_y$, $\dot{y} = H_x$ than (5) would be*

$$\begin{array}{rcl} \dot{x} &=& -H_y + \varepsilon_1 p(x,y,\lambda_1) + \varepsilon_2 p(x,y,\lambda_2) + \cdots + \varepsilon_k p_k(x,y,\lambda_k), \\ \dot{y} &=& H_x + \varepsilon_1 q(x,y,\lambda_1) + \varepsilon_2 q(x,y,\lambda_2) + \cdots + \varepsilon_k q_k(x,y,\lambda_k), \end{array} \tag{6}$$

where ε_i are small parameters and $\lambda_i \in \mathbb{R}^m$ for $i = 1,2,\ldots,k$. Note that if we know $\mathbf{M}(\mathcal{F})$ for some family $\mathcal{F}$, this number also bounds the number of limit cycles that bifurcate from the level curves of $H = \frac{1}{2}(x^2+y^2)$ for each perturbation of the form (6) where $(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_k)$ is on some analytic curve in $\mathbb{R}^k$ passing through zero.

The goal of this paper is to determine the numbers $\mathbf{B}(\mathcal{F})$, $\mathbf{C}(\mathcal{F})$, $\widetilde{\mathbf{M}}^k(\mathcal{F})$, $\mathbf{M}^k(\mathcal{F})$ and $\mathbf{M}(\mathcal{F})$ for several families $\mathcal{F}$.

Our first result is a complete solution of our problem for polynomial Liénard systems.

Theorem A. *Consider the family of Liénard differential equations, $\mathcal{L}_n$, defined by*

$$\begin{array}{rcl} \dot{x} &=& -y + p_n(x), \\ \dot{y} &=& x, \end{array}$$

where the polynomial $p_n(x)$ has no constant or linear terms. If $k \ge 1$, then

$$\mathbf{B}(\mathcal{L}_n) - 1 = \mathbf{C}(\mathcal{L}_n) = \mathbf{M}(\mathcal{L}_n) = \mathbf{M}^k(\mathcal{L}_n) = \left[\frac{n-3}{2}\right].$$

Observe that the above result reinforces the well–known Lins-Melo-Pugh Conjecture that $\left[\frac{n-3}{2}\right]$ is the maximum number of limit cycles for system (4) (see also Remark 1.1).

On the other hand, we have the following result that reduces the relation between the Melnikov and Bautin numbers for $\mathcal{H}_n$ to an algebraic problem.

Theorem B. *Let v_{2k+1} for $k \geq 1$, be the Lyapunov constants of the system $\mathcal{H}_n$ given by*

$$\begin{aligned} \dot{x} &= -y + P_n(x, y), \\ \dot{y} &= \quad x + Q_n(x, y) \end{aligned}$$

where P_n and Q_n are homogeneous polynomials of degree n. Assume that the ideal generated by the Lyapunov constants v_{2k+1} is generated by the first M nonzero constants, i.e.

$$\langle v_n, v_{2n-1}, \dots, v_{n+i(n-1)}, \dots \rangle = \langle v_n, v_{2n-1}, \dots, v_{n+M(n-1)} \rangle$$

if n is odd;

$$\langle v_{2n-1}, v_{4n-3}, \dots, v_{n+(2j-1)(n-1)}, \dots \rangle = \langle v_{2n-1}, v_{4n-3}, \dots, v_{n+(2M-1)(n-1)} \rangle$$

if n is even; and that these constants v_k can take arbitrary values. Then,

$$\mathbf{B}(\mathcal{H}_n) - 1 = \mathbf{C}(\mathcal{H}_n) = \mathbf{M}(\mathcal{H}_n).$$

For $n = 2$ Bautin [1] proved that

$$\langle v_3, v_5, \dots, v_{2k+1}, \dots \rangle = \langle v_3, v_5, v_7 \rangle,$$

and for $n = 3$ K.S. Sibirskii [14] proved that

$$\langle v_3, v_5, \dots, v_{2k+1}, \dots \rangle = \langle v_3, v_5, v_7, v_9, v_{11} \rangle.$$

In both cases expressions for the constants were given. Hence we have the following corollary of the above theorem.

Corollary C. *For the family $\mathcal{H}_n$ of differential equations with homogeneous nonlinearities of degree n,*

$$\begin{aligned} \mathbf{M}(\mathcal{H}_2) = \mathbf{C}(\mathcal{H}_2) = 2, \\ \mathbf{M}(\mathcal{H}_3) = \mathbf{C}(\mathcal{H}_3) = 4. \end{aligned}$$

In Section 4 we will prove Theorem 4.1; it give results analogous to Theorem B for several different families.

In Section 2 we give a generalization of the Françoise algorithm [5], see also [9, 10, 12, 13]. This result is useful to get the Poincaré-Melnikov functions and the Lyapunov constants, see Theorem 2.8. Finally, Theorem 2.3 is the main tool to relate the number of small and big limit cycles, and hence to prove Theorems A and B.

This work was motivated by a guest for a family $\mathcal{F}$ where $\mathbf{M}(\mathcal{F})$ is different than $\mathbf{C}(\mathcal{F})$. However, we have not been able to construct such a family. In fact, in all cases for which we have obtained both numbers, they coincide.

2. General Françoise's Algorithm

Consider the differential equation given by

$$dH + \varepsilon\omega = 0, \tag{7}$$

where $H(x,y) = \frac{1}{2}(x^2+y^2)$ and ω is an analytic 1-form. In every compact region containing the origin, and for ε sufficiently small, it is possible to define, given a transversal section Σ, the map L which associate to each point ρ of Σ the first return $L(\rho,\varepsilon)$ induced by the flow of system (7):

$$L : \rho \longrightarrow L(\rho,\varepsilon).$$

By choosing $H(x,y)$ as a parametrization of Σ, L can be expanded as a series:

$$L(\rho,\varepsilon) = \rho + \varepsilon L_1(\rho) + \varepsilon^2 L_2(\rho) + \cdots + \varepsilon^k L_k(\rho) + O(\varepsilon^{k+1}). \tag{8}$$

Poincaré already proved that the first derivative of $L(\rho,\varepsilon)$ with respect to ε, at $\varepsilon = 0$ is

$$L_1(\rho) = -\int_{H=\rho} \omega.$$

This last integral expression is sometimes called first Poincaré-Melnikov function. Françoise in [5] developed a new method to compute the first nonzero term in the expansion with respect to ε of $L(\rho,\varepsilon)$. The next theorem states his main result.

Theorem 2.1. *Let L denote the return map associated with the solution of system (7) and the transversal section Σ. If L is given as the series (8) and $L_1(\rho) \equiv \cdots \equiv L_{k-1}(\rho) \equiv 0$, then there exist polynomials $g_1,\ldots,g_{k-1}$ and $S_1,\ldots,S_{k-1}$ such that $-\omega = g_1 dH + dS_1$, $-g_1\omega = g_2 dH + dS_2, \ldots, -g_{k-2}\omega = g_{k-1}dH + dS_{k-1}$, and*

$$L_k(\rho) = -\int_{H=\rho} g_{k-1}\omega.$$

We remark that the definition of g_k in Theorem 2.1 does not coincide exactly with the definition in [5]; they differ by a minus sign. We have made this inessential change to have a simpler statement of the following generalization.

Theorem 2.2. *Let $L(\rho,\varepsilon) = \rho + \varepsilon L_1(\rho) + \varepsilon^2 L_2(\rho) + \cdots + \varepsilon^k L_k(\rho) + O(\varepsilon^{k+1})$ be the return map associated with the differential equation*

$$dH + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \cdots + \varepsilon^k\omega_k + \cdots = 0 \tag{9}$$

and the transversal section Σ. If $L_1(\rho) \equiv \cdots \equiv L_{k-1}(\rho) \equiv 0$, then there exist polynomials $h_0 \equiv 1$, $h_1,\ldots,h_{k-1}$, and $S_1,\ldots,S_{k-1}$ such that

$$-\sum_{j=1}^{m} \omega_j h_{m-j} = h_m dH + dS_m$$

for each $m = 1,\ldots,k-1$ and

$$L_k(\rho) = -\int_{H=\rho} \sum_{j=1}^{k} h_{k-j}\omega_j.$$

The proof of the above Theorem uses the same ideas as in the proof of Theorem 2.1 (see also [9, 10, 12, 13]).

Theorem 2.2 gives an algorithm which allows us to determine the Melnikov number at every order. In other words we have, at least theoretically, a method to determine $\widetilde{\mathbf{M}}^k(\mathcal{F})$, and sometimes $\mathbf{M}^k(\mathcal{F})$.

The above two theorems can be related. In fact, we will prove that the computation of the derivatives of a general perturbation of $dH = 0$, as in (9), can be obtained from the expressions given in Theorem 2.1 for system (7). Hence both results are equivalent. This fact is formalized in the next theorem. Before we state it, we note that our proof of the equivalence is based on Theorem 2.2; and our computations for concrete families seem to show that the use of Theorem 2.2 is in general more efficient than the use of Theorem 2.1. Hence, from our point of view, the next theorem is more useful theoretically than computationally.

Theorem 2.3. *For $k \in \mathbb{N}$, let*

$$L^{(1)}(\rho,\varepsilon) = \rho + \varepsilon L_1^{(1)}(\rho) + \varepsilon^2 L_2^{(1)}(\rho) + \cdots + \varepsilon^k L_k^{(1)}(\rho) + O(\varepsilon^{k+1})$$

be the return map (8) associated with the differential equation $dH + \varepsilon\omega = 0$. (Since $L_j^{(1)}(\rho)$ depends on ω, we will denote it by $L_j^{(1)}(\rho,\omega)$.) Also let

$$L^{(2)}(\rho,\varepsilon) = \rho + \varepsilon L_1^{(2)}(\rho) + \varepsilon^2 L_2^{(2)}(\rho) + \cdots + \varepsilon^k L_k^{(2)}(\rho) + O(\varepsilon^{k+1})$$

be the return map associated with the differential equation (9).

Also, suppose that ω_j, $j = 1, 2, \ldots, k$, are arbitrary 1-forms and n is a positive integer. If $\omega = \omega_1 + \varepsilon\omega_2 + \cdots + \varepsilon^{k-1}\omega_k + O(\varepsilon^k)$ and

$$L_n^{(1)}(\rho,\omega) = L_n^{(1)}(\rho,\omega_1 + \omega_2\varepsilon + \cdots) = \sum_{k=0}^{\infty} L_{n,k}(\rho)\varepsilon^k,$$

then

$$L_n^{(2)}(\rho) = \sum_{k=0}^{n-1} L_{n-k,k}(\rho).$$

This last result is the key point for this paper; it will allow us to relate the two problems under consideration. Before we use it, we need to prove some preliminary results.

As in [6], we can decompose a real polynomial 1-form in a very useful way to compute $\int_{H=\rho}\omega$.

Lemma 2.4. *Let ω be a real polynomial 1-form,*

$$\omega = \sum \alpha_{j,k} z^j \overline{z}^k dz + \sum \overline{\alpha}_{j,k} \overline{z}^j z^k d\overline{z},$$

and suppose that ω is decomposed as follows:

$$\begin{aligned}\omega_h &= \sum_{k-j\neq 1} \alpha_{j,k} z^j \overline{z}^k dz + \sum_{k-j\neq 1} \overline{\alpha}_{j,k} \overline{z}^j z^k d\overline{z}, \ \textit{and}\\ \omega_l &= \omega - \omega_h.\end{aligned}$$

Then there exist polynomials h and S such that

(i) $\omega = \omega_l + \omega_h$,
(ii) $\int_{H=\rho} \omega \equiv \int_{H=\rho} \omega_l$,
(iii) $-\omega_h = hdH + dS$.

Proof. From the definition of ω_h and ω_l, the proof of (i) is a simple verification. To prove (ii), we have to see that $\int_{H=\rho} \omega_h \equiv 0$. Note that the expression

$$-\frac{\partial}{\partial \overline{z}} \sum_{k-j\neq 1} \alpha_{j,k} z^j \overline{z}^k + \frac{\partial}{\partial z} \sum_{k-j\neq 1} \overline{\alpha}_{j,k} \overline{z}^j z^k$$

has no terms of the form $(z\overline{z})^k$. This fact, as in [5], allows us to prove the existence of a polynomial function h such that $d(-\omega_h) = d(hdH)$. Hence, there is a polynomial S satisfying $-\omega_h = hdH + dS$. □

In our study of Liénard differential equations we need a more restrictive version of the above lemma in polar coordinates. We state it in the following remark.

Remark 2.5. *Let ω be a 1-form expressed in polar coordinates (r, θ) as*

$$\omega = \alpha(r,\theta)dr + \beta(r,\theta)d\theta,$$

where α and β are 2π-periodic analytic functions in θ and $H = \frac{1}{2}r^2$. If $\int_{H=\rho} \omega = 0$, then there exists a function h given by $h(r,\theta) = -\frac{1}{r}\int_0^\theta \left(\frac{\partial \alpha}{\partial \psi} - \frac{\partial \beta}{\partial r}\right) d\psi$ such that $d(\omega) = d(hdH)$, or in other words, there exist functions $h(r,\theta)$ and $S(r,\theta)$ such that $-\omega = hdH + dS$.

Note that Lemma 2.4 allows us to give the following definition:

Definition 2.6. *For a sequence of polynomial 1-forms, $\omega_1, \omega_2, \omega_3, \ldots$, we define the following sequence of associated polynomials:*

$$l_1(\rho) = -\int_{H=\rho} \omega_1,$$

$$l_2(\rho) = -\int_{H=\rho} (\omega_2 + h_1\omega_1),$$

where $-(\omega_1)_h = h_1 dH + dS_1$, and

$$l_k(\rho) = -\int_{H=\rho} \left(\sum_{j=1}^{k} h_{k-j}\omega_j\right),$$

where $-\left(\sum_{j=1}^{m} h_{m-j}\omega_j\right)_h = h_m dH + dS_m$, *for* $m = 1, \ldots, k-1$, *and* $h_0 = 1$.

Remark 2.7. *(i) In the above definition, and in contrast with Françoise's method, the condition $\int_{H=\rho} \omega \equiv 0$ is not needed to associate the functions h and S with ω.*

(ii) If the above 1-forms, ω_k, coincide with those in Theorem 2.2, and furthermore if $l_1(\rho) \equiv l_2(\rho) \equiv \cdots \equiv l_{k-1}(\rho) \equiv 0$, then the return map associated with Σ for system (9) can be written as

$$L(\rho, \varepsilon) = \rho + \varepsilon^k l_k(\rho) + O(\varepsilon^{k+1}).$$

In other words, $L_1(\rho) \equiv \cdots \equiv L_{k-1}(\rho) \equiv 0$ and $L_k(\rho) = l_k(\rho)$.

Proof of Theorem 2.3. In order to simplify the notation in this proof let us denote by dS every 1-form ω such that $d\omega = 0$. For instance, we will write $dS + dS = dS$.

We will prove the theorem by induction. Consider first the case $n = 1$. From Theorem 2.1 we know that $L_1^{(1)}(\rho) = -\int_{H=\rho} \omega$. Hence, by using the (h,l)-decomposition given in Definition 2.6 we have that $\omega = \omega_h + \omega_l$. Replacing ω by $\omega = \omega_1 + \varepsilon\omega_2 + \cdots$, we get

$$\begin{aligned} L_1^{(1)}(\varepsilon) &= -\int_{H=\rho} (\omega_1 + \varepsilon\omega_2 + \cdots) = -\int_{H=\rho} \omega_1 - \varepsilon \int_{H=\rho} \omega_2 - \cdots \\ &= -\int_{H=\rho} \omega_{1l} - \varepsilon \int_{H=\rho} \omega_{2l} - \cdots = L_{1,0} + \varepsilon L_{1,1} + \cdots . \end{aligned}$$

Furthermore, from the equality $-\omega_h = h_1 dH + dS$, we get

$$-\omega_{1h} - \varepsilon\omega_{2h} - \cdots = h_1(\varepsilon) dH + dS = (h_{1,0} + \varepsilon h_{11} + \cdots) dH + dS.$$

By equating the terms of their ε expansions, we obtain that $-\omega_{jh} = h_{1,j-1} dH + dS$ for each $j \in \mathbb{N}$.

Hence, from Theorem 2.2, it follows that

$$L_1^{(2)}(\rho) = -\int_{H=\rho} \omega_1 = -\int_{H=\rho} \omega_{1l} = L_{1,0}.$$

Furthermore, because $-\omega_{1h} = \tilde{h}_1 dH + dS$, we have that $\tilde{h}_1 = h_{1,0}$. Hence the theorem follows for $n = 1$.

Before we consider the general case, to clarify the proof let us study the case $n = 2$.

From Theorem 2.1 we have that $L_2^{(1)}(\rho) = -\int_{H=\rho} \omega h_1$. By applying the (h,l)-decomposition to $h_1\omega$, it follows that $h_1\omega = (h_1\omega)_h + (h_1\omega)_l$. Putting

$\omega = \omega_1 + \varepsilon\omega_2 + \cdots$ we get

$$\begin{aligned}
L_2^{(1)}(\varepsilon) &= -\int_{H=\rho} (\omega_1 + \varepsilon\omega_2 + \cdots)(h_{1,0} + \varepsilon h_{1,1} + \cdots) \\
&= -\int_{H=\rho} \omega_1 h_{1,0} - \varepsilon \int_{H=\rho} (\omega_2 h_{1,0} + \omega_1 h_{1,1}) \\
&\quad - \varepsilon^2 \int_{H=\rho} (\omega_3 h_{1,0} + \omega_2 h_{1,1} + \omega_1 h_{1,2}) - \cdots \\
&= -\int_{H=\rho} (\omega_1 h_{1,0})_l - \varepsilon \int_{H=\rho} (\omega_2 h_{1,0} + \omega_1 h_{1,1})_l \\
&\quad - \varepsilon^2 \int_{H=\rho} (\omega_3 h_{1,0} + \omega_2 h_{1,1} + \omega_1 h_{1,2})_l - \cdots \\
&= L_{2,0} + \varepsilon L_{2,1} + \varepsilon^2 L_{2,2} + \cdots .
\end{aligned}$$

On the other hand, by using the equality $-(h_1\omega)_h = h_2 dH + dS$, by substituting in this last equation the expression of ω, and by equating the ε terms, we find that

$$-\left(\sum_{k=0}^{j} \omega_{j+1-k} h_{1,k} \right)_h = h_{2,j} dH + dS,$$

for every $j \in \mathbb{N}$.

Define $\tilde{h}_1 := h_{1,0}$ and use Theorem 2.2 to see that

$$L_2^{(2)}(\rho) = -\int_{H=\rho} (\omega_2 + \omega_1 h_1) = -\int_{H=\rho} (\omega_2 + \omega_1 \tilde{h}_1)_l = L_{1,1} + L_{2,0}.$$

Also, from the above decomposition we have that $-(\omega_2+\omega_1\tilde{h}_1)_h = h_{1,1}dH+dS+h_{2,0}dH+dS = (h_{1,1}+h_{2,0})dH+dS = \tilde{h}_2 dH+dS$, and therefore $\tilde{h}_2 = h_{1,1}+h_{2,0}$. Hence our result follows for $n = 2$.

In order to consider the general case, we will make the following induction hypothesis:

$$\tilde{h}_k = \sum_{j=0}^{k-1} h_{k-j,j},$$

$$-\left(\sum_{p+q=j} \omega_p h_{k,q} \right)_h = h_{k+1,j-1} dH + dS \text{ for } j = 1, \ldots, k, \text{ and,}$$

$$L_{k,j} = -\int_{H=\rho} \left(\sum_{p+q=j+1} \omega_p h_{k-1,q} \right)_l \text{ for } j = 0, \ldots, k,$$

for $k = 1, \ldots, n$.

To prove it, first substitute $\omega = \omega_1 + \varepsilon\omega_2 + \cdots$ in the equality

$$L_{n+1}^{(1)}(\rho) = -\int_{H=\rho} \omega h_n,$$

(which follows from Theorem 2.1) and note that

$$\begin{aligned} L^{(1)}_{n+1}(\rho,\varepsilon) &= \int_{H=\rho} (\omega_1+\varepsilon\omega_2+\cdots)(h_{n,0}+h_{n,1}\varepsilon+\cdots) \\ &= \int_{H=\rho} (\omega_1 h_{n,0})_l + \varepsilon \int_{H=\rho} (\omega_2 h_{n,0}+\omega_1 h_{n,1})_l \\ &\quad + \cdots + \varepsilon^k \int_{H=\rho} \left(\sum_{i+j=k+1} \omega_i h_{n,j} \right)_l + \cdots \\ &= \sum_{k=0}^{\infty} L_{n+1,k}\varepsilon^k . \end{aligned}$$

By defining $\tilde{h}_0 = 1$ and by using the induction hypothesis and Theorem 2.2, we have that

$$\begin{aligned} L^{(2)}_{n+1}(\rho) &= -\int_{H=\rho} \sum_{i+j=n+1} \omega_i h_j = -\int_{H=\rho} \sum_{i+j=n+1} \omega_i \tilde{h}_j \\ &= -\int_{H=\rho} (\omega_{n+1})_l + (\omega_n h_{1,0})_l + \cdots + \left(\omega_1 \sum_{i+j=n} h_{i,j}\right)_l \\ &= -\int_{H=\rho} \left(\sum_{k=1}^{n+1} \omega_k \sum_{i+j=n-k+1} h_{i,j} \right)_l = -\int_{H=\rho} \left(\sum_{k+i+j=n+1} \omega_k h_{i,j} \right)_l \\ &= -\int_{H=\rho} \left(\sum_{i=1}^{n+1} \sum_{k+j=i} \omega_k h_{n+1-i,j} \right)_l = \sum_{i=1}^{n+1} L_{n+2-i,i-1}. \end{aligned}$$

Moreover we have that

$$\begin{aligned} -\left(\sum_{i+j=n+1} \omega_i \tilde{h}_j \right)_h &= -\left(\sum_{k=1}^{n+1} \omega_k \sum_{i+j=n-k+1} h_{i,j} \right)_h = -\left(\sum_{k+i+j=n+1} \omega_k h_{i,j} \right)_h \\ &= \left(\sum_{i=1}^{n+1} - \sum_{k+j=i} \omega_k h_{n+1-i,j} \right)_h = \sum_{i=1}^{n+1} (h_{n+2-i,i-1} dH + dS) \\ &= \tilde{h}_{n+1} dH + dS. \end{aligned}$$

Hence $\tilde{h}_{n+1} = \sum\limits_{i+j=n+1} h_{i,j}$, and therefore the theorem is proved. □

Theorem 2.2 can also be used to compute the Lyapunov constants for system (1) with $\alpha = 0$. See [8, Theorem A] for a proof of the following result.

Theorem 2.8. *The differential equation (1) with $\alpha = 0$ can be written as*

$$dH + \omega_1 + \omega_2 + \omega_3 \cdots = 0,$$

where $H = \frac{1}{2}(x^2 + y^2)$ and $\omega_k = \omega_k(x, y)$ are homogeneous polynomial 1-forms of degree $k + 1$.

(i) The K-th Lyapunov constant of this differential equation is given by

$$v_K = -\frac{1}{(\sqrt{2\rho})^{K+1}} \int_{H=\rho} \sum_{l=1}^{K-1} \omega_l h_{K-1-l}$$

where $h_0 = 1$ and, for $m = 1, \ldots, K-1$, the polynomials h_m are defined by the recurrence relation

$$d\left(\sum_{l=1}^{m} \omega_l h_{m-l}\right) = -d(h_m dH).$$

Also,

(ii) $v_{2l} = 0$ for $l \geq 2$.

It can be seen that, although in the expression of the K-th Lyapunov constant given in the first statement of the above theorem there appear the variable ρ, it cancels once the formula is developed, see again [8].

3. Liénard Equations

Consider a new family $\mathcal{F} = \mathcal{G}_n$, which includes the Liénard differential equations $\mathcal{L}_n$. This family is given by the differential equations

$$\begin{array}{rcl} \dot{x} & = & -y + a_1 X_1(x,y) + X_2(x,y) + a_3 X_3(x,y) + \cdots + a_n X_n(x,y), \\ \dot{y} & = & x + a_1 Y_1(x,y) + Y_2(x,y) + a_3 Y_3(x,y) + \cdots + a_n Y_n(x,y), \end{array} \tag{10}$$

where $X_i(x,y)$ and $Y_i(x,y)$ are homogeneous polynomials of degree i. Furthermore, $X_{2j}(-x,y) = X_{2j}(x,y)$ and $Y_{2j}(-x,y) = -Y_{2j}(x,y)$ for each $j = 1, 2, \ldots$ Here $a_n = 1$ if n is even. Note that if $a_1 = a_3 = \cdots = a_{2k+1} = \cdots = 0$, then the origin of (10) is a reversible center.

Theorem 3.1. *Let $\mathcal{G}_n$ denote the family of differential equations defined in (10). Then $\mathbf{M}(\mathcal{G}_n) = [\frac{n-1}{2}]$, where $[\ ]$ denotes the integer part function.*

As an easy corollary of the above result we can prove Theorem A.

Proof of Theorem A. Consider the subfamily $\mathcal{F}_n \subset \mathcal{G}_n$ given by the Liénard equations, where $(X_j(x,y), Y_j(x,y)) = (b_j x^j, 0)$ for $j = 1, \ldots,$ and $b_{2j+1} = 1$ for $j \geq 0$. Taking $a_1 = 0$,(remember that $\mathcal{F}_n$ has no linear terms) we get that $\mathbf{M}(\mathcal{L}_n) = [\frac{n-3}{2}]$. $\square$

Let us prove Theorem 3.1.

Proof of Theorem 3.1. In order to simplify the proof we introduce the operator χ that acts on functions of the form $f(r,\theta) = r^a \cos^b\theta \sin^c\theta$ as follows:

$$\begin{array}{rcl} \chi : \{A r^a \cos^b\theta \sin^c\theta : A \in \mathbb{R}\setminus\{0\}\} & \longrightarrow & \mathbb{N} \times (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z}), \\ f & \longmapsto & (a, b+c, c). \end{array}$$

It has the following properties:

(i) $\chi(fg) = \chi(f) + \chi(g)$;
(ii) $\chi\left(\frac{\partial}{\partial r} f\right) = \chi(f) - (1,0,0)$;
(iii) $\chi\left(\int f d\theta\right) = \chi(f) + (0,0,1)$;
(iv) if $\chi(f) = (*,*,1)$, then $\int_{H=\rho} f d\theta = 0$;

(v) if $\chi(f) = (*, 1, 0)$, then $\int_{H=\rho} f d\theta = 0$;
(vi) if $\chi(f) = (2k, 0, 0)$, then $\int_{H=\rho} f d\theta = \rho^k C_{2k}$, with $C_{2k} \neq 0$.

In the above expressions $*$ denotes an arbitrary integer. Furthermore, if f is given by $f = \sum A_{a,b,c} r^a \cos^b\theta \sin^c\theta$, then

$$\chi(f) = \sum A_{a,b,c}\chi(r^a \cos^b\theta \sin^c\theta) = \sum A_{a,b,c}(a, b+c, c),$$

and if $\omega = f dr + g d\theta$, then

$$\chi(\omega) = \chi(f)dr + \chi(g)d\theta.$$

Let us start the proof. In polar coordinates the differential 1-form associated with system (5) is

$$r dr + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \cdots = 0$$

where

$$\begin{aligned}\omega_i = &\big(\cos\theta Q_i(r\cos\theta, r\sin\theta) - \sin\theta P_i(r\cos\theta, r\sin\theta)\big)dr\\ &- \big(r\cos\theta P_i(r\cos\theta, r\sin\theta) + r\sin\theta Q_i(r\cos\theta, r\sin\theta)\big)d\theta.\end{aligned}$$

For the family $\mathcal{G}_n$ given by (10), we have that

$$P_i(x,y) = a_{1,i}X_{1,i}(x,y) + X_{2,i}(x,y) + a_{3,i}X_{3,i}(x,y) + \cdots + a_{n,i}X_{n,i}(x,y)$$

and

$$Q_i(x,y) = a_{1,i}Y_{1,i}(x,y) + Y_{2,i}(x,y) + a_{3,i}X_{3,i}(x,y) + \cdots + a_{n,i}Y_{n,i}(x,y).$$

Let us study how the function χ acts on the components of the vector field defined by system (5). We have that

$$\begin{aligned}\chi(P_i) = &a_{1,i}(1,1,*) + (2,0,0) + a_{3,i}(3,1,*) + (4,0,0)\\ &+ \cdots + a_{n,i}\left(n, \frac{1-(-1)^n}{2}, \frac{1-(-1)^n}{2}*\right),\\ \chi(Q_i) = &a_{1,i}(1,1,*) + (2,0,1) + a_{3,i}(3,1,*) + (4,0,1)\\ &+ \cdots + a_{n,i}\left(n, \frac{1-(-1)^n}{2}, \frac{1-(-1)^n}{2} * + 1\right).\end{aligned}$$

Hence, for its associated 1-form, we have that

$$\begin{aligned}\chi(\omega_i) = &\Bigg(a_{1,i}(1,0,*) + (2,1,1) + a_{3,i}(3,0,*) + \cdots\\ &+ a_{n,i}\left(n, \frac{1+(-1)^n}{2}, \frac{1-(-1)^n}{2} * + 1\right)\Bigg)dr\\ &+ \Bigg(a_{1,i}(2,0,*) + (3,1,0) + a_{3,i}(4,0,*) + \cdots\\ &+ a_{n,i}\left(n+1, \frac{1+(-1)^n}{2}, \frac{1-(-1)^n}{2}*\right)\Bigg)d\theta.\end{aligned}$$

From Theorem 2.2 and the fact that $\int_{H=\rho} f\,dr = 0$ for every regular function f, we have that

$$\begin{aligned} L_1(\rho) &= \int_{H=\rho} \omega_1 \\ &= a_{1,1}\rho C_2 + a_{3,1}\rho^2 C_4 + a_{5,1}\rho^3 C_6 + \cdots + a_{n,1} C_{[(n+1)/2]}\rho^{[(n+1)/2]}, \end{aligned} \tag{11}$$

and hence $L_1(\rho)$ is a polynomial of degree $[(n+1)/2]$ without constant term. Choosing suitable values for $a_{2k+1,1}$, it is possible to construct examples with ε as small as we like, and with $[(n+1)/2] - 1 = [(n-1)/2]$ hyperbolic limit cycles, as required.

Our objective is to show that for a perturbation of arbitrary order in ε all the polynomials appearing in the computation of the first nonzero Poincaré-Melnikov function $L_k(\rho)$ are like (11). In other words, and since by Theorem 2.2

$$L_k(\rho) = -\int_{H=\rho} \sum_{j=1}^{k} h_{k-j}\omega_j,$$

when $L_1(\rho) \equiv L_2(\rho) \equiv \cdots \equiv L_{k-1}(\rho) \equiv 0$, if we can prove that $\int_{H=\rho} h_j\omega_i = 0$ for all even i and $j \neq 0$, then $L_k = \int_{H=\rho} \omega_k$ for each k and the theorem will follow. To complete the proof, we will show that $L_k = \int_{H=\rho} \omega_k$.

In the proof of the above fact, we will not take into account the degree with respect to r of the involved functions h_j, because this degree is irrelevant to prove that some of the integrals that appear are zero. We also introduce the notation e_* (resp. o_*) for an arbitrary even (resp. odd) number.

Assume that $L_1(\rho) = \int_{H=\rho} \omega_1 \equiv 0$, that is, $a_{i,1} = 0$ for all i. Remark 2.5 states that if $\omega_1 = A_1 dr + B_1 d\theta$, then $h_1 = -\frac{A_1}{r} + \int \frac{B_{1r}}{r} d\theta$. Hence, we have

$$\chi(h_1) = (1,1,1) + (3,1,1) + \cdots + (o_*,1,1).$$

It follows that

$$\begin{aligned} \chi(\omega_1) =& \big((2,1,1) + (4,1,1) + \cdots + (e_*,1,1)\big)dr \\ &+ \big((3,1,0) + (5,1,0) + \cdots + (o_*,1,0)\big)d\theta, \end{aligned}$$

and

$$\begin{aligned} \chi(h_1\omega_1) =& \big((3,0,0) + (5,0,0) + \cdots + (o_*,0,0)\big)dr \\ &+ \big((4,0,1) + (6,0,1) + \cdots + (e_*,0,1)\big)d\theta. \end{aligned}$$

From the properties of the function χ,

$$\int_{H=\rho} h_1\omega_1 = 0$$

and

$$\chi(h_2) = (1,1,1) + (2,0,0) + (3,1,1) + (4,0,0) + \cdots + (o_*,1,1) + (e_*,0,0).$$

Hence we can assume as an induction hypotheses that

$$\chi(h_j) = (1,1,1) + (2,0,0) + (3,1,1) + (4,0,0) + \cdots + (o_*,1,1) + (e_*,0,0),$$

and

$$\begin{aligned}\chi(\omega_j) =&\big((2,1,1)+(4,1,1)+\cdots+(e_*,1,1)\big)dr\\ &+\big((3,1,0)+(5,1,0)+\cdots+(o_*,1,0)\big)d\theta,\end{aligned}$$

for $j=1,\ldots,k-1$. To get $\chi(h_k)$ and $\chi(w_k)$ we will use again Theorem 2.2. Hence we have to study $\chi(\omega_{k-j}h_j)$. We obtain that

$$\begin{aligned}\chi(\omega_{k-j}h_j) =&\big((3,0,0)+(4,1,1)+(5,0,0)+(6,1,1)+\cdots\\ &+(o_*,0,0)+(e_*,1,1)\big)dr\\ &+\big((4,0,1)+(5,1,0)+(6,0,1)+(7,1,0)+\cdots\\ &+(e_*,0,1)+(o_*,1,0)\big)d\theta,\end{aligned}$$

and as a consequence, $\int_{H=\rho}\omega_{k-j}h_j=0$ for every $j=1,\ldots,k-1$. From the above equality we have that

$$L_k=\int_{H=\rho}(\omega_k+\omega_{k-1}h_1+\omega_{k-2}h_2+\cdots+\omega_1h_{k-1})=\int_{H=\rho}\omega_k.$$

Therefore, $L_k(\rho)$ has the same expression as $L_1(\rho)$. Furthermore, it is easy to see that in the case $L_k\equiv 0$, and by using again Remark 2.5, we have

$$\chi(h_k)=(1,1,1)+(2,0,0)+(3,1,1)+(4,0,0)+\cdots+(o_*,1,1)+(e_*,0,0).$$

and

$$\begin{aligned}\chi(\omega_k) =&\big((2,1,1)+(4,1,1)+\cdots+(e_*,1,1)\big)dr\\ &+\big((3,1,0)+(5,1,0)+\cdots+(o_*,1,0)\big)d\theta.\end{aligned}$$

Hence the induction step follows and the proof is complete. □

4. Systems with homogeneous nonlinearities

The next theorem implies our result for $\mathcal{H}_n$ as stated in Theorem B. We introduce the following notation:

Given a sequence of polynomials $l_i(\rho)\in\mathbb{R}[\rho]$, $i\in\mathbb{N}$, we say that they satisfy the property (P_N) if there exist homogeneous polynomials $\widetilde{l}_i(\rho)\in\mathbb{R}[\rho]$ of degree k_i, $i\in\mathbb{N}$, such that

(i) $\tilde{l}_1(\rho)=l_1(\rho)$,
(ii) $\tilde{l}_i(\rho)=l_i(\rho)+\sum_{j<i}p_j(\rho)\tilde{l}_j(\rho)$ for $i\geq 2$, where $p_j(\rho)\in\mathbb{R}[\rho]$.
(iii) $J=\langle l_1(\rho),l_2(\rho),\ldots,l_n(\rho),\ldots\rangle=\langle\tilde{l}_1(\rho),\tilde{l}_2(\rho),\ldots,\tilde{l}_N(\rho)\rangle$,
(iv) $k_1<k_2<\cdots<k_N$.

Theorem 4.1. *Consider a family of differential equations $\mathcal{F}$ and*

$$dH+\varepsilon\omega=0, \tag{12}$$

where ω is a differential 1-form, such that (12) is in $\mathcal{F}$. Assume that the sequence of polynomials $l_k(\rho), k\geq 1$, given in Definition 2.6 and associated to (12), satisfies the property (P_N).

Let $L(\rho,\varepsilon)=\rho+\varepsilon L_1(\rho)+\cdots$ be the return map associated with the solution of

$$dH+\varepsilon\omega_1+\varepsilon^2\omega_2+\cdots=0, \tag{13}$$

where $dH+\varepsilon\omega_k\in\mathcal{F}$ for every k. Then the following holds,

$$L_j(\rho)=\sum_{i=1}^{j}a_{k_i}\rho^{k_i},\ \text{ when } j\le N,$$
$$L_j(\rho)=\sum_{i=1}^{N}a_{k_i}\rho^{k_i},\ \text{ when } j> N,$$

where each a_{k_i} is a polynomial whose variables are the coefficients of (13). In other words, $\widetilde{\mathbf{M}}^j(\mathcal{F})=j-1$ for $j\le N$, and $\widetilde{\mathbf{M}}^j(\mathcal{F})=\widetilde{\mathbf{M}}(\mathcal{F})=N-1$ for $j>N$.

Remark 4.2. *(i) If we add to the hypotheses of the above theorem, the hypothesis that $\tilde{l}_i(\rho)$ can take arbitrary values, then $\widetilde{\mathbf{M}}(\mathcal{F})=\mathbf{M}(\mathcal{F})$.*
(ii) If the family $\mathcal{F}$ has finitely many nonzero coefficients, then the Hilbert Basis Theorem guarantees that the ideal J is finitely generated. In the above theorem we also request that it is generated by the first N elements, and furthermore, that these elements can be replaced by homogeneous polynomials in ρ, with increasing degrees.

As a corollary of the above result we can prove Theorem B.

Proof of Theorem B. Assume first that n is odd. It is easy to see that the only nonzero Lyapunov constants are $v_{n+i(n-1)}$ for $i\ge 0$. If in equation (13) we take all $\omega_i\equiv 0$ except ω_{n-1}, then we get, with $\varepsilon_1:=\varepsilon^{n-1}$ and using Theorem 2.8, that there is the following relation between the return map associated to $dH+\varepsilon_1\omega_{n-1}=0$ and the Lyapunov constants of $dH+\omega_{n-1}=0$:

$$l_i(\rho)=v_{n+(i-1)(n-1)}(2\rho)^{\frac{(n+1)+(i-1)(n-1)}{2}}.$$

Hence, the polynomials $l_i(\rho)$ are homogeneous in the variable ρ, with these degrees, k_i, all different. So we can take $l_i(\rho)\equiv\tilde{l}_i(\rho)$. Therefore, from Theorem 4.1 we have as many nonzero coefficients of the polynomial $L_j(\rho)$ as number of nonzero Lyapunov constants. So, the theorem is proved in this case.

For n even, and using the same computations, we get that the polynomials l_i with i odd are zero. So the ideal J of Theorem 4.1 can be reduced to take just the polynomials l_{2i}. The proof then follows in a similar way. $\square$

Proof of Theorem 4.1. From our hypotheses there exist polynomials $p_{i,j}$ such that

$$
\begin{aligned}
l_1 &= \tilde{l}_1, \\
l_i &= \sum_{j<i} p_{i,j}\tilde{l}_j \text{ for any } i = 2, \dots, N, \\
l_i &= \sum_{j=1}^{N} p_{i,j}\tilde{l}_j \text{ for any } i > N.
\end{aligned}
\tag{14}
$$

As in the proof of Theorem 2.3, we substitute ω by $\omega = \omega_1 + \varepsilon\omega_2 + \varepsilon^2\omega_3 + \cdots$, in the expressions of $l_j(\rho)$ associated to (14). Then the above equalities (14) are given by

$$l_{1,0} + \varepsilon l_{1,1} + \varepsilon^2 l_{1,2} + \cdots = \tilde{l}_{1,0} + \varepsilon\tilde{l}_{1,1} + \varepsilon^2\tilde{l}_{1,2} + \cdots .$$

Hence $l_{1,i} = \tilde{l}_{1,i}$ for any $i = 0, 1, 2, \dots$, and

$$
\begin{aligned}
l_{i,0} + \varepsilon l_{i,1} + \varepsilon^2 l_{i,2} + \cdots = {} & (\tilde{l}_{i,0} + \varepsilon\tilde{l}_{i,1} + \cdots) \\
& + \sum_{j<i} (p^0_{i,j} + p^1_{i,j}\varepsilon + \cdots)(\tilde{l}_{j,0} + \varepsilon\tilde{l}_{j,1} + \cdots).
\end{aligned}
$$

After equating the coefficients with the same ε power we get the following equalities:

$$
\begin{aligned}
l_{i,0} &= \tilde{l}_{i,0} + \sum_{j<i} p^0_{i,j}\tilde{l}_{j,0}, \\
l_{i,1} &= \tilde{l}_{i,1} + \sum_{j<i} p^0_{i,j}\tilde{l}_{j,1} + p^1_{i,j}\tilde{l}_{j,0} = \tilde{l}_{i,1} + \sum_{\substack{0<k<i \\ 0\le m\le 1}} q^{i,1}_{k,m}\tilde{l}_{k,m}.
\end{aligned}
$$

In general, for each $l_{i,j}$ we can write:

$$l_{i,j} = \tilde{l}_{i,j} + \sum_{\substack{0<k<i \\ 0\le m\le j}} q^{i,j}_{k,m}\tilde{l}_{k,m},$$

where the $q^{i,j}_{k,m}$ are again polynomials in the coefficients of the system and ρ. Hence we have for $\tilde{l}_{i,j}$ similar expression to the expressions for $l_{i,j}$. Moreover, in the case $i \ge N$, the $\tilde{l}_{i,j}$ are zero.

Note that the polynomials $\tilde{l}_{i,j}$ are also homogeneous polynomials in ρ with the same degree in ρ, that $l_{i,j}$, l_i, and $\tilde{l}_i$.

From Theorem 2.3 we have that the coefficients of the expansion of the return map associated to (13) are

$$l^{(2)}_k(\rho) = \sum_{\substack{i+j=k \\ j<k}} l_{i,j}(\rho).$$

Hence by using the above expressions for the functions $l_{i,j}$, we get that

$$\begin{aligned}
l_1^{(2)} &= l_{1,0} = \tilde{l}_{1,0}, \\
l_2^{(2)} &= l_{2,0} + l_{1,1} = \tilde{l}_{2,0} + q_{1,0}^{2,0}\tilde{l}_{1,0} + \tilde{l}_{1,1}, \\
l_3^{(2)} &= l_{3,0} + l_{2,1} + l_{1,2} = (\tilde{l}_{3,0} + q_{2,0}^{3,0}\tilde{l}_{2,0} + q_{1,0}^{3,0}\tilde{l}_{1,0}) + (\tilde{l}_{2,1} + q_{1,0}^{2,1}\tilde{l}_{1,0} + q_{1,1}^{2,1}\tilde{l}_{1,1}) + \tilde{l}_{1,2}.
\end{aligned}$$

To determine the first nonzero coefficient of the return map, we need to simplify the above expressions under the assumption that the previous ones are all zero. If $l_1^{(2)} = \tilde{l}_{1,0} \equiv 0$, then $l_2^{(2)} = \tilde{l}_{2,0} + \tilde{l}_{1,1}$, and since both summands are homogeneous in ρ of different degree, $\tilde{l}_2^{(2)}$ is zero if and only if $\tilde{l}_{2,0}$ are $\tilde{l}_{1,1}$ also zero. In this situation $l_3^{(2)} = \tilde{l}_{3,0} + \tilde{l}_{2,1} + \tilde{l}_{1,2}$.

In general, we can prove that $l_k^{(2)} = \sum\limits_{\substack{0\le i+j=k \\ j<k}} \tilde{l}_{i,j}$ when $l_1^{(2)} \equiv l_2^{(2)} \equiv \cdots \equiv l_{k-1}^{(2)} \equiv 0$.

Observe that in each step we get a polynomial in ρ, $l_k^{(2)}$, where each of its monomials in ρ, $\tilde{l}_{i,j}$, is an homogeneous polynomial of degree k_i. Moreover, since $\tilde{l}_{i,j} = 0$, for $i > n$, we have that the polynomial $l_k^{(2)}$ for $k > n$ does not augment its degree. This fact implies that $\widetilde{\mathbf{M}}^k(\mathcal{F}) = k - 1$, if $k \le N$ and $\widetilde{\mathbf{M}}^k(\mathcal{F}) = N - 1$, if $k > N$. Therefore, $\mathbf{M}(\mathcal{F}) \le \widetilde{\mathbf{M}}(\mathcal{F}) = N - 1$ as we wanted to prove. $\square$

5. Other Families

This section is devoted to give the numbers $\mathbf{M}(\mathcal{F})$ and $\mathbf{B}(\mathcal{F})$ for two concrete families $\mathcal{F}$. The first one is a new application of Theorem 4.1. The second one does not satisfy the hypotheses of the theorem; we are just able to compute some values of $\widetilde{\mathbf{M}}^k(\mathcal{F})$ for small k.

Proposition 5.1. *Consider the family $\mathcal{F}$ defined by system*

$$\begin{aligned}
\dot{x} &= -y + a_2x^2 + a_3x^3, \\
\dot{y} &= \quad x + b_3x^3 + b_4x^4.
\end{aligned}$$

Then $\mathbf{M}(\mathcal{F}) = \mathbf{C}(\mathcal{F}) = 1$, *and* $\mathbf{B}(\mathcal{F}) = 2$.

Proof. Its first Lyapunov constants are $v_3 = \frac{3\pi}{4}a_3$ and $v_5 = -\frac{\pi}{2}a_2b_4$. Furthermore, there are two set of solutions of the system $v_3 = v_5 = 0$: $\{a_3 = b_4 = 0\}$ (reversible centers) and $\{a_2 = a_3 = 0\}$ (potential centers).

It is not difficult to see that the ideal $\langle a_3, a_2b_4\rangle$ is radical. From this fact we have that $I = \langle v_3, v_5\rangle$ and so $\mathbf{B}(\mathcal{F}) = 2$.

On the other hand, following the notation of Theorem 4.1 we have that $\tilde{l}_1 = 3a_3\pi\rho^2$ and $\tilde{l}_2 = -2a_2b_4\pi\rho^3$. Since if $\tilde{l}_1 \equiv \tilde{l}_2 \equiv 0$, then $a_3 = a_2b_4 = 0$, by using the above classification of the centers of $\mathcal{F}$, we have that $dH + \varepsilon\omega = 0$ has a center for each ε. Then $\tilde{l}_j \equiv 0$ for $j \ge 3$ if $\tilde{l}_1 = \tilde{l}_2 = 0$. Hence we can apply Theorem 4.1, the radicality of $\langle v_3, v_5\rangle$, and the fact that $\tilde{l}_1 = \frac{1}{4}v_3\rho^2$ and $\tilde{l}_2 = \frac{1}{4}v_5\rho^3$ to conclude that $J = \langle \tilde{l}_1, \tilde{l}_2\rangle$ and $\widetilde{\mathbf{M}}(\mathcal{F}) = \mathbf{B}(\mathcal{F}) - 1$. From the fact that a_3 and a_2b_4 can take arbitrary values it follows that $\widetilde{\mathbf{M}}(\mathcal{F}) = \mathbf{M}(\mathcal{F})$. Finally, it is easy to see that $\mathbf{C}(\mathcal{F}) = 1$. $\square$

Proposition 5.2. *Consider the family, $\mathcal{F}$, defined by*

$$\begin{aligned} \dot{x} &= -y + a_2x^2 + a_3x^3, \\ \dot{y} &= \quad x + b_2y^2 + b_3y^3. \end{aligned}$$

Then $\mathbf{B}(\mathcal{F}) = 4$, $\mathbf{C}(\mathcal{F}) = 2$, *the maximum order of the origin as a weak focus is* 3 *and* $\mathbf{M}^k(\mathcal{F}) = \begin{cases} 0 & \text{if } k = 1, 2, \\ 1 & \text{if } k = 3, 4, 5, \\ 2 & \text{if } k = 6, 7, \ldots, 10. \end{cases}$

Proof. Its first Lyapunov constants are

$$\begin{aligned} v_3 &= \frac{3\pi}{4}(b_3 + a_3), \\ v_5 &= -\frac{\pi}{12}(a_2^2 - b_2^2)(6b_2a_2 + 5b_3), \\ v_7 &= -\frac{5\pi}{8}b_3(a_2^4 - b_2^4), \\ v_9 &= -\frac{382\pi}{125}a_2^4b_3(a_2^2 - b_2^2). \end{aligned}$$

In [7] it is proved that if $v_3 = v_5 = v_7 = 0$, then the origin of $\mathcal{F}$ is a center. Since v_9 is not zero when $\{v_3 = v_5 = v_7 = 0\}$ in $\mathbb{C}[a_2, b_2, a_3, b_3]$, it follows that

$$\langle v_3, v_5, v_7 \rangle \neq \langle v_3, v_5, v_7, v_9 \rangle.$$

Furthermore, the fact that $\langle v_3, v_5, v_7, v_9 \rangle$ is a radical ideal (this is tested by using the algebraic package MAGMA), and the fact that its zero set coincides with the set of centers of $\mathcal{F}$, we have that, for $n \geq 10$, $v_n \in \mathrm{rad}(v_3, v_5, v_7, v_9) = \langle v_3, v_5, v_7, v_9 \rangle$ and hence $\mathbf{B}(\mathcal{F}) = 4$.

Note also that on any real solution of $v_3 = v_5 = v_7 = 0$, v_9 is also 0. Then the maximum order of the origin is 3.

The fact that v_3, v_5, v_7 can take arbitrary values and $v_7v_9 \geq 0$ implies that $\mathbf{C}(\mathcal{F}) = 2$.

On the other hand, to compute the Melnikov number we have to study the expression for the return map associated to $dH + \varepsilon\omega = 0$. We obtain that

$$\begin{aligned} \tilde{l}_1 &= 3\pi(b_3 + a_3)\rho^2, \\ \tilde{l}_2 &= 0, \\ \tilde{l}_3 &= \frac{5\pi}{3}b_3(a_2^2 - b_2^2)\rho^3, \\ \tilde{l}_4 &= -2\pi a_2b_2(a_2^2 - b_2^2)\rho^3. \end{aligned}$$

Since the degrees in ρ of $\tilde{l}_3$ and $\tilde{l}_4$ coincide, we can not apply Theorem 4.1. So we study directly the equation $dH + \varepsilon\omega + \varepsilon^2\omega + \cdots = 0$ and we get the values of $\widetilde{\mathbf{M}}^k(\mathcal{F})$ given in the statement. We do not give here the details of the computations due to their length. $\square$

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