CENTER-FOCUS PROBLEM FOR DISCONTINUOUS PLANAR DIFFERENTIAL EQUATIONS

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ABSTRACT. We study the center-focus problem as well as the number of limit cycles which bifurcate from a weak focus for several families of planar discontinuous ordinary differential equations. Our computations of the return map near the critical point are performed with a new method based on a suitable decomposition of certain one forms associated to the expression of the system in polar coordinates. This decomposition simplifies all the expressions involved in the procedure. Finally we apply our results to study a mathematical model of a mechanical problem, the movement of a ball between two elastic walls

1. INTRODUCTION

There are many problems in science, and particularly in mechanics and in engineries, where their mathematical modelization is given by a dynamical system whose right-hand side is not continuous or not differentiable, see for instance the classical book [AVK87] or the new one [Kun00] and the references therein.

In this paper we study the following class of discontinuous planar systems of ordinary differential equations

$$(\dot{x}, \dot{y}) = \begin{cases} (-y + P^+(x,y), x + Q^+(x,y)) & \text{if } y \geq 0, \\ (-y + P^-(x,y), x + Q^-(x,y)) & \text{if } y \leq 0, \end{cases} \tag{1}$$

where P^+, Q^+, P^-, Q^- are analytic functions starting at least with second order terms.

The above system has the origin as a monodromic critical point. We are interested in the following two problems:

- The center-focus problem, i.e. to determine if the origin of Sys. (1) is either a center, an attractor or a repeller.
- The cyclicity problem, that is, fix a class of systems of type (1) and determine the maximum number of limit cycles which bifurcate from the origin under the variation of the parameters inside this class of systems.

Our main contribution is the development of a new method to compute the Lyapunov constants -defined in next section- for systems of type (1). This method gives

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a tool to solve the above problems. It is based in a suitable decomposition of certain one forms associated to the expression of (1) in polar coordinates. Our decomposition is done in such a way that it simplifies the computations needed in all the procedure. Furthermore, it is easy to be implemented in a computer algebra system. The main difference, between the computation of the Lyapunov constants for systems of type (1), and for smooth differential equations is that, in the second case there appear some cancellations -due to a symmetry which relates the solutions in the upper plane with the solutions in the lower plane- which make the computations shorter and simpler. These cancellations are not present when we consider general discontinuous systems of type (1).

The precise statement of our method as well as the proofs involved to develop it are given in Sec. 2. Subsection 3.1 is devoted to study a family of quadratic systems of type (1). More specifically, we study the case where $(-y+P^+(x,y), x+Q^+(x,y))$ is an arbitrary quadratic vector field and $P^- \equiv P^+ \equiv 0$. For these systems we solve the center-focus problem, see Theorem 3.1, and we construct examples with 5 small amplitude limit cycles. We note that just examples with 3 limit cycles, see [Kun00, Chap. 7], and with 4 limit cycles, see [CGP01], were known. Note that this last result also shows that discontinuous planar differential equations have richer dynamics than smooth dynamical systems, because it is well known that when we consider smooth quadratic systems just 3 small amplitude limit cycles appear, see [Bau54], and our system, having the same number of free parameters, has 5 limit cycles. In Subsec. 3.2 we study the center problem for Sys. (1) of Liénard type, improving some results of [CPG99]. In the last subsection we study a mechanical problem, the movement of a ball between two elastic walls, modeled by a Kukles system. We consider just the quadratic case, and for it we study both the cyclicity and the center-focus problems.

2. Definitions and main results

This section is devoted to prove the main results of the paper. We start by giving some definitions.

The expression of (1) in polar coordinates, $(x,y)=(r\cos\theta, r\sin\theta)$, is given by:

$$\begin{cases} dH + \sum\limits_{i\geq 1}\omega_i^+ = 0 & \text{if } \theta\in[0,\pi], \\ dH + \sum\limits_{i\geq 1}\omega_i^- = 0 & \text{if } \theta\in[\pi,2\pi], \end{cases} \tag{2}$$

where $H(r)=r^2/2$, and $\omega_i^\pm=\omega_i^\pm(r,\theta)$ are analytic one forms, 2π–periodic in θ and polynomial in r.

Let $r^+(\rho,\theta)$ (resp. $r^-(\rho,\theta)$) be the solution of Sys. (2) such that $r^+(\rho,0)=\rho$ (resp. $r^-(\rho,\pi)=\rho$). Then, we can define the positive half-return map as

$$\Pi^+(\rho)=r^+(\rho,\pi)=\rho+\sum_{i\geq 2}p_i^+\rho^i,$$

and the negative half-return map as

$$\Pi^-(\rho) = r^-(\rho, 2\pi) = \rho + \sum_{i\geq 2} p_i^- \rho^i.$$

The complete return map associated to Sys. (1) -or equivalently to Sys. (2)- is given by the composition of these two maps, see also Fig. 1,

$$\Pi(\rho) = \Pi^-(\Pi^+(\rho)) := \rho + \sum_{i\geq 2} p_i \rho^i. \tag{3}$$

When we want to stress that a return or a half-return map $\Pi^\pm$ is associated to a vector field X, we write Π_X or $\Pi_X^\pm$. The first non zero p_k is called the *k–Lyapunov constant* of Sys. (1), and is denoted by V_k. The above definition coincides with the usual one when (1) is a smooth system. A main difference between the smooth case and general Sys. (1) is that while in the first case the first non zero p_k occurs always for k an odd number, see [ALGM73, p. 243], for the second case k can be any natural number bigger than 1.

Given a family of systems, we remark that the expressions of $V_2, V_3, \ldots, V_m$ in terms of the coefficients of the system have to be understood in the following way: the expression of a V_k has just meaning for systems with $V_2 = V_3 = \cdots = V_{k-1} = 0$.

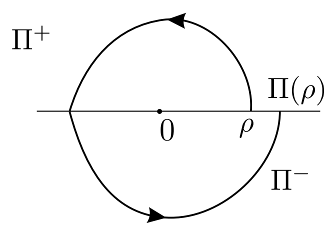

FIGURE 1. Return map of Sys. (1).

To obtain the Lyapunov constants following (3) we need a method to compute Π^+ and Π^-, and afterwards to compose them. Next lemma shows a way to simplify both problems. We use the following notation: X^+ (resp. X^-) denotes the vector field $(-y + P^+(x, y), x + Q^+(x, y))$ (resp. $(-y + P^-(x, y), x + Q^-(x, y))$) and X denotes the vector field associated to (1).

Lemma 2.1. *The first non zero term of the map $\Pi_X(\rho) - \rho$ defined in (3) coincides with the first non-zero term of the map*

$$\Pi_{X^+}^+(\rho) - (\Pi_{X^-}^-)^{-1}(\rho) = \Pi_{X^+}^+(\rho) - \Pi_{-X^-(x,-y)}^+(\rho),$$

see also Fig. 2.

Proof. By performing the change of variables $(x, y, t) \to (x, -y, -t)$ in the differential equation associated to X^-, it is easy to prove that $\Pi_{X^-(x,y)}^- = \Pi_{-X^-(x,-y)}^+$. At this point, the lemma follows just by proving that the leading terms of the next two expressions

$$g(f(\rho)) - \rho \quad \text{and} \quad f(\rho) - g^{-1}(\rho),$$

where f and g are analytic functions vanishing at zero and such that $f'(0) = g'(0) = 1$, coincide. This is done in [CPG99, Lemma 3.2]. $\square$

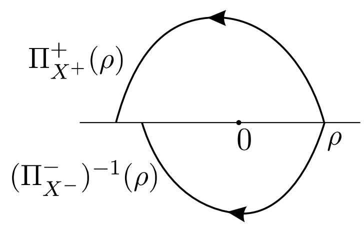

FIGURE 2. Half-return maps Π^+ i $(\Pi^-)^{-1}$ of Sys. (1).

The above lemma reduces the problem of the computation of the Lyapunov constants to the study of the half-positive return map, Π^+, of arbitrary smooth planar differential equation of the form

$$dH + \sum_{i\geq 1} \omega_i = 0.$$

Next lemma reduces this second problem to study the perturbation of the Hamiltonian $dH = 0$, where $H = H(r) = r^2/2$.

Lemma 2.2. *Let*

$$\Pi^+(\rho) = \rho + \sum_{i\geq 2} p_i^+ \rho^i,$$

be the positive half-return map associated to the polar expression of a smooth system of type (1),

$$dH + \sum_{i\geq 1} \omega_i = 0. \tag{4}$$

Let $r(\theta, \varepsilon, \rho) = \sum_{i\geq 0} r_i(\theta, \rho)\varepsilon^i$ *the solution of the initial value problem*

$$\begin{cases} dH + \sum\limits_{i\geq 1} \varepsilon^i \omega_i = 0, \\ r(0, \varepsilon, \rho) = \rho. \end{cases}$$

Then $\Pi^+(\varepsilon\rho) = \varepsilon r(\pi, \varepsilon, \rho)$, *and as a consequence for* $i \geq 2$, $p_i^+ = r_{i-1}(\rho, \theta)/\rho^i$.

Proof. The proof follows just by considering the effect of the scaling $r \to \varepsilon r$ in Sys. (4). $\square$

To state our main result we also need the following technical result, which decomposes an arbitrary one form. It is reminiscent of the decompositions used by Françoise in [Fra96, Fra97, Fra98]. Its proof is straightforward.

Lemma 2.3. *Let $\Omega = \alpha(r,\theta)dr + \beta(r,\theta)d\theta$, be an arbitrary analytic one form, $2\pi-$periodic in θ and $H(r) = r^2/2$. Then there exist functions $h(r,\theta)$, $S(r,\theta)$ and $F(r)$ also $2\pi-$periodic in θ and defined by $F(r) = \frac{1}{2\pi}\int_0^{2\pi} \beta(r,\psi)d\psi$, $S(r,\theta) = \int_0^{\theta} \beta(r,\psi)\, d\psi$ $-F(r)\theta$ and $h(r,\theta) = (\alpha(r,\theta) - \frac{\partial S(r,\theta)}{\partial r})/H'(r)$, and such that*

$$\Omega = \Omega^0 + \Omega^1 \ \text{ where } \ \Omega^0 = hdH + dS,\ \Omega^1 = F(r)d\theta,$$

and

$$\int_{H=\rho} \Omega^0 = 0, \quad \int_{H=\rho} \Omega^1 = \int_{H=\rho} \Omega.$$

Theorem 2.4. *Let $r(\theta,\varepsilon,\rho)$ be the solution of the initial value problem*

$$\begin{cases} dH + \sum\limits_{i\geq 1} \varepsilon^i \omega_i = 0, \\ r(0,\varepsilon,\rho) = \rho, \end{cases} \tag{5}$$

where $H(r) = r^2/2$ and $\omega_i = \omega_i(r,\theta)$ are one forms $2\pi-$periodic in θ. Then for any $n \in \mathbb{N}$, $r(\theta,\varepsilon,\rho)$ satisfies the following implicit equation

$$\frac{r^2(\theta,\varepsilon,\rho) - \rho^2}{2} + O(\varepsilon^{n+1}) =$$
$$\sum_{i=1}^{n} \varepsilon^i \left[\int_0^{\theta} F_i(r(\psi,\varepsilon,\rho))d\psi + S_i(r(\psi,\varepsilon,\rho),\psi)|_{\psi=0}^{\psi=\theta} \right],$$

where the one forms Ω_i and the functions $F_i(r)$, $h_i(r,\theta)$ and $S_i(r,\theta)$ are defined inductively in the following way: $h_0 = 1$,

$$-\Omega_1 := -\omega_1 h_0 = h_1 dH + dS_1 + F_1 d\theta,$$

and

$$-\Omega_i := -\sum_{j=1}^{i} \omega_i h_{i-j} = h_i dH + dS_i + F_i d\theta,$$

for $i = 1, 2, \ldots, n$ and we have used the decomposition given in Lemma 2.3 for the forms $-\Omega_i$.

Proof. Denote by $\gamma_\varepsilon = \gamma_\varepsilon(\theta,\rho)$ the curve $\{r(\psi,\varepsilon,\rho),\ \psi \in [0,\theta]\}$ solution of (5). Consider the one forms $-\Omega_i$, $i = 1,\ldots,n$ and their decompositions given in Lemma 2.3. Therefore

$$\begin{aligned} 0 &= \int_{\gamma_\varepsilon} \left(1 + \sum_{i=1}^{n} \varepsilon^i h_i\right)\left(dH + \sum_{i\geq 1} \varepsilon^i \omega_i\right) \\ &= \int_{\gamma_\varepsilon} dH + \sum_{i=1}^{n} (\Omega_i + h_i dH)\,\varepsilon^i + O(\varepsilon^{n+1}) \\ &= \int_{\gamma_\varepsilon} dH - \sum_{i=1}^{n} (F_i(r)d\theta + dS_i(r,\theta))\,\varepsilon^i + O(\varepsilon^{n+1}). \end{aligned}$$

By making the integration, and taking into account that $H(r) = r^2/2$ the theorem follows. $\square$

Corollary 2.5. *Let $r(\theta, \varepsilon, \rho) = \sum_{i\geq 0} r_i(\theta, \rho)\varepsilon^i$ the solution of the initial value problem (5). Assume that the functions $r_0(\theta, \rho) = \rho$, $r_1(\theta, \rho)$, $r_2(\theta, \rho)$, $\ldots$, $r_{n-1}(\theta, \rho)$ are known. Then $r_n(\theta, \rho)$ can be obtained by equating the ε^n–terms of the implicit expression of $r(\theta, \varepsilon, \rho)$ given in Theorem 2.4. In fact the equation looks like*

$$\rho r_n(\theta, \rho) = \mathcal{F}_n(\rho, r_1(\theta, \rho), \ldots, r_{n-1}(\theta, \rho)),$$

where $\mathcal{F}_n$ depends on the one forms $\omega_1, \omega_2, \ldots, \omega_n$, through the corresponding F_i and S_i, $i = 1, 2, \ldots, n$.

In particular $\mathcal{F}_1 = F_1(\rho)\theta + S_1(\rho, \theta) - S_1(\rho, 0)$ and

$$\mathcal{F}_2 \;=\; [F_2(\psi) + S_2(\rho, \psi) + S_1'(\rho) r(\psi, \rho)]|_{\psi=0}^{\psi=\theta} - \frac{1}{2} r_1^2(\theta, \rho) + F_1'(\rho) \int_0^\theta r_1(\psi, \rho) d\psi.$$

Poggiale [Pog94], Françoise [Fra96], Iliev [Ili98], Roussarie [Rou98], and Iliev & Perko [IP99] give similar result to the above theorem, see also [GT99]. This result allows to compute $r_n(2\pi, \rho)$ under the assumption that $r_2(2\pi, \rho) = r_3(2\pi, \rho) = \cdots = r_{n-1}(2\pi, \rho) = 0$. Note that Theorem 2.4 is an improvement of that result. It allows to calculate $r_n(\theta, \rho)$ for any θ and without any further assumption.

As a survey of this section we explain the steps needed to find the Lyapunov constants of Sys. (1).

Method of computation of the Lyapunov constants:

1. Write Sys. (1) in the polar form (2).
2. Consider the expression of (2) in the upper half-plane and denote by X^+ its associated vector field.
3. Associate to the vector field the initial value problem (5) and calculate $r(\theta, \varepsilon, \rho)$ by using Corollary 2.5.
4. Use Lemma 2.2 to obtain $\Pi_X^+(\rho)$ from $r(\pi, \varepsilon, \rho)$.
5. Consider the expression (2) in the lower half-plane and denote by X^- its associated vector field. Take $-X^-(x, -y)$ and reproduce for it the steps 3 and 4. We get $\Pi^+_{-X^-(x,-y)}(\rho)$.
6. Make the computation $\Pi^+_{X^+}(\rho) - \Pi^+_{-X^-(x,-y)}(\rho)$. From Lemma 2.1, its series expansion gives the Lyapunov constants for Sys. (1).

3. Applications

We apply the above method for solving the center-focus problem and for obtaining small amplitude limit cycles for the following cases of discontinuous differential systems: quadratic, Liénard and a class of Kukles systems.

Before starting with the concrete systems there is a general result, proved in [PS73], that we want to comment, see also [CGP00] for a different proof. The result asserts that if a system of the form (1) has a center at the origin, and the smooth system

$$(\dot{x}, \dot{y}) = \big(-y + P^-(x,y), x + Q^-(x,y)\big),$$

considered in the whole plane, has also a center at the origin, then the system

$$(\dot{x}, \dot{y}) = \big(-y + P^+(x,y), x + Q^+(x,y)\big),$$

also has to have a center in the whole plane. Although we will not use this result it helps to understand why all the systems, with a center at the origin, studied in next section have a computable first integral.

3.1. **Quadratic Systems.** In this section, we classify the centers of a family of discontinuous quadratic systems, considered in [Kun00, Chap. 7] and also in [CGP01]. For this family we also find an example with five limit cycles, obtaining a limit cycle more than in previous approaches.

Theorem 3.1. *Consider the system*

$$\left\{\begin{array}{l}\left\{\begin{array}{rcl}\dot{x} &=& -y+p_{20}x^2+p_{11}xy+p_{02}y^2, \\ \dot{y} &=& x+q_{20}x^2+q_{11}xy+q_{02}y^2, \quad y \geq 0,\end{array}\right. \\ \left\{\begin{array}{rcl}\dot{x} &=& -y, \\ \dot{y} &=& x, \qquad\qquad\qquad\qquad\quad y \leq 0.\end{array}\right.\end{array}\right.$$

Then, it has a center at the origin if and only if one of the following conditions holds:

(i) $p_{11} = q_{20} = q_{02} = 0$,
(ii) $p_{20} = p_{11} + q_{20} = p_{02} + q_{11} = q_{02} = 0$,
(iii) $2p_{20} + q_{11} = p_{11} + 2q_{02} = q_{20} = 0$,
(iv) $p_{20} = -p_{11} + q_{20} = q_{02} + q_{20} = p_{02} = 0$,
(v) $2p_{11}q_{20}+3p_{20}^2-2q_{20}^2=2q_{11}+5p_{20}=8p_{02}q_{20}^2-3p_{20}^2+8q_{20}^2=4q_{02}q_{20} - 3p_{20}^2 + 4q_{20}^2 = 0.$

Proof. Firstly, let us prove that in each one of the cases the systems has a center at the origin. Since in the lower half-plane the return map is the identity map it suffices to prove that $\Pi^+(\rho) = \rho$.

For the first case, this is a consequence of the invariance of the equation by the change $(x, y, t) \to (-x, y, -t)$.

To study the other cases we compute the first integrals $H_i = H_i(x, y)$ associated to case (i), $i = 2, 3, 4, 5$. We have obtained them by direct exploration, but we can also get them from [LS82]. They are

$$\begin{aligned}
H_2 =& x^2 + y^2, \\
H_3 =& \frac{1}{2}(x^2 + y^2) + \frac{q_{11}}{2}x^2y + q_{02}xy^2 - \frac{p_{02}}{3}y^3, \\
H_4 =& (q_{20}x - 1)\left(q_{20}x + \frac{1}{2}(q_{11} - \gamma)y + 1\right)^{\alpha}\left(q_{20}x + \frac{1}{2}(q_{11} + \gamma)y + 1\right)^{(1-\alpha)}, \\
H_5 =& (-2q_{20}x + p_{20}y + 2)^2\left(4(q_{20}x + 1)^2 + (-4p_{20} - 12p_{20}q_{20}x)y + (3p_{20}^2 - 8q_{20}^2)y^2\right),
\end{aligned}$$

with $\alpha = \frac{4q_{20}^2}{\gamma(\gamma+q_{11})}$ and $\gamma = \sqrt{q_{11}^2 + 8q_{20}^2}$.

The fact that in all the cases $H_i(x,0) = H_i(-x,0)$ proves the result.

To show that there are no more centers inside this family we compute several Lyapunov constants by using the method developed in Sec. 2. We get

$$\begin{aligned}
V_2 =& \frac{2}{3}(p_{11} + q_{20} + 2q_{02}),\\
V_3 =& -\frac{\pi}{8}(2p_{20}q_{02} + q_{02}q_{11} + 3p_{20}q_{20} + q_{11}q_{20} + p_{02}q_{20}),\\
V_4 =& \frac{1}{15}(2q_{20}^3 - 2p_{11}^2 q20 - 18p_{20}^2 q_{20} + 6p_{11}p_{20}q_{11} + 12p_{11}p_{20}^2 - 6q_{11}p_{20}q_{20}),\\
V_5 =& \frac{\pi}{64}q_{20}p_{20}(p_{20}^2 - 2q_{11}p_{20} + 4p_{11}q_{20} - 4q_{20}^2),\\
V_6 =& \frac{8}{105}q_{20}(p_{11} - q_{20})(p_{11} + q_{20})(-5p_{11}q_{20} + 5q_{20}^2 + 3q_{11}p_{20}).
\end{aligned}$$

Solving the non linear system $\{V_2 = V_3 = V_4 = V_5 = V_6 = 0\}$ we just obtain the families of the statement and therefore the theorem follows. □

By using the expressions of the Lyapunov constants obtained in the above theorem, we can get a discontinuous quadratic system with five small amplitude limit cycles.

Theorem 3.2. *Consider the system*

$$(\dot{x}, \dot{y}) = \begin{cases} (-y+w_1x+x^2+p_{11}xy+p_{02}y^2, x+w_1y+x^2+q_{11}xy+q_{02}y^2) & \text{if } y \geq 0,\\ (-y, x) & \text{if } y \leq 0, \end{cases} \tag{6}$$

where $p_{11} = \frac{7}{5} + \alpha$,

$$\begin{aligned}
p_{02} =& -\frac{17}{50} + \frac{3}{20}\alpha - \frac{99}{40}w_2 + \frac{32}{25}w_5 + \frac{16}{5}\alpha w_5 + \frac{3}{2}w_4 - \frac{3}{2}\alpha w_2 + 24w_2w_5 - 8w_3,\\
q_{11} =& \frac{13}{10} + 2\alpha - 32w_3, \quad \text{and}\\
q_{02} =& -\frac{6}{5} - \frac{1}{2}\alpha + \frac{3}{4}w_2,
\end{aligned}$$

being $\alpha = \alpha(w_4, w_5)$ *the solution of the quadratic equation* $50\alpha^2 + (-960w_5 + 95)\alpha - 75w_4 - 384w_5 = 0$, *such that* $\alpha(0,0) = 0$. *Then, if we choose* $w_1, w_2, w_3, w_4,$ *and* w_5 *such that* $w_1 < 0$, $w_2 > 0$, $w_3 < 0$, $w_4 > 0$, $w_5 < 0$ *and* $|w_1| \ll |w_2| \ll |w_3| \ll |w_4| \ll |w_5| \ll 1$, *Sys. (6) has five small amplitude limit cycles.*

Proof. If $w_1 = 0$, we can use the formulas of the Lyapunov constants obtained in the previous theorem. We get that $V_i = w_i$ for $i = 2, 3, 4, 5$. Therefore, the return map $\Pi(\rho)$ in a neighbourhood of the origin writes as

$$\begin{aligned}
\Pi(\rho, w_1, w_2, w_3, w_4, w_5) =& e^{w_1\pi}\rho + (w_2 + f_2(w_1, w_2, w_3, w_4, w_5))\rho^2\\
&+ (w_3 + f_3(w_1, w_2, w_3, w_4, w_5))\rho^3 + (w_4 + f_4(w_1, w_2, w_3, w_4, w_5))\rho^4\\
&+ (w_5 + f_5(w_1, w_2, w_3, w_4, w_5))\rho^5 + (\frac{608}{4375} + f_6(w_1, w_2, w_3, w_4, w_5))\rho^6 + O(\rho^7),
\end{aligned}$$

where f_i, $i = 2, \ldots, 6$, are continuous functions satisfying $f_2(0, w_2, w_3, w_4, w_5) \equiv 0$, $f_3(0, 0, w_3, w_4, w_5) \equiv 0$, $f_4(0, 0, 0, w_4, w_5) \equiv 0$, $f_5(0, 0, 0, 0, w_5) \equiv 0$ and $f_6(0, 0, 0, 0, 0) \equiv 0$. By choosing the parameters as in the statement of the theorem, it is not difficult to see that, in a neighbourhood of the origin, the function $\Pi(\rho) - \rho$ changes sign six times, and therefore Π has at least five fix points. This fact ends the proof. □

We think that no more than five limit cycles can bifurcate from the origin for the systems studied in this section. The main reason is that the highest order of degeneracy of a weak focus, inside this family, is six.

3.2. **Liénard Equations.** In this section, we study the center-focus problem for Liénard discontinuous systems of the form

$$(\dot{x}, \dot{y}) = \begin{cases} \left(-y + \sum\limits_{i=2}^{n} a_i x^i, x\right) & \text{if } y \geq 0, \\ \left(-y + \sum\limits_{i=2}^{n} b_i x^i, x\right) & \text{if } y \leq 0. \end{cases} \tag{7}$$

These systems have been already studied in [CPG99]. In that paper, it is proved that the following families of systems of this type have a center at the origin:

(i) $a_{2k+1} = b_{2k+1} = 0$, or
(ii) $a_k + b_k = 0$.

for all $k \in \mathbb{N}$. Also the authors try to prove that the above families are the only centers inside (7). In particular, they show that if for the following particular systems

$$(\dot{x}, \dot{y}) = \begin{cases} \left(-y + x^{2j+1} + x^{2(k-j)}, x\right) & \text{if } y \geq 0, \\ \left(-y - x^{2j+1}, x\right) & \text{if } y \leq 0, \end{cases}$$

for $1 \leq j < k$, the Lyapunov constant $V_{2k} = C_{k,j}$ is not zero, then it is true that the above families are the only centers for Sys. (7). With this aim they compute some $C_{k,j}$ for $k \leq 6$, obtaining in all cases a number different from zero. Our methods allows to compute these values for larger values of k. In particular in Tab. 1, the values of $C_{k,j}$ for $1 \leq j < k \leq 10$ are given. It is worth to say that the sign of the value $C_{k,j}$ gives the stability of the associated system considered.

By using the results of Tab. 1, and some more computations, we have also checked that

$$\begin{aligned} C_{k,k-1} &= \frac{-10 + 12k}{3 + 6k}, \quad \text{for } 2 \leq k \leq 10, \\ C_{k,k-2} &= \frac{-262 - 80k + 120k^2}{45 + 120k + 60k^2}, \quad \text{for } 3 \leq k \leq 11, \\ C_{k,k-3} &= \frac{-4022 - 3948k + 280k^2 + 560k^3}{525 + 1610k + 1260k^2 + 280k^3}, \quad \text{for } 4 \leq k \leq 12, \\ C_{k,k-4} &= \frac{-101278 - 142784k - 40656k^2 + 8960k^3 + 3360k^4}{11025 + 36960k + 36120k^2 + 13440k^3 + 1680k^4}, \quad \text{for } 5 \leq k \leq 15. \end{aligned}$$

For the moment we have not find a proof of the above equalities for arbitrary k.

$k\backslash j$	1	2	3	4	5	6	7	8	9
2	$\frac{14}{15}$								
3	$\frac{58}{525}$	$\frac{26}{21}$							
4	$-\frac{702}{1225}$	$\frac{578}{945}$	$\frac{38}{27}$						
5	$-\frac{84806}{72765}$	$\frac{1774}{24255}$	$\frac{446}{495}$	$\frac{50}{33}$					
6	$-\frac{2516806}{1486485}$	$-\frac{381454}{945945}$	$\frac{20506}{45045}$	$\frac{2338}{2145}$	$\frac{62}{39}$				
7	$-\frac{10882038}{5010005}$	$-\frac{5201926}{6243237}$	$\frac{22138}{405405}$	$\frac{17746}{25025}$	$\frac{3578}{2925}$	$\frac{74}{45}$			
8	$-\frac{390159442}{149324175}$	$-\frac{80616454}{65702637}$	$-\frac{1571630}{5054049}$	$\frac{1388402}{3828825}$	$\frac{20666}{23205}$	$\frac{562}{425}$	$\frac{86}{51}$		
9	$-\frac{123363871018}{40811445675}$	$-\frac{3311635214}{2080583505}$	$-\frac{809489458}{1248350103}$	$\frac{6970394}{160044885}$	$\frac{3868322}{6613425}$	$\frac{174142}{169575}$	$\frac{6778}{4845}$	$\frac{98}{57}$	
10	$-\frac{36344996758}{10667118605}$	$-\frac{788914022}{4134146445}$	$-\frac{-2575685746}{2675035935}$	$-\frac{58491854}{231175945}$	$\frac{13153354}{43648605}$	$\frac{1609546}{2136645}$	$\frac{89678}{79135}$	$\frac{8738}{5985}$	$\frac{110}{63}$

TABLE 1. The values of $C_{k,j}$ for $1 \le j < k \le 10$.

To end this subsection we want to comment that our method also allows to solve the center-focus problem for concrete values of n, without using the results of [CPG99]. For instance we have the following result:

Proposition 3.3. *For $n = 7$ the only centers of Sys. (7) are the ones satisfying*

(i) either $a_3 = b_3 = a_5 = b_5 = a_7 = b_7 = 0$, or
(ii) $a_i + b_i = 0$, $i = 2, \ldots, 7$.

Proof. We already know that both families have a center at the origin. We have to prove that there are no more centers. By using our method and some algebraic manipulations we get the following Lyapunov constants:

$$
\begin{aligned}
V_2 &= 0,\\
V_3 &= \frac{3}{8}\pi(a_3 + b_3),\\
V_4 &= -\frac{14}{15}b_3(b_2 + a_2),\\
V_5 &= \frac{5}{16}\pi(a_5 + b_5),\\
V_6 &= -\frac{58}{525}b_3(a_4 + b_4) - \frac{26}{21}b_5(a_2 + b_2),\\
V_7 &= \frac{6825}{24960}\pi(a_7 + b_7),\\
V_8 &= \frac{702}{1225}b_3(a_6 + b_6) - \frac{578}{945}b_5(a_4 + b_4) - \frac{38}{27}b_2(a_7 + b_7),\\
V_9 &= 0,\\
V_{10} &= -\frac{1774}{24255}b_5(a_6 + b_6) - \frac{446}{495}b_7(a_4 + b_4),
\end{aligned}
$$

$$
\begin{aligned}
V_{11} &= 0, \\
V_{12} &= -\frac{20506}{45045} b_7(a_6+b_6).
\end{aligned}
$$

By solving the system $\{V_2 = V_3 = \cdots = V_{11} = V_{12} = 0\}$ the proposition follows. Of course the rational numbers involved in the even Lyapunov constants are the ones given in Tab. 1. □

3.3. **Kukles systems: a mechanical example.** The planar systems associated to the second order differential equation $\ddot{y} = f(y, \dot{y})$ are usually called Kukles systems. The center problem for these systems, when the function f is a polynomial, has been intensively studied. In this subsection we consider the easiest case of discontinuous Kukles systems: the quadratic case. More specifically we study the following two systems:

$$
(\dot{x}, \dot{y}) = \begin{cases} (-y+p_{20}x^2+p_{11}xy+p_{02}y^2, x) & \text{if } y \geq 0, \\ (-y+q_{20}x^2+q_{11}xy+q_{02}y^2, x) & \text{if } y \leq 0, \end{cases} \tag{8}
$$

and

$$
(\dot{x}, \dot{y}) = \begin{cases} (-y, x+q_{20}x^2+q_{11}xy+q_{02}y^2) & \text{if } y \geq 0, \\ (-y, x+p_{20}x^2+p_{11}xy+p_{02}y^2) & \text{if } y \leq 0. \end{cases} \tag{9}
$$

Note that the second case correspond to a Kukles system by making the change of variables $(x, y) \to (y, x)$.

As we will see, Sys. (8) is a mathematical model of the movement of a ball between two elastic walls. Therefore their periodic orbits will correspond to periodic movement of the ball. Before describing the modelization, we solve the center-focus problem for both systems.

Theorem 3.4. *Consider the Sys. (8). Then it has a center at the origin if and only if one of these conditions holds:*

(i) $p_{11} = q_{11} = 0$*,*
(ii) $p_{11} - q_{11} = p_{20} - q_{20} = p_{02} + q_{20} = q_{02} + q_{20} = 0$*,*
(iii) $p_{11} - q_{11} = p_{02} + q_{02} = p_{20} + q_{20} = 0$*.*

Proof. The proof is similar to the one of Theorem 3.1. Firstly, we prove that the families have a center at the origin. For the case (i), this is due to the fact that it is a reversible system, invariant with respect to the change of variables $(x, y, t) \to (-x, y, -t)$.

The case (ii) corresponds to a smooth center with first integral

$$
H(x, y) = (2p_{20}x + (p_{11} - \gamma)y + 2)^{\alpha}(2p_{20}x + (p_{11} + \gamma)y + 2)^{\beta} e^{-2p_{20}x},
$$

where $\gamma = \sqrt{p_{11}^2 + 4p_{20}^2}$ and $\alpha + \beta = 2$.

For the case (iii), the system is invariant with respect the change of variables $(x, y, t) \to (x, -y, -t)$. This property also forces the origin to be a center.

To prove that there no other centers, as usual, we compute several Lyapunov constants:

$$\begin{aligned} V_2 &= \frac{2}{3}(p_{11} - q_{11}), \\ V_3 &= \frac{1}{8}\pi q_{11}(q_{20} + q_{02} + p_{02} + p_{20}), \\ V_4 &= \frac{2}{45}q_{11}(q_{02} + p_{02})(3p_{02} + 2q_{20} - q_{02}), \\ V_5 &= \frac{1}{864}\pi q_{11}(q_{20} + q_{02})^2(q_{02} + p_{02}). \end{aligned}$$

Solving the system $\{V_2 = V_3 = V_4 = V_5 = 0\}$ we just obtain the families given in the statement. Therefore the theorem follows. □

In next proposition we get examples of Sys. (8) with four limit cycles. We remark that from the expressions of the Lyapunov constants obtained in the previous theorem it can be deduced that inside this family the highest order of degeneracy of the weak focus is five, and so it seems that no examples with five limit cycles can be obtained.

Proposition 3.5. *Consider the system*

$$(\dot{x}, \dot{y}) = \begin{cases} (-y + w_1 x + p_{20}x^2 + xy, x) & \text{if } y \geq 0, \\ (-y + q_{20}x^2 + q_{11}xy + 2y^2, x) & \text{if } y \leq 0, \end{cases} \tag{10}$$

where $p_{20} = (-3+8w_3-\frac{45}{8}w_4)$, $q_{20} = 1+\frac{45}{8}w_4$ *and* $q_{11} = 1+\frac{3}{2}w_2$, *where* w_1, w_2, w_3, w_4 *are chosen such that* $w_1 > 0$, $w_2 < 0$, $w_3 > 0$, $w_4 < 0$ *and* $|w_1| \ll |w_2| \ll |w_3| \ll |w_4| \ll 1$. *Then (10) has four small amplitude limit cycles.*

The proof of the above result is like the proof of Theorem 3.2. The main difference, due to the different shape of the linear perturbation, is that

$$\Pi(\rho, w_1, w_2, w_3, w_4) = \rho\, e^{\pi\frac{w_1}{\sqrt{4-w_1^2}}} + O(\rho^2).$$

Finally, we classify all the centers for Sys. (9). This result also has been proved in [Lun68, Thm. 4]. The proof that we present is shorter. The study for this system is more difficult than the study of Sys. (8) because it can be proved that inside this family there are weak focus of order eight. Our result is:

Theorem 3.6. *Consider Sys. (9). Then it has a center at the origin if and only if one of these conditions holds:*

(i) $q_{20} - p_{20} = p_{11} + q_{11} = q_{02} - p_{02} = 0$,
(ii) $q_{20} - p_{20} = q_{02} + q_{20} = p_{02} + q_{20} = 0$,
(iii) $q_{20} + 2p_{02} = q_{02} - 2p_{02} = p_{11} = p_{20} = 0$,
(iv) $p_{20} + 2q_{02} = p_{02} - 2q_{02} = q_{11} = q_{20} = 0$.

Proof. First we proof the sufficient condition, that is, that all the families described in the statement have a center at the origin.

The first family has a center at the origin because their solutions are invariant with respect the change of variables $(x, y, t) \to (x, -y, -t)$.

For the rest of families, the origin is also a center when we extend the system of the upper (or the lower) half-plane to the whole plane. These centers are of the following two types:

$$(\dot{x}, \dot{y}) = \left(-y, x + ay^2\right),$$

or

$$(\dot{x}, \dot{y}) = \left(-y, x + bx^2 + cxy - by^2\right).$$

The first system has the first integral

$$H_1(x, y) = \left(2ax - 1 + 2a^2y^2\right) e^{2ax}.$$

The second one has the first integral

$$H_2(x, y) = (2bx + (c - \gamma)y + 2)^{\alpha} (2bx + (c + \gamma)y + 2)^{\beta} e^{-2bx},$$

where $\gamma = \sqrt{c^2 + 4b^2}$, and $\alpha + \beta = 2$.

Denote by H_i^+ (resp. H_i^-) the first integral of the system restricted to the upper (resp. lower) half-plane. First, we consider the case (ii). Since H_2^+ (resp. H_2^-) is a first integral of the system in the positive (resp. negative) half-plane, then for any x positive, the orbit trough $(x, 0)$ has to satisfy the following equalities

$$H_2^+(x, 0) = H_2^+\left(-\Pi^+_{X^+}(x), 0\right), \text{ and } H_2^-(x, 0) = H_2^-\left(-(\Pi^-_{X^-})^{-1}(x), 0\right).$$

On the other hand

$$H_2^+(x, 0) = H_2^-(x, 0) = 4(1 + q_{20}x)^2 e^{-2q_{20}x} := h(x).$$

Since the equation $h(x) = h(z)$, for z near zero, has a unique negative solution it follows that $\Pi^+_{X^+}(x) = (\Pi^-_{X^-})^{-1}(x)$, and therefore, the origin is a center.

The proof of case (iii) follows in a similar way, using the fact that $H_2^+(x, 0) = 4(H_1^-(x, 0))^2$.

The last case coincides with the case (iii) after interchanging the system in the upper half plane with the system in the lower half plane.

To prove the necessary condition we compute the Lyapunov constants until V_8. By solving, with the help of a computer algebra system, the associated equations we get the desired result. We omit the details. We just comment that the Lyapunov constants are large polynomial expressions, involving rational numbers with large numerators and denominators.

□

To end this subsection we study the differential equations associated to the mechanical system showed in Fig. 3. It can be modelled by the system in 3 zones:

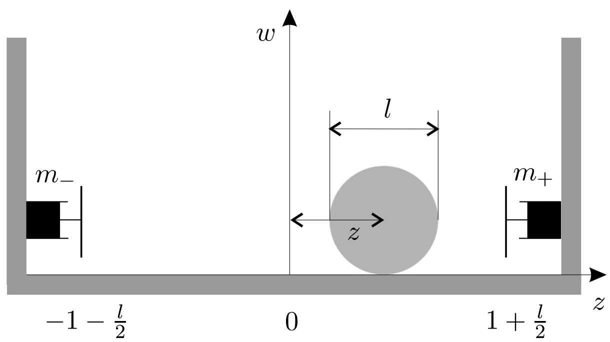

FIGURE 3. Ball moving between two elastic walls.

$$\begin{cases} m^- : \ddot{z}^- = f^-(z^-, \dot{z}^-) & \text{if } z \leq -1, z^- = z+1, \\ m^0 : \ddot{z} = -k\dot{z} & \text{if } -1 \leq z \leq 1, \\ m^+ : \ddot{z}^+ = f^+(z^+, \dot{z}^+) & \text{if } z \geq 1, z^+ = z-1, \end{cases} \tag{11}$$

We assume that the distance between the two walls is normalized to $2+l$, where $l \ll 1$ is the diameter of the ball, and that it moves between the two walls with friction ($k > 0$). The functions f^- and f^+ describe the reaction of the ball when it interacts with the walls. In our model we assume that the function f^+ (resp. f^-) in mode m^+(resp. mode m^-) is such that the point $(1, -\frac{k}{2})$ (resp. $(-1, \frac{k}{2})$) is a critical point of center-focus type. The phase portrait for these type of systems, in the case of having a continuum of periodic orbits, is plotted in Fig. 4. Note that to study the periodic solutions of (11), we can first remove the middle zone, and afterwards, translate the system in mode m^+ from $(1, -\frac{k}{2})$ to $(0,0)$, and the system in mode m^- from $(-1, \frac{k}{2})$ to $(0,0)$. Finally, by making a rotation of $\frac{\pi}{2}$ radians we get a system of Kukles type,

$$(\dot{x}, \dot{y}) = \begin{cases} (g^+(x,y), x) & \text{if } y \geq 0, \\ (g^-(x,y), x) & \text{if } y \leq 0. \end{cases} \tag{12}$$

Notice also that Sys. (8) corresponds to the above system with $g^{\pm}$ quadratic polynomials. Each closed trajectory of Sys. (11) has it corresponding closed trajectory of Sys. (12), see Figs. 4-5.

Observe that the closed trajectories of Sys. (12) inside the dashed curve do not correspond to periodic movements of the ball. On the other hand periodic orbits outside this dashed curve do correspond to periodic movement of the ball. We think that this problem gives a nice application of the method of the Lyapunov constants, because from the local study of the critical point -which does not correspond to real movement of the ball- we get, by analyticity of the return map, a center which gives closed orbits of Sys. (12), which actually correspond to real periodic movement of the ball.

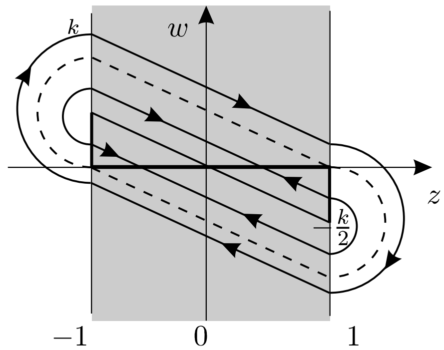

FIGURE 4. Periodic orbits for (11). The thick lines $\{(-1,w) : 0 < w \le \frac{k}{2}\} \cup \{(z,0) : |z| \le 1\} \cup \{(1,w) : -\frac{k}{2} \le w < 0\}$ are full of critical points.

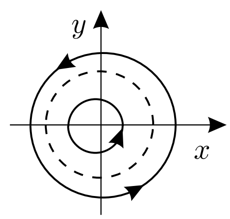

FIGURE 5. Periodic orbits for (12).

From the above point of view, the results of Theorem 3.4 can be reinterpreted as conditions on the functions $f^{\pm}$, which describe the reaction of the ball when it impacts with the walls, to get periodic movement of the ball.

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