THE NUMBER OF POLYNOMIAL SOLUTIONS OF POLYNOMIAL RICCATI EQUATIONS

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Abstract. Consider real or complex polynomial Riccati differential equations $a(x)\dot{y} = b_0(x) + b_1(x)y + b_2(x)y^2$ with all the involved functions being polynomials of degree at most η. We prove that the maximum number of polynomial solutions is $\eta + 1$ (resp. 2) when $\eta \geq 1$ (resp. $\eta = 0$) and that these bounds are sharp.

For real trigonometric polynomial Riccati differential equations with all the functions being trigonometric polynomials of degree at most $\eta \geq 1$ we prove a similar result. In this case, the maximum number of trigonometric polynomial solutions is 2η (resp. 3) when $\eta \geq 2$ (resp. $\eta = 1$) and, again, these bounds are sharp.

Although the proof of both results has the same starting point, the classical result that asserts that the cross ratio of four different solutions of a Riccati differential equation is constant, the trigonometric case is much more involved. The main reason is that the ring of trigonometric polynomials is not a unique factorization domain.

1. Introduction and statement of the main results

Riccati differential equations

$$a(x)\dot{y} = b_0(x) + b_1(x)y + b_2(x)y^2, \tag{1}$$

where the dot denotes the derivative with respect to the independent variable x, appear in all text books of ordinary differential equations as first examples of nonlinear equations. It is renowned that if one explicit solution is known then they can be totally solved, by transforming them into linear differential equations, see for instance [17]. One of their more remarkable properties is that given any four solutions defined on an open set $\mathcal{I} \subset \mathbb{R}$, where $y_1(x) < y_2(x) < y_3(x) < y_4(x)$, there exists a constant c such that

$$\frac{(y_4(x) - y_1(x))(y_3(x) - y_2(x))}{(y_3(x) - y_1(x))(y_4(x) - y_2(x))} = c \quad \text{for all} \quad x \in \mathcal{I}. \tag{2}$$

Geometrically this result says that the cross ratio of the four functions is constant. Analytically, it implies that once we know three solutions any other solution can be given in a closed form from these three. We will often use an equivalent version of this fact, see Lemmas 6 and 15.

Historically, Riccati equation has played a very important role in the pioneer work of D. Bernoulli about the smallpox vaccination, see [1, 9] and also appears in many mathematical and applied problems, see [13, 15, 16]. The main motivation of this paper came to us reading the works of Campbell and Golomb ([6, 7]), where the authors

2010 *Mathematics Subject Classification.* Primary 34A05. Secondary 34C05; 37C10.

Key words and phrases. Riccati differential equations, polynomial differential equations, trigonometric polynomial differential equations, number of polynomial solutions, explicit solutions.

present examples of polynomial Riccati differential equations with 4 and 5 polynomial solutions. The respective degrees of the polynomials a, b_0, b_1 and b_2 in these examples are $3, 3, 2, 0$ and $4, 4, 3, 0$, respectively. At that point we wonder about the maximum number of polynomial solutions that Riccati differential equations can have. Quickly, we realize that an upper bound for this maximum number does not exist for linear differential equations. For instance the equation $\dot{y} = x$ has all its solutions polynomials $y = x^2/2 + c$. In fact it is very easy to prove that linear equations have 0, 1 or all its solutions being polynomials.

Looking at some other papers we found that Rainville [16] in 1936 proved the existence of one or two polynomial solutions for a subclass of (1). Bhargava and Kaufman [4, 5] obtained some sufficient conditions for equation (1) to have polynomial solutions. Campbell and Golomb [6, 7] provided some criteria determining the degree of polynomial solutions of equation (1). Bhargava and Kaufman [3] considered a more general form of equations than (1), and got some criteria on the degree of polynomial solutions of the equations. Giné, Grau and Llibre [11] proved that polynomial differential equations

$$\dot{y} = b_0(x) + b_1(x)y + b_2(x)y^2 + \ldots + b_n(x)y^n, \tag{3}$$

with $b_i(x) \in \mathbb{R}[x]$, $i = 0, 1, \ldots, n$, and $b_n(x) \not\equiv 0$ have at most n polynomial solutions and they also prove that this bound is sharp.

In short, to be best of our knowledge, the question of knowing the maximum number of polynomial solutions of Riccati polynomial differential equations when $a(x)$ is nonconstant is open. We believe that it is also interesting because it is reminiscent of a similar question of Poincaré about the degree and number of algebraic solutions of autonomous planar polynomial differential equations in terms of their degrees, when these systems have finitely many algebraic solutions. Recall that in this situation, similarly that for the linear case, there are planar polynomial equations having rational first integrals for which all solutions are algebraic. As far as we know, Poincaré's question is open even for planar quadratic differential equations.

The first result of this paper solves completely our question for real or complex polynomial Riccati differential equations. To be more precise, we say that equation (1) is a *polynomial Riccati differential equation of degree* η when a, b_0, b_1, $b_2 \in \mathbb{F}[x]$, the ring of polynomials in x with coefficients in $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, and $\eta := \max\{\alpha, \beta_0, \beta_1, \beta_2\}$, where $\alpha = \deg a$, and $\beta_i = \deg b_i$ for $i = 0, 1, 2$.

Theorem 1. *Real or complex polynomial Riccati differential equations* (1) *of degree* η *and* $b_2(x) \not\equiv 0$, *have at most* $\eta + 1$ (*resp.* 2) *polynomial solutions when* $\eta \geq 1$ (*resp.* $\eta = 0$). *Moreover, there are equations of this type having any given number of polynomial solutions smaller than or equal to these upper bounds.*

Notice that this theorem shows that the examples of Campbell and Golomb quoted in the introduction precisely correspond to Riccati equations of degrees $\eta \in \{3, 4\}$ with the maximum number $(\eta + 1)$ of polynomial solutions.

As we will see in Proposition 10, the simple Riccati equation $a(x)\dot{y} = \dot{a}(x)y + y^2$, with $a(x)$ a polynomial of degree η and simple roots has exactly $\eta + 1$ polynomial solutions.

When the functions appearing in (1) are real trigonometric polynomials of degree at most η we will say that we have a *trigonometric polynomial Riccati differential equation of degree* η. Recall that the degree of a real trigonometric polynomial is defined as the degree of its corresponding Fourier series. We are interested in them, not as a simple generalization of the polynomial ones, but because they appear together with Abel equations ($n = 3$ in (3)) in the study of the number of limit cycles on planar polynomial differential equations with homogeneous nonlinearities, see for instance [10, 14]. In particular, periodic orbits surrounding the origin of these planar systems correspond to 2π-periodic solutions of the corresponding Abel or Riccati equations.

It is an easy consequence of relation (2) that when $a(x)$ does not vanish, T-periodic Riccati equations of class $\mathcal{C}^1$ have either continua of T-periodic solutions or at most two T-periodic solutions. When $a(x)$ vanishes the situation is totally different. These equations are singular at the zeros of a. Equations with this property are called *constrained differential equation* and the zero set of a is named *impasse set*, see [18]. In particular, the Cauchy problem has no uniqueness on the impasse set and the behavior of the solutions of the Riccati equation is more complicated. As we will see, there are real trigonometric polynomial Riccati differential equations with an arbitrary large number of trigonometric polynomial solutions, which are 2π-periodic solutions of the equation. As we will see this number is bounded above in terms of the degrees of the corresponding trigonometric polynomials defining the equation.

In general, we will write real trigonometric polynomial Riccati differential equations as

$$A(\theta)Y' = B_0(\theta) + B_1(\theta)Y + B_2(\theta)Y^2, \tag{4}$$

where the prime denotes the derivative with respect to θ. To be more precise A, B_0, B_1, $B_2 \in \mathbb{R}_t[\theta] := \mathbb{R}[\cos\theta, \sin\theta]$ the ring of trigonometric polynomials in $\cos\theta, \sin\theta$ with coefficients in $\mathbb{R}$. In this case we have also $\eta := \max\{\alpha, \beta_0, \beta_1, \beta_2\}$, where $\alpha = \deg A$, and $\beta_i = \deg B_i$ for $i = 0, 1, 2$. Our second main result is:

Theorem 2. *Real trigonometric polynomial Riccati differential equations* (4) *of degree* $\eta \geq 1$ *and* $B_2(\theta) \not\equiv 0$*, have at most* 2η (*resp.* 3) *trigonometric polynomial solutions when* $\eta \geq 2$ (*resp.* $\eta = 1$). *Moreover, there are equations of this type having any given number of trigonometric polynomial solutions smaller than or equal to these upper bounds.*

For instance, consider the degree 3 trigonometric polynomial Riccati equation (4) with $A(\theta) = 5\sin\theta + 8\sin(2\theta) + 5\sin(3\theta)$, $B_0(\theta) = 0$, $B_1(\theta) = 2 + 6\cos\theta + 18\cos(2\theta) + 10\cos(3\theta)$, and $B_2(\theta) = -1$. It has exactly six trigonometric polynomial solutions $Y_1(\theta) = 0$,

$$\begin{aligned}
Y_2(\theta) &= 10 + 16\cos\theta + 10\cos(2\theta),\\
Y_3(\theta) &= 1 - 2\cos\theta + 3\sin(2\theta) + \cos(2\theta),\\
Y_4(\theta) &= 1 - 2\cos\theta - 3\sin(2\theta) + \cos(2\theta),\\
Y_5(\theta) &= -3 - 8\sin\theta - 2\cos\theta - 5\sin(2\theta) + 5\cos(2\theta),\\
Y_6(\theta) &= -3 + 8\sin\theta - 2\cos\theta + 5\sin(2\theta) + 5\cos(2\theta).
\end{aligned}$$

This example is constructed following the procedure described after the proof of Theorem 2. From our general study of (4) it can be understood why these solutions cross the impasse set, see Lemma 14. This phenomenon is illustrated in Figure 1.

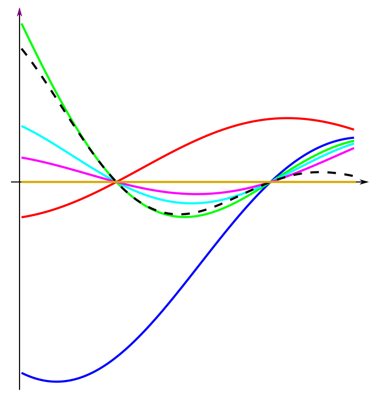

FIGURE 1. The function $A(\theta)$ and the six trigonometric polynomial solutions for a trigonometric polynomial Riccati equation of degree 3. The graph of $A(\theta)$ is drawn as a dashed line.

As we will see, many steps of the proof of Theorem 1 are based on divisibility arguments in the ring of polynomials. A major difference for proving Theorem 2 is that the ring of trigonometric polynomials is no more a *Unique Factorization Domain*. This can be seen for instance using the celebrated identity $\cos^2\theta = 1-\sin^2\theta = (1-\sin\theta)(1+\sin\theta)$. It holds that $\cos\theta$ divides the right hand expression but it does not divide either $1-\sin\theta$ nor $1+\sin\theta$. Fortunately, divisibility reasonings in this ring can be addressed by using the isomorphism

$$\begin{array}{rccc} \Phi: & \mathbb{R}_t(\theta) & \longrightarrow & \mathbb{R}(x) \\ & (\cos\theta,\, \sin\theta) & \longmapsto & \left(\dfrac{1-x^2}{1+x^2}, \dfrac{2x}{1+x^2}\right), \end{array} \tag{5}$$

between the two fields $\mathbb{R}_t(\theta) = \mathbb{R}(\cos\theta, \sin\theta)$ and $\mathbb{R}(x)$. In fact, it can be seen, see Lemma 11, that trigonometric polynomials of degree ν correspond in $\mathbb{R}(x)$ to rational functions of the form $f(x)/(1+x^2)^\nu$, with $f(x)$ a polynomial of degree at most 2ν and coprime with $1+x^2$. Then, for instance, the previous example $\cos^2\theta$ moves to $(1-x^2)^2/(1+x^2)^2$. This rational function decomposes as one of the two products

$$\frac{1-x^2}{1+x^2} \times \frac{1-x^2}{1+x^2}, \quad \frac{(1-x)^2}{1+x^2} \times \frac{(1+x)^2}{1+x^2},$$

which precisely corresponds with the only two decompositions of $\cos^2\theta$ as a product of trigonometric polynomials, which are $\cos\theta \times \cos\theta$ and $(1-\sin\theta)\times(1+\sin\theta)$, respectively.

The results for the polynomial Riccati equation are proved in Section 2. The proof of Theorem 2 about the trigonometric case will be given in Section 3. As we will see, both proofs provide also more detailed information about the number and degrees of the polynomial or trigonometric polynomial solutions.

2. Polynomial Riccati equations

This section is devoted to prove Theorem 1. Firstly we give some technical results and secondly we organize the proof in three parts: the upper bound for $\eta \geq 2$ (Proposition 8), the upper bound for $\eta \leq 1$ (Proposition 9), and the example that provides the concrete number of polynomial solutions up to reach the upper bound (Proposition 10). Recall that $a(x), b_0(x), b_1(x), b_2(x)$ in equation (1) are real or complex polynomials of degrees $\alpha, \beta_0, \beta_1, \beta_2$, respectively.

Next result provides an upper bound on the degree of the polynomial solutions of this equation.

Lemma 3. *If $y_0(x)$ is a polynomial solution of equation* (1), *then* $\deg y_0 \leq \eta - \deg b_2$.

Proof. By the assumption we have

$$a(x)\dot{y}_0(x) \equiv b_0(x) + b_1(x)y_0(x) + b_2(x)y_0^2(x), \quad x \in \mathbb{R}. \tag{6}$$

Set $\delta = \deg y_0$. We will prove that $\deg(b_2 y_0) \leq \eta$. If $\delta = 0$ the result is trivial. Assume that $\delta > 0$, then the four components in (6) have respectively the degrees

$$(\alpha + \delta - 1;\ \beta_0,\ \beta_1 + \delta,\ \beta_2 + 2\delta). \tag{7}$$

Clearly we have

$$\max\{\beta_0, \beta_1 + \delta, \beta_2 + 2\delta\} \geq \alpha + \delta - 1.$$

Additionally, when the above maximum is taken only by one of the three numbers then the equality holds.

- [*Case* $\max\{\beta_0, \beta_1 + \delta, \beta_2 + 2\delta\} = \alpha + \delta - 1$]: Then $\beta_2 + \delta \leq \alpha - 1 < \eta$. That is $\deg(b_2\, y_0) = \beta_2 + \delta < \eta$.
- [*Case* $\max\{\beta_0, \beta_1 + \delta, \beta_2 + 2\delta\} > \alpha + \delta - 1$]: We must have

$$\beta_1 + \delta = \beta_2 + 2\delta, \quad \text{or} \quad \beta_0 = \beta_2 + 2\delta, \quad \text{or} \quad \beta_0 = \beta_1 + \delta.$$

When $\beta_1 + \delta = \beta_2 + 2\delta$, we have $\beta_2 + \delta = \beta_1 \leq \eta$. If $\beta_2 + 2\delta = \beta_0$ then $\beta_2 + \delta < \beta_0 \leq \eta$. Finally, when $\beta_0 = \beta_1 + \delta$, we must have $\beta_2 + 2\delta \leq \beta_0 = \beta_1 + \delta$. Then again $\beta_2 + \delta < \eta$. $\square$

Lemma 3 has a useful consequence in proving Theorem 1 for a reduced case.

Lemma 4. *If a polynomial equation* (1) *of degree η has a polynomial solution, then it is equivalent to an equation with $b_0(x) \equiv 0$ and the same degree:*

$$a(x)\dot{y} = b_1(x)y + b_2(x)y^2. \tag{8}$$

Moreover, each polynomial solution of equation (1) *corresponds to a polynomial solution of the new equation* (8).

Proof. If y_0 is a solution of (1), the proof follows taking the change of variable $w(x) = y(x) - y_0(x)$ and applying Lemma 3. More concretely, equation (1) becomes

$$a(x)\dot{w} = (b_1(x) + 2\, b_2(x)\, y_0(x))w + b_2(x)\, w^2,$$

and $\deg(b_1 + 2b_2 y_0) \leq \max\{\deg b_1, \deg(b_2 y_0)\} \leq \eta$. $\square$

The next result gives a criterion to relate the zeros of a with the zeros of polynomial solutions of equation (8).

Lemma 5. *If x^* is a zero of a nonzero polynomial solution of equation* (8)*, then $a(x)$ also vanishes at $x = x^*$.*

Proof. Let y_1 be a polynomial solution of (8). Then we have

$$a(x)\dot{y}_1(x) = (b_1(x) + b_2(x)y_1(x))y_1(x).$$

The statement follows by equating the power of $(x - x^*)$ in the above equation. □

We remark that in the later result even when x^* is a zero of a solution with multiplicity $\nu \geq 2$, it may be a simple zero of $a(x)$. For example, the equation $x\dot{y} = (\nu - x^\nu)y + y^2$ has the solution $y = x^\nu$ with $x = 0$ a zero of multiplicity ν, but $a(x) = x$ has $x = 0$ as a simple zero. This example also illustrates that the upper bound of the degree of polynomial solutions in Lemma 3 can be achieved.

Next lemma is the version of (2) that we will use. From now on, in this work, given two polynomials f and g we will denote by $\gcd(f, g)$ their monic greater common divisor.

Lemma 6. *Let $y_0 = 0$, $y_1(x)$, $y_2(x)$ be polynomial solutions of* (8) *such that $y_1(x) \not\equiv 0$ and $y_2(x) \not\equiv 0$. Set $y_1(x) = g(x)\tilde{y}_1(x)$ and $y_2(x) = g(x)\tilde{y}_2(x)$, where $g = \gcd(y_1, y_2)$, and $\gcd(\tilde{y}_1, \tilde{y}_2) = 1$. Except the solution $y = 0$, all the other solutions of equation* (8) *can be expressed as*

$$y(x; c) = \frac{\tilde{y}_1(x)\,\tilde{y}_2(x)g(x)}{c\tilde{y}_1(x) + (1 - c)\tilde{y}_2(x)}, \tag{9}$$

where c is an arbitrary constant.

Proof. Let y be a nonzero solution of equation (8). The functions $z = 1/y$, $z_1 = 1/y_1$, and $z_2 = 1/y_2$ are solutions of a linear differential equation and satisfy

$$-a(x)\dot{z} = b_1(x)z + b_2(x), \quad -a(x)\dot{z}_1 = b_1(x)z_1 + b_2(x), \quad -a(x)\dot{z}_2 = b_1(x)z_2 + b_2(x).$$

It follows that

$$\frac{\dot{z}(x) - \dot{z}_1(x)}{z(x) - z_1(x)} = \frac{\dot{z}_2(x) - \dot{z}_1(x)}{z_2(x) - z_1(x)}.$$

Consequently

$$z(x) = z_1(x) + c(z_2(x) - z_1(x)),$$

with c an arbitrary constant. So, the general solution of (8) is

$$y(x; c) = \frac{y_1(x)\,y_2(x)}{cy_1(x) + (1 - c)y_2(x)},$$

with c an arbitrary constant. The proof ends substituting $y_1(x) = g(x)\,\tilde{y}_1(x)$ and $y_2(x) = g\tilde{y}_2(x)$ into this last expression. □

Lemma 7. *Denote by $N(g)$ the number of different zeros of g. The following statements hold.*

(*i*) *If for any value of c, $c\tilde{y}_1(x) + (1 - c)\tilde{y}_2(x)$ is not a constant, then equation* (8) *has at most $N(g) + 3$ polynomial solutions.*

(*ii*) *If there exists a value c_0 such that $c_0\tilde{y}_1(x) + (1 - c_0)\tilde{y}_2(x)$ is a constant, then equation* (8) *has at most $N(g) + 4$ polynomial solutions.*

Proof. From Lemma 6 the general solution of equation (8) is (9). Since $\gcd(\tilde{y}_1, \tilde{y}_2) = 1$, in order that $y(x; c)$ is a polynomial solution of equation (8) we must have $c = 0$, or $c = 1$, or $c \neq 0$, 1 and $c\tilde{y}_1 + (1-c)\tilde{y}_2$ divides g. Since $\deg g \leq \eta$, it follows that g has at most η zeros. We distinguish two cases depending on whether the denominator of (9) can be a constant or not.

• [*Case of denominator never being a constant*]: For any c, the polynomial $c\tilde{y}_1(x) + (1-c)\tilde{y}_2(x)$ is never a constant. Let x^* be a root of g. In order that $c\tilde{y}_1 + (1-c)\tilde{y}_2$ divides g for some c, we must have

$$c\tilde{y}_1(x^*) + (1-c)\tilde{y}_2(x^*) = 0.$$

If $\tilde{y}_1(x^*) = \tilde{y}_2(x^*)$ then from the above relation we get that both must be zero, but this is impossible because $\gcd(\tilde{y}_1, \tilde{y}_2) = 1$. If $\tilde{y}_1(x^*) - \tilde{y}_2(x^*) \neq 0$, solving this last equation gives $c^* = \tilde{y}_2(x^*)/(\tilde{y}_2(x^*) - \tilde{y}_1(x^*))$. If $c^*\tilde{y}_1 + (1-c^*)\tilde{y}_2$ divides g, we have the polynomial solution $y(x; c^*)$ given in (9). Thus we have at most $N(g)$ different values of c such that $c\tilde{y}_1 + (1-c)\tilde{y}_2$ divides g. As a consequence, (1) has at most $N(g)$ number of polynomial solutions together with the solutions 0, y_1 and y_2.

• [*Case of some constant denominator*]: There exists a c_0 such that $c_0\tilde{y}_1(x) + (1-c_0)\tilde{y}_2(x)$ is constant. First we claim that this c_0 is unique. If not, there are two numbers c_1, c_2 such that

$$c_1\tilde{y}_1(x) + (1-c_1)\tilde{y}_2(x) \equiv d_1,$$
$$c_2\tilde{y}_1(x) + (1-c_2)\tilde{y}_2(x) \equiv d_2,$$

for some constants d_1 and d_2. Then we have

$$(c_1 d_2 - c_2 d_1)\tilde{y}_1(x) + (d_2 - d_1 - c_1 d_2 + c_2 d_1)\tilde{y}_2(x) \equiv 0. \tag{10}$$

If the coefficients of $\tilde{y}_1$ and $\tilde{y}_2$ in (10) are not all zero, then one of $\tilde{y}_1$ and $\tilde{y}_2$ is a multiple of the other. This is in contradiction with the fact that $\gcd(\tilde{y}_1, \tilde{y}_2) = 1$. If the coefficients in (10) are both zero, we have $d_1 = d_2$ and $c_1 = c_2$. This proves the claim.

As in the proof of the previous case, we now have $N(g) + 1$ possible values of c for which $g/(c\tilde{y}_1 + (1-c)\tilde{y}_2)$ are polynomials. This implies that equation (8) has at most $N(g) + 4$ nonzero polynomial solutions. □

Proposition 8. *Let $a(x), b_i(x)$ be polynomials of degrees $\alpha = \deg a$, $\beta_i = \deg b_i$, $i = 1, 2$, and $\eta = \max\{\alpha, \beta_1, \beta_2\} \geq 2$ with $b_2(x) \not\equiv 0$. Then equation (8) has at most $\eta + 1$ polynomial solutions.*

Proof. We assume that equation (8) has three different polynomial solutions $0, y_1, y_2$. Otherwise the statement follows. Recall that $g = \gcd(y_1, y_2)$ and it is monic, see Lemma 6. Then, $\deg g \leq \max\{\deg y_1, \deg y_2\} \leq \eta$ and $N(g) \leq \eta$. We will split our study in the three cases: $N(g) = \eta$, $N(g) = \eta - 1$, or $N(g) \leq \eta - 2$.

• [*Case $N(g) = \eta$*]: Then $\deg g = \deg y_1 = \deg y_2 = \eta$. Hence

$$y_1(x) = p\, g(x), \qquad y_2(x) = q\, g(x), \qquad p, q \in \mathbb{F}^* = \mathbb{F} \setminus \{0\}.$$

It follows from Lemma 6 that all solutions of equation (8) are of the form

$$y(x; c) = k_c\, g(x), \qquad k_c \in \mathbb{F}.$$

This forces that $b_2(x) \equiv 0$, a contradiction with the assumption $b_2(x) \not\equiv 0$.

• [*Case* $N(g) = \eta - 1$]: We split the proof in two cases:

(i) $\deg g = \eta$ *or* $\deg y_1 = \deg y_2 = \eta - 1$: We have $y_1 = p\,g$, $y_2 = q\,g$ with $p, q \in \mathbb{F}^*$. Similarly with the case $N(g) = \eta$, we get $b_2(x) \equiv 0$, again a contradiction.

(ii) $\deg g = \eta - 1$ *and* $\max(\deg y_1, \deg y_2) = \eta$: We have that $y_i = g\,\tilde{y}_i$ with $\deg \tilde{y}_i \leq 1$, $i = 1, 2$. Moreover, by Lemma 5 it holds that $a = g\,\tilde{a}$ with $\deg \tilde{a} \leq 1$.

Since y_1 and y_2 are solutions of equation (8), we get from

$$a(x)\dot{y}_1 = b_1(x)y_1 + b_2(x)y_1^2, \quad a(x)\dot{y}_2 = b_1(x)y_2 + b_2(x)y_2^2,$$

that

$$b_2(x) = \frac{a(x)\big(y_1(x)\dot{y}_2(x) - y_2(x)\dot{y}_1(x)\big)}{y_1(x)y_2(x)(y_2(x) - y_1(x))} = \frac{\tilde{a}(x)\big(\tilde{y}_1(x)\dot{\tilde{y}}_2(x) - \tilde{y}_2(x)\dot{\tilde{y}}_1(x)\big)}{\tilde{y}_1(x)\tilde{y}_2(x)(\tilde{y}_2(x) - \tilde{y}_1(x))}. \tag{11}$$

By Lemma 3, $b_2(x) \equiv p$ for some constant number. Thus

$$p\,\tilde{y}_1(x)\tilde{y}_2(x)(\tilde{y}_2(x) - \tilde{y}_1(x)) = \tilde{a}(x)\big(\tilde{y}_1(x)\dot{\tilde{y}}_2(x) - \tilde{y}_2(x)\dot{\tilde{y}}_1(x)\big).$$

Notice that $\tilde{y}_1\dot{\tilde{y}}_2 - \tilde{y}_2\dot{\tilde{y}}_1$ is always a nonzero constant, because $\max(\deg \tilde{y}_1, \deg \tilde{y}_2) = 1$ and $\gcd(\tilde{y}_1, \tilde{y}_2) = 1$. Hence, in the above formula, the left-hand side has degree at least two while the right-hand side has degree at most one, a contradiction.

Hence under all the above hypotheses the maximum number of polynomial solutions is $2 \leq \eta + 1$, for $\eta \geq 1$, as we wanted to prove.

• [*Case* $N(g) \leq \eta - 2$]: If $c\,\tilde{y}_1(x) + (1 - c)\,\tilde{y}_2(x)$ is nonconstant, for any value of c, the result holds from statement (i) of Lemma 7. If $N(g) < \eta - 2$ the result also follows from Lemma 7.

Hence we assume that $N(g) = \eta - 2$ and that there exists a constant c_0, with $c_0 \neq 0$, 1, such that

$$c_0\tilde{y}_1(x) + (1 - c_0)\tilde{y}_2(x) \equiv q \in \mathbb{F}^*.$$

Note that if $q = 0$, then $\tilde{y}_1$ and $\tilde{y}_2$ will have a common factor, it is in contradiction with $\gcd(\tilde{y}_1, \tilde{y}_2) = 1$. We recall that $y(x; 0) = y_1(x)$ and $y(x; 1) = y_2(x)$. Now equation (8) also has the polynomial solution

$$y_3(x) = \tilde{y}_1(x)\tilde{y}_2(x)g(x)/q.$$

From $N(g) = \eta - 2$ we get that $\deg g \geq \eta - 2$.

Since any polynomial solution of equation (1) has degree at most η, by the assumptions and the expressions of y_1 and y_2 we get that $\tilde{y}_1$ and $\tilde{y}_2$ are polynomials of the same degree $d \in \{0, 1, 2\}$. But it follows from the expression of y_3 that $d = 2$ is not possible, otherwise the polynomial solution $y_3(x)$ would have degree at least $\eta + 2$, a contradiction.

If $d = 0$, then y_1 and y_2 are constant multiples of g. Similarly to the proof of case $N(g) = \eta$ we get that $b_2(x) \equiv 0$, again a contradiction.

When $d = 1$, if $\tilde{y}_1$ or $\tilde{y}_2$ has a zero which is not a zero of g, we assume without loss of generality that $\tilde{y}_1$ satisfies this condition. Set

$$h(x) := \gcd(y_1(x), y_3(x)) = \tilde{y}_1(x)\,g(x).$$

Then $N(h) = \eta - 1$. In this situation we can start again our proof taking y_1 and y_3 instead of y_1 and y_2 and we are done.

Finally, assume that $d = 1$ and each zero of $\tilde{y}_1$ and $\tilde{y}_2$ is a zero of g. Then, since any polynomial solution of equation (8) has degree at most η, we get from the expression of y_3 that $\deg g \leq \eta - 2$. Thus $\deg g = \eta - 2$. So we have

$$g(x) = (x - x_1) \cdots (x - x_{\eta-2}) \quad \text{with } x_i \neq x_j \text{ for } 1 \leq i \neq j \leq \eta - 2. \tag{12}$$

Since the zeros of $\tilde{y}_1$ and $\tilde{y}_2$ are zeros of $g(x)$, we can assume that

$$\tilde{y}_1 = p_1(x - x_1), \quad \tilde{y}_2 = p_2(x - x_2) \quad \text{with } p_1, p_2 \text{ nonzero constants.} \tag{13}$$

Thus in order that $y(x; c)$ with $c \neq 0$, 1, is a polynomial solution of equation (8), we get from (9), (12), and (13) that

$$c\tilde{y}_1(x) + (1 - c)\tilde{y}_2(x) = r_i(x - x_i) \quad \text{with } i \in \{3, \ldots, \eta - 2\},$$

where each r_i is a nonzero constant. Clearly the number of possible values of c satisfying these conditions is exactly $\eta - 4$. This proves that equation (8) has η polynomial solutions, including 0, y_1, y_2 and y_3. $\square$

Proposition 9. *Let $a(x), b_i(x)$ be polynomials of degrees $\alpha = \deg a$, $\beta_i = \deg b_i$, $i = 1, 2$, and $\eta = \max\{\alpha, \beta_1, \beta_2\} \leq 1$ with $b_2(x) \not\equiv 0$. Then equation* (8) *has at most* 2 *polynomial solutions.*

Proof. We prove the statement in two different cases: $\eta = 0$ and $\eta = 1$.

• [*Case* $\eta = 0$]: Equation (8) has constant coefficients, and it can be written as

$$\dot{y} = p\, y(y - q),$$

with $p \neq 0$ and q constants. This equation has two constant solutions for $q \neq 0$ and one constant solution for $q = 0$. The proof of this case finishes, using Lemma 3, because any polynomial solution should be constant.

• [*Case* $\eta = 1$]: By contradiction we assume that equation (8) has three polynomial solutions 0, y_1, and y_2. Lemma 3 shows that $\max\{\deg y_1, \deg y_2\} \leq 1$. The proof is done with a case by case study on the degrees of y_1 and y_2.

(*i*) If y_1 and y_2 are both linear, then they cannot have a common zero, otherwise it follows from Lemma 6 that all solutions are constant multiple of y_1, and so $b_2(x) \equiv 0$, a contradiction. Since y_1 and y_2 have different zeros, it follows from Lemma 5 that a has two different zeros. This means that a has degree at least 2, again a contradiction with $\eta = 1$. These contradictions imply that equation (8) has never two nonzero polynomial solutions of the given form.

(*ii*) If y_1 and y_2 are both constants, we get from Lemma 6 that all solutions of equation (1) are constants. And so $b_2(x) \equiv 0$, again a contradiction.

(*iii*) Assume that one of y_1 and y_2 is linear and another is a constant. Without loss of generality we assume that y_1 is linear and $y_2(x) = p \neq 0$ a constant. Now we have by Lemma 5 that $a(x) = r\, y_1(x)$ with r a nonzero constant. From (11) we have

$$b_2(x) = \frac{r\dot{y}_1(x)}{y_1(x) - p},$$

that is not a polynomial, because $\dot{y}_1$ is a nonzero constant. This contradiction shows that equation (1) has neither two nonzero polynomial solutions of the given form. $\square$

Proposition 10. *Let x_i, $i = 1, \ldots, \eta$ be different values in $\mathbb{F}$. For each $j \in \{2, \ldots, \eta+1\}$, consider the polynomial $a(x) = -(x - x_1)^{\eta+2-j}(x - x_2)\cdots(x - x_{j-1})$ of degree η. Then the differential equation* (1) *with $b_0(x) \equiv 0$, $b_1(x) = \dot{a}(x)$, and $b_2(x) \equiv 1$ has exactly j polynomial solutions.*

Proof. Equation (1) with $b_0(x) \equiv 0$, $b_1(x) = \dot{a}(x)$ and $b_2(x) \equiv 1$ has the solution $y = 0$. When $y \neq 0$, (1) can be written as

$$\frac{d}{dx}\left(\frac{a(x)}{y}\right) = -1.$$

It has a general solution

$$y(x) = \frac{-a(x)}{x - c} = \frac{(x - x_1)^{\eta+2-j}(x - x_2)\cdots(x - x_{j-1})}{x - c},$$

with c an arbitrary constant. For each $j \in \{2, \ldots, \eta+1\}$, choosing $c = x_i$, $i = 1, \ldots, j-1$, we get exactly $j - 1$ polynomial solutions

$$y_i = (x - x_1)^{\eta+2-j}\cdots(x - x_{i-1})\widehat{(x - x_i)}(x - x_{i+1})\cdots(x - x_{j-1}), \quad i = 1, \ldots, j-1,$$

where $\widehat{(x - x_i)}$ denotes the absence of the factor. Then equation (1) has exactly j polynomial solutions including the trivial one $y = 0$. $\square$

Proof of Theorem 1. When equation (1) has no polynomial solutions we are done. Otherwise we can apply Lemma 4 and restrict our analysis to equation (8). Propositions 8 and 9 provide the first part of the statement, which corresponds to the upper bound of the statement when the degree is $\eta \geq 2$ and $\eta \leq 1$, respectively. Proposition 10 gives a concrete polynomial differential equation with exactly j polynomial solutions for each $j \in \{2, \ldots, \eta + 1\}$. The cases $j \in \{0, 1\}$ are trivial. These facts prove the second part of the statement. $\square$

3. Trigonometric polynomial Riccati equations

This section is devoted to prove Theorem 2. As in the previous section, first we give some technical results. Since the proof of Theorem 2 is different for $\eta \geq 2$ and $\eta = 1$, we distinguish these two cases.

Let $F(\theta) \in \mathbb{R}_t[\theta]$ be a real trigonometric polynomial, recall that it is said that $F(\theta)$ has *degree* ν if ν is the degree of its associated Fourier series, i.e.

$$F(\theta) = \sum_{k=-\nu}^{\nu} f_k e^{k\theta\mathbf{i}}, \quad \text{with } \mathbf{i} = \sqrt{-1},$$

where $f_k = \overline{f}_{-k} \in \mathbb{C}$, $k \in \{1, \ldots, \nu\}$, and f_ν is nonzero, see for instance [2, 8]. As we have already explained, $\mathbb{R}_t[\theta]$ is not a unique factorization domain and this fact complicates our proofs. The next lemma provides the image, by the isomorphism Φ given in (5), of the ring $\mathbb{R}_t[\theta]$ in $\mathbb{R}(x)$, see for instance [8, Lemma 10] and [12, Lemma 17]. In fact, it is one of the key points in the proof of Theorem 2 because the map Φ moves a polynomial in $\sin\theta, \cos\theta$ into a rational function in x, such that the numerator has a unique decomposition as a product of irreducible polynomials. Another minor difference between this case and the polynomial one is that the degree remains invariant by derivation when we consider trigonometric polynomials.

For sake of shortness, in this paper we do not treat the case of complex trigonometric polynomial Riccati equations.

Lemma 11. *Set* $F(\theta) \in \mathbb{R}_t[\theta]$ *with* $\deg F = \nu$. *Then*

$$\Phi(F(\theta)) = \frac{f(x)}{(1+x^2)^\nu},$$

with $\gcd(f(x), 1+x^2) = 1$ *and* $\deg f \leq 2\nu$. *Conversely, any rational function* $g(x)/(1+x^2)^\nu$ *with* $g(x)$ *an arbitrary polynomial of degree no more than* 2ν *can be written as a trigonometric polynomial through the inverse change,* Φ^{-1}.

In equation (4), set

$$A(\theta) = \frac{a(x)}{(1+x^2)^\alpha}, \quad B_i(\theta) = \frac{b_i(x)}{(1+x^2)^{\beta_i}}, \tag{14}$$

where $\alpha = \deg A$, $\deg a \leq 2\alpha$, $\beta_i = \deg B_i$, and $\deg b_i \leq 2\beta_i$ for $i = 0, 1, 2$. By the assumption of Theorem 2 we have $\eta = \max\{\alpha, \beta_0, \beta_1, \beta_2\}$.

Using Lemma 11 we can transform the trigonometric polynomial Riccati differential equation (4) into a polynomial Riccati differential equation (1), see the proof of Lemma 14. Then we can apply Theorem 1 to this polynomial differential equation. It can be seen that in this way we can prove (we omit the details) that an upper bound of the trigonometric polynomial solutions of equation (4) is $6\eta + 1$. This upper bound is much higher than the actual one given in Theorem 2.

Although the outline for proving Theorem 2 will be the same that we have followed in Section 2, we will see that in this case the proof is much more involved.

We start providing an upper bound on the degree of trigonometric polynomial solutions of the trigonometric polynomial equation (4). Notice that, for this first result, we do not use the map Φ.

Lemma 12. *If* $Y_0(\theta)$ *is a real trigonometric polynomial solution of the real trigonometric polynomial equation* (4) *of degree* η, *then* $\deg(Y_0(\theta)) \leq \eta - \deg(B_2(\theta))$.

Proof. Given two trigonometric polynomials P and Q it holds that $\deg(P(\theta)Q(\theta)) = \deg(P(\theta)) + \deg(Q(\theta))$ and $\deg(P'(\theta)) = \deg(P(\theta))$. Set $\delta = \deg Y_0$ then the degrees of the four components in (4) are respectively

$$(\alpha + \delta;\ \beta_0,\ \beta_1 + \delta,\ \beta_2 + 2\delta).$$

The proof follows, as in Lemma 3, comparing the degrees of the four terms in both sides of equation (4). Notice that the only difference is that the first degree of the above list is one larger than the corresponding list given in (7). □

Next two results are the equivalent versions of Lemmas 4 and 5 for trigonometric Riccati equations. We only prove the second one because the proof of the first one is essentially the same.

Lemma 13. *If a trigonometric polynomial equation* (4) *of degree* η *has a trigonometric polynomial solution, then it is equivalent to an equation with* $B_0(\theta) \equiv 0$ *and the same degree:*

$$A(\theta)Y' = B_1(\theta)Y + B_2(\theta)Y^2. \tag{15}$$

Moreover, each trigonometric polynomial solution of equation (4) *corresponds to a trigonometric polynomial solution of the new equation* (15).

Lemma 14. *If* $Y_1(\theta)$ *is a nonconstant real trigonometric polynomial solution of equation* (15)*, set*

$$Y_1(\theta) = \frac{y_1(x)}{(1+x^2)^{\eta_1}} \quad \text{with} \quad \gcd(y_1(x), 1+x^2) = 1,$$

where η_1 *is the degree of* Y_1 *and* $\deg y_1 \le 2\eta_1$*, then any irreducible factor of* $y_1(x)$ *is a factor of the polynomial* $a(x)$ *defined in* (14).

Proof. From the transformation Φ, we get

$$x' = \frac{dx}{d\theta} = \frac{1+x^2}{2},$$

it follows that

$$Y_1{}'(\theta) = \frac{\dot{y}_1(x)(1+x^2) - 2\eta_1 x y_1(x)}{2(1+x^2)^{\eta_1}},$$

where the dot and prime denote the derivative with respect to x and θ, respectively. So equation (15) becomes

$$\frac{a(x)}{2(1+x^2)^{\alpha}}\left(\dot{y}_1(x)(1+x^2) - 2\eta_1 x y_1(x)\right) = \frac{b_1(x)y_1(x)}{(1+x^2)^{\beta_1}} + \frac{b_2(x)(y_1(x))^2}{(1+x^2)^{\beta_2+\eta_1}}.$$

This last equality shows that each irreducible factor of $y_1(x)$ is a factor of $a(x)$, where we have used the fact that $\gcd(y_1(x), 1+x^2) = 1$. The lemma follows. $\square$

Next five lemmas split the essential difficulties for proving Theorem 2.

Lemma 15. (*i*) *Assume that equation* (15) *has at least three different trigonometric polynomial solutions* $Y_0(\theta) \equiv 0$, $Y_1(\theta)$, $Y_2(\theta)$. *Set*

$$Y_1(\theta) = \frac{y_1(x)}{(1+x^2)^{\eta_1}}, \quad Y_2(\theta) = \frac{y_2(x)}{(1+x^2)^{\eta_2}}, \tag{16}$$

where $\eta_1 = \deg Y_1$, $\eta_2 = \deg Y_2$, $\deg y_i \le 2\eta_i$, $\eta_1 \le \eta_2$ *and* $\gcd(y_i(x), 1+x^2) = 1$ *for* $i = 1, 2$. *Except the solution* $Y_0(\theta) \equiv 0$*, all the other solutions of equation* (15) *can be expressed as*

$$\begin{aligned} Y(\theta; c) =& \frac{y_1(x)\, y_2(x)}{c y_1(x)(1+x^2)^{\eta_2} + (1-c) y_2(x)(1+x^2)^{\eta_1}} \\ =& \frac{g(x)\, \tilde{y}_1(x)\, \tilde{y}_2(x)}{(1+x^2)^{\eta_1}\left(c\tilde{y}_1(x)(1+x^2)^{\eta_2-\eta_1} + (1-c)\tilde{y}_2(x)\right)}, \quad c \in \mathbb{R}, \end{aligned}$$

where $g = \gcd(y_1, y_2)$*, i.e.* $y_i = g\, \tilde{y}_i$*, with* $\gcd(\tilde{y}_1, \tilde{y}_2) = 1$.

(ii) Assume that equation (15) *has at least four different trigonometric polynomial solutions. Then we can always choose two of them, say* $Y_1(\theta)$ *and* $Y_2(\theta)$*, of the same degree* $\eta_1 = \eta_2$*, and then all the solutions of* (15)*, except* $Y_0(\theta) \equiv 0$*, are*

$$Y(\theta; c) = \frac{y_1(x)\, y_2(x)}{(1+x^2)^{\eta_1}\big(cy_1(x) + (1-c)y_2(x)\big)} = \frac{g(x)\,\tilde{y}_1(x)\,\tilde{y}_2(x)}{(1+x^2)^{\eta_1} r_c(x)}, \quad c \in \mathbb{R}, \tag{17}$$

where $r_c(x) := c\tilde{y}_1(x) + (1-c)\tilde{y}_2(x)$*.*

Proof. (i) It follows similarly as Lemma 6 and the expressions (16) of $Y_1(\theta)$ and $Y_2(\theta)$. The details are omitted.

(ii) We can assume that we are in the situation of item (i) with $\eta_1 < \eta_2$, because if $\eta_1 = \eta_2$, we are done. Observe that

$$Y(\theta; 0) = Y_1(\theta) \quad \text{and} \quad Y(\theta; 1) = Y_2(\theta),$$

where none of the polynomials g, $\tilde{y}_1$ and $\tilde{y}_2$ can have the factor $1 + x^2$.

Therefore in order for $Y(\theta; c)$ to be a real trigonometric polynomial solution of equation (15) we must have $c = 0$, or $c = 1$, or $c \neq 0, 1$ and it is such that

- (a) either $s_c(x) := c\tilde{y}_1(x)(1+x^2)^{\eta_2-\eta_1} + (1-c)\tilde{y}_2(x)$ has no the factor $1 + x^2$ and it divides $g(x)$;
- (b) or $s_c(x) = \widehat{s}_c(x)(1+x^2)^{\sigma}$ with $0 < \sigma \in \mathbb{N}$, $\gcd(\widehat{s}_c(x), 1+x^2) = 1$ and $\widehat{s}_c(x)$ divides $g(x)$.

Let us prove that case (b) never happens. Otherwise, notice that since $\eta_2 - \eta_1 > 0$, the polynomials $s_c(x)$ and $c\tilde{y}_1(x)(1+x^2)^{\eta_2-\eta_1}$, both have the factor $(1 + x^2)$. Then $\tilde{y}_2$ would also have the factor $1 + x^2$, in contradiction with its definition.

Therefore, only case (a) happens and the degree of $Y(\theta, c), c \neq 1$ must be η_1. As a consequence, taking one of the solutions $Y(\theta; c), c \neq 0, 1$, together with $Y_1(\theta)$ both have the same degree, that is $\eta_2 = \eta_1$. Thus the expression proved in item (i) reduces to (17), as want to prove. □

Lemma 16. *Assume that equation* (15) *has at least four different trigonometric polynomial solutions and all the notations of Lemma 15. Then all the trigonometric polynomial solutions of this equation, different from* $Y_0(\theta) \equiv 0$*, have degree* η_1*, except maybe one that can have higher degree. If this solution exists, then it corresponds to a unique value of* c*,* $c = \check{c}$ *such that*

$$r_{\check{c}}(x) = \widehat{r}_{\check{c}}(x)(1+x^2)^{\nu}, \quad 0 < \nu \in \mathbb{N}.$$

Moreover, if $Y(\theta; c)$*,* $c = 0, 1, c_1, c_2 \ldots, c_k$ *denote all its trigonometric polynomial solutions of degree* η_1 *and* $Y(\theta; \check{c})$ *its trigonometric polynomial solution of higher degree* $\eta_1 + \nu$ *(if it exists). Then*

$$g(x) = \widehat{r}_{\check{c}}(x) \prod_{j=1}^{k} r_{c_j}(x)\check{g}(x), \tag{18}$$

for some polynomial $\check{g}$*. Furthermore, for all* r_{c_j} *except maybe one of them, say* $r_{c_k} = r_{\hat{c}}$ *which can be a nonzero constant polynomial, it holds that*

$$\deg r_{c_j} = \rho \geq 1, \quad j = 1, 2, \ldots, k-1. \tag{19}$$

Proof. To study the degrees of all the trigonometric polynomial solutions, observe that from equation (17) given in Lemma 15 all the solutions have degree η_1 except the ones corresponding to the values of c such that r_c has the factor $1+x^2$ and for these values of c their degrees will be greater than η_1. Let us prove that this value of c, if exists, is unique. If there were two values of $c \notin \{0,1\}$, say $c_1 \neq c_2$ then r_{c_1} and r_{c_2} evaluated at $x = \pm\mathbf{i}$ would vanish simultaneously. Some simple computations will give that also both $\tilde{y}_1$ and $\tilde{y}_2$ would vanish at $x = \pm\mathbf{i}$, a contradiction with $\gcd(\tilde{y}_1, \tilde{y}_2) = 1$.

In fact, the above reasoning proves that for $c_1 \neq c_2$, the corresponding r_{c_1} and r_{c_2} have no common roots. This shows that (18) holds. Hence the lemma follows. □

Lemma 17. *Assume that equation* (15) *has at least four different trigonometric polynomial solutions and all the notations of Lemmas 15 and 16. Denote by $N(g)$ the number of different factors of $g(x)$ decomposed into linear complex polynomial factors and let ρ be given in* (19)*. The following statements hold.*

(*i*) *If for any $c \in \mathbb{R}\backslash\{0,1\}$, r_c is never a constant multiple of $(1+x^2)^\sigma$, where $0 \leq \sigma \in \mathbb{N}$, then equation* (15) *has at most* $\min\big(N(g)+3, [\deg(g)/\rho]+3\big)$ *trigonometric polynomial solutions, where* [ ] *denotes the integer part function.*

(*ii*) *If there is exactly one $c \notin \{0,1\}$ such that r_c is a nonzero constant multiple of $(1+x^2)^\sigma$, with $0 \leq \sigma \in \mathbb{N}$, then equation* (15) *has at most* $\min\big(N(g)+4, [\deg(g)/\rho]+4\big)$ *trigonometric polynomial solutions.*

(*iii*) *If there is one c, $c = \check{c}$ such that $r_{\check{c}}(x) = p(1+x^2)^\sigma, 0 < \sigma \in \mathbb{N}$ and one c, $c = \hat{c}$ such that $r_{\hat{c}}(x) = q$ for some nonzero constants p and q, with $\check{c}, \hat{c} \notin \{0,1\}$, then equation* (15) *has at most $\eta + 2$ trigonometric polynomial solutions.*

Proof. By item (*ii*) of Lemma 15, along this proof we can assume that all nonzero solutions of the Riccati equation (15) are given by the expression (17).

(*i*) Set $r_c(x) = \widehat{r}_c(x)(1+x^2)^\nu$ with ν a nonnegative integer and $\widehat{r}_c(x)$ a nonconstant polynomial satisfying $\gcd(\widehat{r}_c(x), 1+x^2) = 1$. Since $g(x)$ is a real polynomial and $\gcd(g(x), 1+x^2) = 1$, in order that $g(x)/r_c(x)$ with $\nu = 0$ or $g(x)/\widehat{r}_c(x)$ with $\nu \neq 0$, is a polynomial, each zero x_0 of $g(x)$ must be a zero of $r_c(x)$, i.e.

$$r_c(x_0) = c\tilde{y}_1(x_0)(1+x_0^2)^{\eta_2-\eta_1} + (1-c)\tilde{y}_2(x_0) = 0.$$

Since $\gcd(\tilde{y}_1, \tilde{y}_2) = 1$ and $\gcd(\tilde{y}_2, 1+x^2) = 1$, this last equation has a unique solution c_0. Hence we have proved that there are at most $N(g)$ values of c for which $g(x)/r_c(x)$ with $\nu = 0$ or $g(x)/\widehat{r}_c(x)$ with $\nu \neq 0$ can be real polynomials. Hence the general solution (17) of equation (15) contains at most $N(g)+3$ real trigonometric polynomial solutions including the trivial one $Y_0 = 0$, and Y_1, Y_2. This proves the first part of item (*i*). The second bound given by $[\deg(g)/\rho]+3$ follows from (18) and (19).

(*ii*) The proof of this item follows adding to the maximum number of trigonometric polynomial solutions given in item (*i*) the extra one corresponding to this special value of c.

(*iii*) We have that $r_{\check{c}}(x) = p\,(1+x^2)^\sigma, 0 < \sigma \in \mathbb{N}$ and $r_{\hat{c}}(x) = q$, with p, q nonzero real numbers. Then

$$\tilde{y}_1(x) = \frac{(1-\hat{c})\,p\,(1+x^2)^\sigma - (1-\check{c})\,q}{\check{c}-\hat{c}}, \quad \tilde{y}_2(x) = \frac{-\hat{c}\,p\,(1+x^2)^\sigma + \check{c}\,q}{\check{c}-\hat{c}}.$$

As a consequence
$$r_c(x) = \frac{(c-\hat{c})\, p\,(1+x^2)^\sigma - (c-\check{c})\, q}{\check{c}-\hat{c}},$$
and then $\deg r_c = 2\sigma \geq 2$, for all $c \neq \hat{c}$. Since we are assuming the existence of the value $c = \check{c}$ for which the equation has a trigonometric polynomial solution of degree strictly greater than the degree η_1 of all the other trigonometric polynomial solutions, we know that $\eta_1 \leq \eta - \sigma \leq \eta - 1$. By Lemmas 15 and 16,
$$Y(\theta; c) = \frac{g(x)\, \tilde{y}_1(x)\, \tilde{y}_2(x)}{(1+x^2)^{\eta_1} r_c(x)} = \frac{\check{g}(x)\, \tilde{y}_1(x)\, \tilde{y}_2(x)\, \prod_{j=1}^k r_{c_j}(x)}{(1+x^2)^{\eta_1} r_c(x)},$$
where k is the number of trigonometric polynomial solutions different from $0, Y_1(\theta), Y_2(\theta)$, $Y(\theta;\check{c})$ and $Y(\theta;\hat{c})$. We claim that $k \leq \eta - 3$. If the claim holds then the maximum number of trigonometric polynomial solutions is $(\eta - 3) + 5 = \eta + 2$ as we wanted to prove.

To prove the claim, notice that by imposing that for $c = \hat{c}$, the function $Y(\theta;\hat{c})$ is a trigonometric polynomial. Since $r_{\hat{c}}(x) = q$, we get that
$$D := \deg\Big(\check{g}\, \tilde{y}_1\, \tilde{y}_2 \prod_{j=1}^k r_{c_j}\Big) \leq 2\eta_1 \leq 2\eta - 2.$$
By using that for all $j = 1, ..k$, $\deg r_{c_j} \geq 2$ and that $\deg \tilde{y}_1 = \deg \tilde{y}_2 \geq 2$, we get that $D \geq 2(k+2)$. Thus $2(k+2) \leq 2\eta - 2$ and $k \leq \eta - 3$, as we wanted to prove. ☐

Lemma 18. (i) *Consider an equation of the form* (15) *and all the notations introduced in Lemma 15. Assume that it has two trigonometric polynomial solutions* $Y_1(\theta)$ *and* $Y_2(\theta)$*, both of degree* η_1*. Then the function* $b_2(x)$ *appearing in the expression of* $B_2(\theta)$ *in the* x*-variables is*
$$b_2(x) = (1+x^2)^{\beta_2+\eta_1+1-\alpha} \frac{\big(\dot{\tilde{y}}_1(x)\tilde{y}_2(x) - \tilde{y}_1(x)\dot{\tilde{y}}_2(x)\big)a(x)}{2\, g(x)\, \tilde{y}_1(x)\tilde{y}_2(x)(\tilde{y}_1(x) - \tilde{y}_2(x))}. \tag{20}$$
(ii) *If* $\eta_1 = \eta$ *and the polynomial* $g = \gcd(y_1, y_2)$ *has degree greater than or equal to* $2\eta - 1$ *then* $b_2(x) \equiv 0$*. As a consequence, there are no trigonometric polynomial Riccati equations of the form* (15) *and degree* η *with* $B_2(\theta) \not\equiv 0$ *having two trigonometric polynomial solutions of degree* η *and such that their corresponding* g *satisfies* $\deg g \geq 2\eta - 1$*.*

Proof. (i) Recall that $Y_1(\theta) = y_1(x)/(1+x^2)^{\eta_1}$ and $Y_2(\theta) = y_2(x)/(1+x^2)^{\eta_1}$ are the solutions of equation (15). Then, we have for $i = 1, 2$,
$$\frac{a(x)}{2(1+x^2)^\alpha}\big(\dot{y}_i(x)(1+x^2) - 2\eta_1 x y_i(x)\big) = \frac{b_1(x) y_i(x)}{(1+x^2)^{\beta_1}} + \frac{b_2(x) y_i^2(x)}{(1+x^2)^{\beta_2+\eta_1}}.$$
Solving these two equations gives
$$b_2(x) = (1+x^2)^{\beta_2+\eta_1+1-\alpha} \frac{(\dot{y}_1(x) y_2(x) - y_1(x)\dot{y}_2(x))\, a(x)}{2\, y_1(x) y_2(x)\, (y_1(x) - y_2(x))}.$$
By using that $y_i(x) = g(x)\, \tilde{y}_i(x), i = 1, 2$, the desired expression for $b_2(x)$ follows.

(ii) Since $\deg Y_j = \eta_j = \eta$ it follows from Lemma 12 that $\deg B_2 = \beta_2 = 0$. Moreover since $y_j = g\tilde{y}_j$, $\deg g \geq 2\eta - 1$ and $\deg y_j \leq 2\eta$ it holds that $\deg \tilde{y}_j \leq 1$. Furthermore, by

Lemma 14 we get that $\deg a \geq 2\eta - 1$ and as a consequence $\deg A = \alpha = \eta$. Putting all together, by using item (i) and that $B_2(\theta)$ must be a constant, $B_2(\theta) \equiv p \in \mathbb{R}$, we get that

$$2\, p\, g(x)\, \tilde{y}_1(x)\tilde{y}_2(x)(\tilde{y}_1(x) - \tilde{y}_2(x)) = (1+x^2)\big(\dot{\tilde{y}}_1(x)\tilde{y}_2(x) - \tilde{y}_1(x)\dot{\tilde{y}}_2(x)\big)a(x). \tag{21}$$

Recall that the polynomial g has no the factor $1 + x^2$. Moreover $\tilde{y}_1$, $\tilde{y}_2$, and $\tilde{y}_1 - \tilde{y}_2$ have at most degree 1, so they neither have this quadratic factor. Finally, notice that $\dot{\tilde{y}}_1(x)\tilde{y}_2(x) - \tilde{y}_1(x)\dot{\tilde{y}}_2(x)$ is a real constant, which is not zero unless both $\tilde{y}_j$ are of degree zero, because recall that $\gcd(\tilde{y}_1, \tilde{y}_2) = 1$.

Hence the above equation (21) has only the solution $p = 0$ when both $\tilde{y}_j$ are of degree zero and otherwise it is not possible. □

Lemma 19. *Consider an equation of the form* (15) *with* $\eta \geq 2$, $B_2(\theta) \not\equiv 0$, *having at least four different trigonometric polynomial solutions, and all the notations introduced in Lemma 15. Assume that it has two trigonometric polynomial solutions* $Y_1(\theta)$ *and* $Y_2(\theta)$, *both of degree* η. *Moreover suppose that there exists* $c_0 \notin \{0, 1\}$ *such that* $r_{c_0}(x) = c_0\tilde{y}_1(x) + (1 - c_0)\tilde{y}_2(x) = q(1 + x^2)^\sigma$ *for some* $0 \neq q \in \mathbb{R}$, $0 \leq \sigma \in \mathbb{N}$ *and that* $\deg g \in \{2\eta - 2, 2\eta - 3\}$. *Then the number of trigonometric polynomial solutions of* (15) *is at most* $\eta + 2$.

Proof. Since $\deg Y_j = \eta$, by Lemma 12, $\deg B_2 = 0$ and $b_2(x) \equiv p \neq 0$. In this case, using the same ideas as in the proof of item (i) of Lemma 18, we get the expression

$$2\, p\, g(x)\, \tilde{y}_1(x)\tilde{y}_2(x)(\tilde{y}_1(x) - \tilde{y}_2(x)) = (1+x^2)^{\eta-\alpha+1}\big(\dot{\tilde{y}}_1(x)\tilde{y}_2(x) - \tilde{y}_1(x)\dot{\tilde{y}}_2(x)\big)a(x).$$

that is essentially the same that in (21). Since none of the functions $\tilde{y}_1$, $\tilde{y}_2$, g has the factor $1+x^2$, by imposing that the above equality holds we get that there exists a nonzero polynomial $k(x)$, of degree at most one, such that $\tilde{y}_1(x) - \tilde{y}_2(x) = k(x)(1 + x^2)$. This equality together with the assumption $r_{c_0}(x) = c_0\tilde{y}_1(x) + (1 - c_0)\tilde{y}_2(x) = q(1 + x^2)^\sigma$ implies that

$$\tilde{y}_2(x) = q(1 + x^2)^\sigma - c_0 k(x)(1 + x^2).$$

Since $\gcd(\tilde{y}_2, 1 + x^2) = 1$ the above equality implies that $\sigma = 0$ and as a consequence

$$\tilde{y}_2(x) = q - c_0 k(x)(1 + x^2) \quad \text{and} \quad \tilde{y}_1(x) = q + (1 - c_0)k(x)(1 + x^2).$$

Hence, by Lemma 15 the general solution of the Riccati equation is

$$Y(\theta; c) = \frac{g(x)\tilde{y}_1(x)\tilde{y}_2(x)}{(1 + x^2)^\eta r_c(x)} = \frac{g(x)\tilde{y}_1(x)\tilde{y}_2(x)}{(1 + x^2)^\eta\big(q + (c - c_0)k(x)(1 + x^2)\big)}.$$

Notice that the solution corresponding to $c = c_0$ is never a polynomial because the degree of the numerator is at least $2\eta-3+4 > 2\eta$. Hence, following the notations of Lemma 17, for $c \neq c_0$, $\deg r_c = \rho \geq 2$, and then the maximum number of polynomial solutions is $[\deg(g)/2] + 3 \leq \eta + 2$, where the 3 corresponds to the solutions $Y = 0$, $Y_1(\theta)$, and $Y_2(\theta)$. □

Lemma 20. *Consider an equation of the form* (15) *with* $B_2(\theta) \not\equiv 0$, *having at least four different trigonometric polynomial solutions, and all the notations introduced in Lemma 15. Assume that it has two solutions* $Y_1(\theta)$ *and* $Y_2(\theta)$ *of degree* $\eta \geq 1$ *and such that* $\deg g = 2\eta - 2$. *Then* $\eta \geq 2$ *and the number of trigonometric polynomial solutions of* (15) *is at most* $\eta + 2$.

Proof. Set
$$y_1(x) = g(x)\tilde y_1(x), \quad y_2(x) = g(x)\tilde y_2(x), \quad a(x) = g(x)\tilde a(x)$$
where $\gcd(\tilde y_1, \tilde y_2) = 1$, $\deg \tilde y_1, \deg \tilde y_2, \deg \tilde a \le 2$, and none of these three polynomials have the factor $1+x^2$. Set $\tilde a(x) = a_2x^2 + a_1x + a_0$.

Notice that it is not possible that $\deg \tilde y_1 = \deg \tilde y_2 = 0$ because by (20) in Lemma 18 this fact implies that $B_2(\theta) \equiv 0$, a contradiction.

Since $\deg(Y_1) = \deg(Y_2) = \eta$ we know by Lemma 12 that $B_2(\theta) \equiv p \in \mathbb{R} \setminus \{0\}$. Then, by Lemma 18,
$$2\,p\,g(x)\,\tilde y_1(x)\tilde y_2(x)(\tilde y_1(x) - \tilde y_2(x)) = (1+x^2)^{\eta-\alpha+1}\big(\dot{\tilde y}_1(x)\tilde y_2(x) - \tilde y_1(x)\dot{\tilde y}_2(x)\big)a(x).$$
Moreover, since g and $\tilde y_i$ have no the factor $1+x^2$, we get that $\alpha = \eta$ and
$$\tilde y_1(x) - \tilde y_2(x) = q\,(1+x^2) \tag{22}$$
for some $q \in \mathbb{R} \setminus \{0\}$ and the above equality simplifies to
$$2\,p\,q\,\tilde y_1(x)\tilde y_2(x) = \big(\dot{\tilde y}_1(x)\tilde y_2(x) - \tilde y_1(x)\dot{\tilde y}_2(x)\big)\tilde a(x).$$
Using (22) we get
$$2\,p\,\big(\tilde y_2(x) + q(1+x^2)\big)\tilde y_2(x) = \big(2\,x\,\tilde y_2(x) - (1+x^2)\dot{\tilde y}_2(x)\big)\tilde a(x),$$
or equivalently,
$$\big(2\,p\,\tilde y_2(x) - 2\,x\,\tilde a(x)\big)\tilde y_2(x) = -(1+x^2)\big(\dot{\tilde y}_2(x)\tilde a(x) + 2\,p\,q\tilde y_2(x)\big). \tag{23}$$
Since $\gcd(\tilde y_2, 1+x^2) = 1$, the above equality can only happen if
$$2\,p\,\tilde y_2(x) - 2\,x\,\tilde a(x) = s(x)\,(1+x^2), \quad \text{where} \quad s(x) = s_1x + s_0,\ s_0, s_1 \in \mathbb{R}.$$
From it we get that
$$\tilde y_2(x) = \frac{s(x)}{2p}(1+x^2) + \frac{x}{p}\,\tilde a(x), \quad \tilde y_1(x) = \Big(\frac{s(x)}{2p} + q\Big)(1+x^2) + \frac{x}{p}\,\tilde a(x),$$
and moreover $s_1 = -2a_2$ because $\deg \tilde y_i \le 2, i = 1, 2$. Then
$$r_c(x) = \Big(\frac{s(x)}{2p} + c\,q\Big)(1+x^2) + \frac{x}{p}\,\tilde a(x), \qquad \deg(r_c) \le 2,$$
and by Lemma 15,
$$Y(\theta; c) = \frac{g(x)\tilde y_1(x)\tilde y_2(x)}{(1+x^2)^\eta r_c(x)}.$$
Notice that for all $c \ne c_0 := -(2a_1 + s_0)/(2pq)$, $\deg r_c = 2$ and $\deg r_{c_0} < 2$. Moreover $\gcd(r_c(x), 1+x^2) = 1$, because otherwise, $1+x^2$ would be a factor of $\tilde a$. Since the degree of g is $2\eta - 2$, by item *(ii)* of Lemma 17 the maximum number of trigonometric polynomial solutions is $(2\eta-2)/2 + 4 = \eta + 3$.

To reduce this upper bound by 1 we have to continue our study. First notice that if the degree of r_{c_0} is 1 then r_{c_0} should also divide $g(x)$ and then
$$Y(\theta; c) = \frac{\breve g(x) r_{c_0}(x)\tilde y_1(x)\tilde y_2(x)}{(1+x^2)^\eta r_c(x)},$$
with $\deg(\breve g) = 2\eta - 3$. As a consequence, $\eta \ge 2$. By using the same arguments that in the proof of Lemma 17 we get that the maximum number of trigonometric polynomials

solutions in this case is at most $[\deg(\breve{g})/2]+4=[(2\eta-3)/2]+4=\eta+2$, where the 4 counts the solution 0 and the ones corresponding to $c\in\{0,1,c_0\}$. Then the result is proved when $\deg(r_{c_0})=1$.

Finally, we will prove that there is no $c_0\notin\{0,1\}$ such that $\deg r_{c_0}=0$. Imposing that r_{c_0} is constant we get that

$$s_0=-2(c_0\,p\,q+a_1),\quad a_2=a_0,\quad\text{and}\quad r_{c_0}(x)=-a_1/p.$$

Then $\tilde{a}(x)=a_0\,x^2+a_1\,x+a_0$ and $s(x)=-2\,a_0\,x-2(c_0\,p\,q+a_1)$. Substituting the above expressions into the function

$$W(x):=\big(2\,p\,\tilde{y}_2(x)-2\,x\,\tilde{a}(x)\big)\tilde{y}_2(x)+(1+x^2)\big(\dot{\tilde{y}}_2(x)\tilde{a}(x)+2\,p\,q\tilde{y}_2(x)\big),$$

obtained from (23), we get that

$$W(x)=\frac{2}{p}(1+x^2)\Big(c_0\,(c_0-1)\,p^2q^2x^2+a_0\,a_1\,x+((c_0-1)\,p\,q+a_1)\,(c_0\,p\,q+a_1)\Big)$$

and we know that it must vanish identically. The only solutions compatible with our hypotheses are either $a_1=p\,q$ and $a_0=c_0=0$ or $a_1=-p\,q$, $a_0=0$ and $c_0=1$. In both situations the value c_0 is either 0 or 1 and hence there are no solutions different from Y_1 or Y_2 such that r_{c_0} is constant.

Therefore, we have shown that in all the situations the Riccati equation has at most $\eta+2$ trigonometric polynomial solutions. Hence $\eta\geq 2$ and the lemma follows. □

Proof of the upper bound given in Theorem 2 when $\eta\geq 2$. Recall that we want to prove that trigonometric Riccati equations of degree $\eta\geq 2$ have at most 2η trigonometric polynomial solutions. By using Lemma 13 we can restrict our attention to equation (15). Moreover, we can assume that this equation has at least four trigonometric polynomial solutions ($\eta\geq 2$), because if not we are done. Therefore we are always under the assumptions of Lemma 17 and, apart of the solution $Y_0(\theta)\equiv 0$, we can suppose that the equation has two trigonometric polynomial solutions $Y_1(\theta)$ and $Y_2(\theta)$, both of degree $\eta_1\leq\eta$. Then the corresponding y_1 and y_2 given in expressions (16) have degree at most $2\eta_1$. By the definition of g in Lemma 15 we have $\deg g\leq 2\eta_1$ and so $N(g)\leq 2\eta_1\leq 2\eta$.

In item (iii) of Lemma 17 we have also proved that when the situation described there happens (that is the existence of one c, $c=\check{c}$, such that $r_{\check{c}}(x)=p(1+x^2)^\sigma$, $0<\sigma\in\mathbb{N}$ and one c, $c=\hat{c}$, such that $r_{\hat{c}}(x)=q$ for some nonzero constants p and q) then equation (15) has at most $\eta+2$ trigonometric polynomial solutions. Since for $\eta\geq 2$ it holds that $\eta+2\leq 2\eta$ and we do not need to consider this situation anymore. Hence, by items (i) and (ii) of Lemma 17, when $N(g)\leq 2\eta-4$ we have proved the upper bound given in the statement.

Moreover, since $\deg g\geq N(g)$, by using Lemma 18 we also know that the upper bound holds when $N(g)\in\{2\eta,2\eta-1\}$, because either they correspond to $B_2(\theta)\equiv 0$ or with a Riccati equation with at most 4 trigonometric polynomial solutions.

We will prove the result for the remaining situations by a case by case study, according whether $N(g)=2\eta-2$ or $N(g)=2\eta-3$.

Observe also that we never have to consider the situations where $\deg\tilde{y}_1=\deg\tilde{y}_2=0$ because by (20) in Lemma 18 this fact implies that $B_2(\theta)\equiv 0$, a contradiction.

• [*Case* $N(g) = 2\eta - 2$]: We have that $\eta_1 = \eta$. Let $g_1(x), \dots, g_{2\eta-2}(x)$ be the $2\eta - 2$ different linear factors of $g(x)$. Again by Lemma 18 we do not need to consider the cases $\deg g \geq 2\eta - 1$. Then $g(x) = g_1(x) \cdots g_{2\eta-2}(x)$. We are precisely under the situation of Lemma 20 and the upper bound is at most $\eta + 2 \leq 2\eta$, as we wanted to see.

• [*Case* $N(g) = 2\eta - 3$]: By item (i) of Lemma 17, if for any $c \in \mathbb{R}$, $c\tilde{y}_1(x) + (1-c)\tilde{y}_2(x)$ is not a constant multiple of $(1+x^2)^\sigma$ with σ is a nonnegative integer, then equation (15) has at most 2η real trigonometric polynomial solutions. Therefore, we can assume that there exists a $c_0 \in \mathbb{R} \setminus \{0, 1\}$ such that $c_0\tilde{y}_1(x) + (1-c_0)\tilde{y}_2(x) = q(1+x^2)^\sigma$ with $q \neq 0$ a nonzero constant and σ a nonnegative integer and that this c_0 is unique, see Lemma 16 and item (iii) of Lemma 17.

Notice that $\deg g \geq N(g) = 2\eta - 3$ and recall that we only need to consider the cases $\deg g \leq 2\eta - 2$. Then $\eta_1 \in \{\eta, \eta - 1\}$.

▶ [*Subcase* $\deg(g) = 2\eta - 2$ *and* $\eta_1 = \eta - 1$]: It never holds because these hypotheses imply that $\deg \tilde{y}_1 = \deg \tilde{y}_2 = 0$.

▶ [*Subcase* $\deg(g) = 2\eta - 2$ *and* $\eta_1 = \eta$]: We can assume that $\max(\deg \tilde{y}_1, \deg \tilde{y}_2) > 0$. Then, by Lemma 19 the maximum number of trigonometric polynomial solution is $\eta + 2 \leq 2\eta$, for $\eta \geq 2$, as we wanted to prove.

▶ [*Subcase* $\deg(g) = 2\eta - 3$ *and* $\eta_1 = \eta - 1$]: We know that $\max(\deg \tilde{y}_1, \deg \tilde{y}_2) = 1$. In fact, the existence of $c_0 \notin \{0, 1\}$ such that $r_{c_0}(x) = q(1 + x^2)^\sigma, q \neq 0$ implies that $\sigma = 0$ and $\deg \tilde{y}_1 = \deg \tilde{y}_2 = 1$. Now, by item (ii) of Lemma 15,

$$Y(\theta; c) = \frac{g(x)\,\tilde{y}_1(x)\,\tilde{y}_2(x)}{(1 + x^2)^{\eta-1} r_c(x)}.$$

In this situation, for this special value $c = c_0$, the corresponding $Y(\theta; c_0)$ is not a trigonometric polynomial because $\deg(g\,\tilde{y}_1\,\tilde{y}_2) = 2\eta - 3 + 2 = 2\eta - 1 > 2(\eta - 1)$. Then by item (i) of Lemma 17 the number of trigonometric polynomial solutions in this case is at most $N(g) + 3 = 2\eta$.

▶ [*Subcase* $\deg(g) = 2\eta - 3$ *and* $\eta_1 = \eta$]: Once more, we know that $\max(\deg \tilde{y}_1, \deg \tilde{y}_2) = 1$. This situation is again covered by Lemma 19 and the maximum number of trigonometric polynomial solutions is $\eta + 2 \leq 2\eta$, for $\eta \geq 2$. □

Examples for proving Theorem 2 with $\eta \geq 2$. The above proofs show that equation (15) can have 2η trigonometric polynomial solutions only in the case $N(g) = \deg g = 2\eta - 3$.

Next we provide examples showing that there exist equations (15) which have exactly $k + 3$ trigonometric polynomial solutions with $k \in \{1, \dots, 2\eta - 3\}$. Taking $g(x) = \prod_{i=1}^{k}(x - c_i)$ with $c_i \in \mathbb{R}$ and $c_i \neq c_j$ for $1 \leq i \neq j \leq k$. Set

$$A(\theta) = \frac{2(x - d_1)(x - d_2)g(x)}{(1 + x^2)^\eta}, \quad B_1(\theta) = \frac{g(x)h_1(x) + \dot{g}(x)h_2(x)}{(1 + x^2)^\eta}, \quad B_2(\theta) = -1, \tag{24}$$

where d_1 and d_2 are two different real constants and are different from c_i for $i = 1, \dots, k$, and $h_1(x) = (1 + x^2)(2x - d_1 - d_2) - 2(\eta - 1)x(x - d_1)(x - d_2)$ and $h_2(x) = (1 + x^2)(x - d_1)(x - d_2)$. We can check that equation (15) with the prescribed A, B_1, B_2 given in

(24) has the trigonometric polynomial solutions

$$Y_1(\theta) = \frac{g(x)(x-d_1)}{(1+x^2)^{\eta-1}}, \qquad Y_2(\theta) = \frac{g(x)(x-d_2)}{(1+x^2)^{\eta-1}}.$$

Then, by Lemma 15,

$$Y(\theta;c) = \frac{y_1(x)y_2(x)}{cy_1(x)+(1-c)y_2(x)} = \frac{\prod_{i=1}^{k}(x-c_i)(x-d_1)(x-d_2)}{(1+x^2)^{\eta-1}(x-(cd_1+(1-c)d_2))}.$$

Taking

$$c=0, \quad 1, \quad \frac{c_i-d_2}{d_1-d_2}, \quad i=1,\ldots,k,$$

we get $k+3$ trigonometric polynomial solutions of equation (15) counting also the trivial solution $Y_0=0$. Clearly for any other c, the solution $Y(\theta;c)$ cannot be a trigonometric polynomial.

Of course, in the above construction if we take g a nonzero constant, then the equation has exactly three trigonometric polynomial solutions. □

Proof of Theorem 2 when $\eta=1$. The simple Riccati equation

$$\sin\theta\, Y' = 2\cos\theta\, Y - Y^2,$$

has three trigonometric polynomial solutions $Y=0$, $Y=1+\cos\theta$, and $Y=-1+\cos\theta$. Hence we know that when $\eta=1$ the number of trigonometric polynomial solutions is at least 3.

Let us prove that 3 is also the upper bound. Otherwise, assume that there are Riccati equations with $\eta=1$ and four trigonometric polynomial solutions to arrive to a contradiction. Therefore, we are under the hypotheses of item (ii) of Lemma 15 and we can suppose that the equation has two nonzero solutions of the same degree $\eta_1\in\{0,1\}$. If $\eta_1=0$, then y_1 and y_2 are both constants. By Lemma 18, this forces that $B_2(\theta)\equiv 0$, a contradiction.

Hence we can assume that $\eta_1=1$. By Lemma 12, $B_2(\theta)\equiv p\neq 0$. By using once more the same ideas that in the proof of item (i) of Lemma 18, we get that

$$2\,p\,g(x)\,\tilde y_1(x)\tilde y_2(x)(\tilde y_1(x)-\tilde y_2(x)) = (1+x^2)^{2-\alpha}\big(\dot{\tilde y}_1(x)\tilde y_2(x)-\tilde y_1(x)\dot{\tilde y}_2(x)\big)a(x).$$

Then $\alpha=1$, and $\tilde y_1-\tilde y_2$ is a constant multiple of $1+x^2$. In particular, one of $\tilde y_1$ and $\tilde y_2$ has degree 2 and so $\deg g=0$. Therefore, we are under the hypotheses of Lemma 20 and we get that $\eta\geq 2$, again a contradiction. Then the result follows. □

Acknowledgments

The first and second authors are partially supported by the MINECO MTM2013-40998-P and AGAUR 2014SGR568 grants. The third author is partially supported by the NNSF of China 11271252 and the Innovation program of Shanghai Municipal Education Commission of China 15ZZ02. All authors are partially supported by the European Community FP7-PEOPLE-2012-IRSES-316338 grant.

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