PIECEWISE LINEAR DIFFERENTIAL SYSTEMS WITH AN ALGEBRAIC LINE OF SEPARATION

ARMENGOL GASULL, JOAN TORREGROSA, AND XIANG ZHANG

Abstract. We study the number of limit cycles of planar piecewise linear differential systems separated by a branch of an algebraic curve. We show that for each $n \in \mathbb{N}$ there exist piecewise linear differential systems separated by an algebraic curve of degree n having $[n/2]$ hyperbolic limit cycles. Moreover, when $n = 2, 3$ we study in more detail the problem, considering a perturbation of a center and constructing examples with 4 and 5 limit cycles, respectively. These results follow by proving that the set of functions generating the first order averaged function associated to the problem is an extended complete Chebyshev system in a suitable interval.

1. Introduction and statement of the main results

For planar polynomial differential systems there still stands the unsolved Hilbert's 16th problem, whose second part asks for the maximum number of limit cycles and their distribution in terms of their degree, see for instance [9, 11]. As we know, planar linear differential systems have no limit cycle, whereas the situation is different for piecewise linear differential systems. For them, in the case when they are defined in two zones separated by a straight line, it is known that there are examples with 3 limit cycles, see [4, 7, 8, 13]. It is yet an open problem to know the maximum number of limit cycles that this type of systems can have.

We address to a very related question. Let $\mathcal{PL}_n$ be the set of planar piecewise linear differential systems with two zones separated by a branch of an algebraic curve of degree n, C_n, where in each of the two zones the differential system is linear. As usual, limit cycles will be periodic solutions isolated in the set of all periodic solutions. Here, the definition of solution in the two zones scenario is the one given in [5, 6, 15]. We remark that in this setting there are crossing and sliding limit cycles. In this paper, when we refer to limit cycles we will mean the ones of crossing type, even when we do not explicitly mention it. Recall that this type of periodic solutions cut the discontinuity curve only at finitely many points, both vector fields are transversal to this curve at each of these points and, moreover, both vectors fields point towards the same component of $\mathbb{R}^2 \setminus C_n$ at each of them.

Let $L_n \in \mathbb{N} \cup \{\infty\}$ be the maximum number of limit cycles that systems inside class $\mathcal{PL}_n$ can have. Our first main result, proved in Section 2, implies that L_n tends to infinity with n.

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Theorem 1. *For $n \in \mathbb{N}$, there exists a polynomial $f_n(x)$ of degree n and a piecewise linear differential system with two zones separated by the curve $y = f_n(x)$, such that the corresponding piecewise linear differential system has at least $[n/2]$ hyperbolic limit cycles, where $[\cdot]$ denotes the integer part function. As a consequence, $L_n \geq [n/2]$.*

The above theorem does not aim to give the optimal lower bound for L_n, but simply to prove that L_n goes to infinity with n. As far as we know, this result is new.

It is worthwhile to comment here that there are already results proving that systems with two zones can have an arbitrary large number of limit cycles (or even infinitely many), see [3, 16, 18], but in all these previous works the separation curve is neither polynomial, nor analytic.

Using similar arguments as in the proof of Theorem 1 we can get the next result.

Proposition 2. *There exists an analytic function $f(x)$ and a piecewise linear differential system with two zones separated by the curve $y = f(x)$, such that the corresponding piecewise linear differential system has infinitely many limit cycles.*

Notice that in particular, Theorem 1 gives no information when $n = 1$, although it is known that $L_1 \geq 3$. Next, we study in more detail the cases where the separation curve is a branch C_n of a quadratic curve ($n = 2$) or a cubic curve ($n = 3$). This curve is contained in the set $\{(x, y) \mid F_n(x, y) = 0\}$, for some irreducible polynomial F_n, of degree n.

For tackling this problem, and to fix a generic situation, we treat the case in which C_n is tangent to the x–axis at the origin, and it is locally convex, see Figure 3. The region above C_n is denoted by Ω_n^+ and the one below is denoted by Ω_n^-. To improve the lower bounds for L_n, $n = 2, 3$, we consider the piecewise linear differential system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} y + \varepsilon(a_0^+ + a_1^+ x + a_2^+ y) \\ -x + \varepsilon(b_0^+ + b_1^+ x + b_2^+ y) \end{pmatrix}, & \text{for } (x, y) \in \Omega_n^+, \\ \begin{pmatrix} y + \varepsilon(a_0^- + a_1^- x + a_2^- y) \\ -x + \varepsilon(b_0^- + b_1^- x + b_2^- y) \end{pmatrix}, & \text{for } (x, y) \in \Omega_n^-, \end{cases} \tag{1}$$

which is a perturbation of a global linear center. As we will see in Proposition 5, given any C_n as above, if we denote by M_{C_n} the first order Melnikov function associated to this vector field, it is a linear combination, with free parameters, of 6 functions. The expression of M_{C_n} is already obtained in [12].

Our main result is based on proving that the functions defining M_{C_n} form an extended complete Chebyshev system (ECT-system). In Section 3 we recall this concept and its utility in our context.

Theorem 3. *Let the separation curve C_n be tangent to the x–axis at the origin and locally convex. Let M_{C_n} be the first order Melnikov function associated to* (1)*.*

(*a*) *If $n = 2$ and C_2 is a symmetric conic with respect to the y–axis then the functions defining M_{C_2} form an ECT-system formed by 4 functions in the whole interval of definition of M_{C_2}. Then $L_2 \geq 3$.*

(*b*) *There exist C_2 non symmetric conics with respect to the y–axis, being an ellipse, a parabola or a hyperbola, such that the corresponding functions defining*

M_{C_2} form an ECT-system formed by 5 functions in some interval $(0,\delta]$, for δ small enough. Consequently $L_2 \geq 4$.

(c) *There exists C_3, non symmetric cubic with respect to the y–axis, such that the functions defining M_{C_3} form an ECT-system formed by 6 functions in some interval $(0,\delta]$, for δ small enough. Then $L_3 \geq 5$.*

The case in which the separation curve is the symmetric parabola, $F_2(x,y) = y - x^2$, is also studied in [14]. In that paper the authors also prove that 3 limit cycles appear. They use that the 4 functions defining the first order Melnikov function are linearly independent. Recently, in [2], a second order study has been developed and applied to the symmetric cubic curve, $F_3(x,y) = y - x^3$. In that work the lower bound for L_3 has been increased to 7.

2. Proof of Theorem 1 and Proposition 2.

2.1. **Proof of Theorem 1.** Take $f_n(x,\varepsilon) = \varepsilon T_n(x)$, with $\varepsilon > 0$ small enough, where $T_n(x)$ is the n-th Chebyshev polynomial of the first kind, i.e. for $|x| \leq 1$,

$$T_n(x) = \cos(n \arccos x),$$

and for $|x| > 1$, its analytic extension. It is known that $T_n(x)$ has degree n and all its roots are in $[-1,1]$. Now $y = f_n(x,\varepsilon)$ separates the (x,y) plane in two zones, denoted by Ω^+ when $y \geq f_n(x,\varepsilon)$ and Ω^- when $y \leq f_n(x,\varepsilon)$.

Inside Ω^+ and Ω^- we consider respectively the linear differential system

$$(\dot{x},\dot{y}) = \begin{cases} (x - 4y - 2, x/2 - y), & \text{in } \Omega^+, \\ (-y+1, x), & \text{in } \Omega^-, \end{cases} \tag{2}$$

which form a piecewise linear differential system in the full plane. Some easy calculations show that system (2) has the first integrals

$$H^+(x,y) = 8y + x^2 - 4xy + 8y^2 \quad \text{and} \quad H^-(x,y) = -2y + x^2 + y^2,$$

in Ω^+ and Ω^-, respectively.

Notice that the level curves $H^+(x,y) = a^2$ and $H^-(x,y) = b^2$, with $a > 0, b > 0$, are ellipses and circles, respectively. Their intersection points with the x–axis are both symmetric with respect to the origin. So the piecewise linear differential system (2) with the separation curve $y - f_n(x,0) = 0$ has a center at the origin. We can check that the ellipses $H^+(x,y) = a^2$ have positive slopes at the intersection points with $y = 0$, whereas the circles $H^-(x,y) = b^2$ are perpendicular to $y = 0$.

Recall that the Chebyshev polynomial $T_n(x)$ has exactly n zeros, all located in the interval $[-1,1]$. Moreover, for n even, the polynomial $T_n(x)$ is even, and its zeros are symmetric with respect to the origin. For n odd, the polynomial $T_n(x)$ is odd, and its $n-1$ zeros except $x = 0$ are also symmetric with respect to the origin.

We claim that for $\varepsilon > 0$ small enough, the piecewise linear differential system (2) with the separation curve $y - f_n(x,\varepsilon) = 0$ has at least $[n/2]$ hyperbolic limit cycles. Moreover, when n is even, the system has exactly $[n/2]$ limit cycles, all them hyperbolic. Figure 1 illustrates the separation curve and the limit cycles for n even.

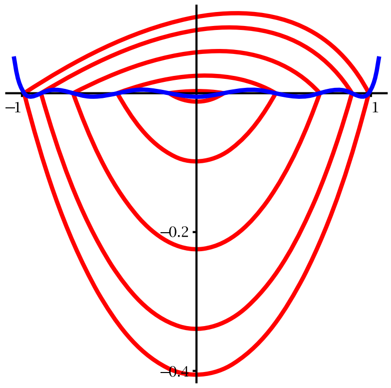

FIGURE 1. Separation curve defined by a Chebyshev polynomial and the limit cycles for $n = 10$.

We now prove this claim. Set $m = [(n-2)/2]$ and let $\pm x_0, \dots, \pm x_m$ be the $2(m+1)$ not null zeros of $f_n(x, \varepsilon)$. Then

$$x_k = \cos\left(\frac{2k+1}{2n}\pi\right), \qquad k = 0, 1, \dots, m.$$

Clearly, for each $k \in \{0, 1, \dots, m\}$,

$$\Gamma_k := \{(x,y)|\ H^+(x,y) = H^+(P_{\pm k}),\ y \geq 0\} \cup \{(x,y)|\ H^-(x,y) = H^-(P_{\pm k}),\ y \leq 0\},$$

is a periodic orbit of system (2), where $P_{\pm k} = (\pm x_k, 0)$ for $k \in \{0, 1, \dots, m\}$.

We now prove that Γ_k is a hyperbolic limit cycle of the piecewise linear differential system (2). For doing so, we compute the derivative of the Poincaré map associated to the periodic orbit Γ_k. We take the transversal sections of the flows of the two linear differential systems defined in (2) near P_k and P_{-k} as two small enough segments Σ_k and Σ_{-k} in $y = f_n(x, \varepsilon)$ centered at P_k and P_{-k}, respectively. Notice that for $\varepsilon > 0$ suitably small, both segments Σ_k and Σ_{-k} are transversal to the flows of the two linear systems. Denote by Π_k^- the Poincaré map from Σ_{-k} to Σ_k along the flow of the linear system in Ω^-, and by Π_k^+ the Poincaré map from Σ_k to Σ_{-k} along the flow of the linear system in Ω^+. Then the composition

$$\Pi_k = \Pi_k^+ \circ \Pi_k^-$$

is the Poincaré map of the piecewise linear differential system (2) near the periodic orbit Γ_k defined on Σ_{-k}.

For computing the derivative of the Poincaré map Π_k, we use the next general formula (see [1])

$$\Pi'(0) = \frac{\langle X(0), (\gamma_0'(0))^\perp \rangle}{\langle X(T), (\gamma_1'(0))^\perp \rangle} \exp\left(\int_0^T \operatorname{div} X(\varphi(t))\, dt\right), \tag{3}$$

where $\gamma_0(s)$ and $\gamma_1(s)$ are the local expressions of two transversal sections Σ_0 and Σ_1 to a class $\mathcal{C}^1$ vector field X, T is the flying time from $P = \gamma_0(0)$ to $\Pi(P) = \gamma_1(0)$, and $\Pi : \Sigma_0 \to \Sigma_1$ is the map induced by the flow φ of X, see Figure 2. Here, as usual, $\langle \cdot, \cdot \rangle$ is the inner product of two vectors and $(u,v)^\perp = (-v, u)$.

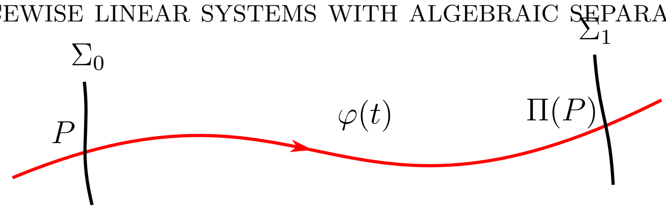

FIGURE 2. The map Π

Denote by X^+ and X^- the two vector fields associated to systems defined in (2), respectively. We parameterize the two segments Σ_k and Σ_{-k} by $(x, f_n(x,\varepsilon))$ with $x \in (x_k-\delta, x_k+\delta)$ and $x \in (-x_k-\delta, -x_k+\delta)$, respectively, where $\delta>0$ is suitably small. Since the divergences of the two linear systems defined in (2) identically vanish, by using (3) we have

$$
\begin{aligned}
(\Pi_k^+)'(x_k) &= \frac{\langle X^+(x_k,0),(1,f_n'(x_k,\varepsilon))^\perp\rangle}{\langle X^+(-x_k,0),(1,f_n'(-x_k,\varepsilon))^\perp\rangle} \\
&= \frac{\langle (x_k-2, x_k/2),(\varepsilon T_n'(x_k),-1)\rangle}{\langle(-x_k-2,-x_k/2),(\varepsilon T_n'(-x_k),-1)\rangle} = \frac{(x_k-2)\varepsilon T_n'(x_k)-x_k/2}{-(x_k+2)\varepsilon T_n'(-x_k)+x_k/2},
\end{aligned}
$$

and

$$
\begin{aligned}
(\Pi_k^-)'(-x_k) &= \frac{\langle X^-(-x_k,0),(1,f_n'(-x_k,\varepsilon))^\perp\rangle}{\langle X^-(x_k,0),(1,f_n'(x_k,\varepsilon))^\perp\rangle} \\
&= \frac{\langle(1,-x_k),(\varepsilon T_n'(-x_k),-1)\rangle}{\langle(1,x_k),(\varepsilon T_n'(x_k),-1)\rangle} = \frac{\varepsilon T_n'(-x_k)+x_k}{\varepsilon T_n'(x_k)-x_k}.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\Pi_k'(x_{-k}) &= (\Pi_k^+)'(x_k)(\Pi_k^-)'(x_{-k}) \\
&= \frac{\varepsilon(x_k-2)T_n'(x_k)-x_k/2}{-\varepsilon(x_k+2)T_n'(-x_k)+x_k/2}\cdot\frac{\varepsilon T_n'(-x_k)+x_k}{\varepsilon T_n'(x_k)-x_k} \\
&= 1-\varepsilon\left(T_n'(x_k)\left(2-\frac{5}{x_k}\right)-T_n'(-x_k)\left(2+\frac{5}{x_k}\right)\right)+O(\varepsilon^2).
\end{aligned}
$$

By using the expression of x_k, one gets that for $k\in\{0,1,\dots,m\}$,

$$
\begin{aligned}
T_n'(x_k) &= \frac{n\sin(n\arccos x_k)}{\sqrt{1-x_k^2}} = \frac{(-1)^k n}{\sqrt{1-x_k^2}}, \\
T_n'(-x_k) &= \frac{n\sin(n\arccos(-x_k))}{\sqrt{1-x_k^2}} = \frac{(-1)^k(-1)^{n+1}n}{\sqrt{1-x_k^2}}.
\end{aligned}
$$

As a consequence,

$$\begin{aligned}\Pi_k'(x_{-k}) &= 1+\varepsilon\frac{n(-1)^k\big(2\left(\left(-1\right)^{n+1}-1\right)x_k+5\left(\left(-1\right)^{n+1}+1\right)\big)}{x_k\sqrt{1-x_k^2}}+O(\varepsilon^2)\\ &= \begin{cases} 1-\varepsilon\dfrac{4n(-1)^k}{\sqrt{1-x_k^2}}+O(\varepsilon^2), & \text{for } n \text{ even},\\ 1+\varepsilon\dfrac{10n(-1)^k}{x_k\sqrt{1-x_k^2}}+O(\varepsilon^2), & \text{for } n \text{ odd}.\end{cases}\end{aligned}$$

The above computations imply:

- For each $k\in\{0,1,\ldots,m\}$ and $\varepsilon>0$ suitably small, Γ_k is a hyperbolic limit cycle of system (2).
- For n even, Γ_{2s} are stable, and $\Gamma_{2\ell+1}$ are unstable for $s,\ell\in\mathbb{Z}$ and $0\le 2s, 2\ell+1\le m$.
- For n odd, Γ_{2s} are unstable, and $\Gamma_{2\ell+1}$ are stable for $s,\ell\in\mathbb{Z}$ and $0\le 2s,2\ell+1\le m$.

These facts prove the first part of the claim.

Let us prove the second part, i.e. that when n is even and $\varepsilon>0$ is small enough, system (2), with the separation curve $y-f_n(x,\varepsilon)=0$, has exactly $[n/2]$ limit cycles.

It suffices to prove that for n even system (2) has no more limit cycles than the $[n/2]$ ones given above. Let Γ^* be a limit cycle of the considered piecewise linear system. Then, Γ^* intersects the separation curve $y=\varepsilon T_n(x)$ at exactly two points, provided $\varepsilon>0$ suitably small. Denote these two points by $(v,\varepsilon T_n(v))$ and $(u,\varepsilon T_n(u))$ with $v<u$. We know that Γ^* is given by one piece of the level curve $H^+(x,y)=a_0^2$ and one piece of $H^-(x,y)=b_0^2$, for some $a_0,b_0\in\mathbb{R}$. Define

$$\begin{aligned} W^+(x,\varepsilon) :=& x^2+(8-4x)\varepsilon T_n(x)+8\varepsilon^2T_n^2(x),\\ W^-(x,\varepsilon) :=& x^2-2\varepsilon T_n(x)+\varepsilon^2T_n^2(x).\end{aligned}$$

With the above definitions, we have

$$W^+(u,\varepsilon)=W^+(v,\varepsilon)\quad\text{and}\quad W^-(u,\varepsilon)=W^-(v,\varepsilon).$$

On the other hand, since $H^-(x,y)=b_0^2$ and $y=\varepsilon T_n(x)$ are both symmetric with respect to the y–axis, we must have $v=-u$. This implies that

$$0=W^+(u,\varepsilon)-W^+(-u,\varepsilon)=-4u\varepsilon(T_n(u)+T_n(-u))=-8u\varepsilon T_n(u).$$

Hence u is a zero of T_n and Γ^* is one of the hyperbolic limit cycles Γ_k's, as we wanted to prove. This completes the proof of the theorem.

2.2. **Proof of Proposition 2.** We choose the separation curve as $y=\varepsilon\cos x$ separating the (x,y) plane in two zones with $\Omega^+:=\{(x,y)|\ y\ge\varepsilon\cos x\}$ and $\Omega^-:=\{(x,y)|\ y\le\varepsilon\cos x\}$. The piecewise linear differential system with two zones separated by $y=\varepsilon\cos x$ is again (2).

Notice that at the intersection points on the x–axis of $y = \varepsilon \cos x$ with the level curves $H^+(x, y) = a^2$ and $H^-(x, y) = b^2$, the slopes of the two curves $H^+(x, y) = a^2$ and $H^-(x, y) = b^2$ at the intersection points are

$$\frac{dy}{dx} = \frac{x}{2x-4}, \quad \frac{dy}{dx} = x,$$

whose absolute values are always larger than a positive constant. Using this fact together with similar arguments as in the proof of Theorem 1 we get that for $\varepsilon > 0$ small enough the piecewise linear differential system has infinitely many limit cycles, which pass through the points $(-\pi/2 - k\pi, \pi/2 + k\pi)$ for $k \in \mathbb{N}$. Thus, the proposition follows.

3. Preliminary results and proof of Theorem 3.

Recall that given $n+1$ smooth functions $g_0, \ldots, g_n$ on an open interval I, it is said that $(g_0, \ldots, g_n)$ is an ECT-system on I if for all $k = 0, 1, 2, \ldots, n$, any nontrivial linear combination

$$\gamma_0 g_0(x) + \cdots + \gamma_k g_k(x) \tag{4}$$

with $\gamma_j \in \mathbb{R}$ has at most k isolated zeros on I counting multiplicities. We remark that it is easy to see that given a linear combination of $n+1$ functions, as in (4) with $k = n$, there always exist γ_j such that it has at least n zeros in I. When the functions form an ECT-system, we can moreover ensure that these n zeros are simple. When the function (4) corresponds to a first order Melnikov function, the condition that its zeros are simple is precisely the one needed to ensure that they give rise to limit cycles of the perturbed system.

A very useful characterization of ECT-systems is the following (see [10]):

Proposition 4. *$(g_0, \ldots, g_n)$ is an ECT-system on I if and only if for all $k = 0, 1, \ldots n$, $W[g_0, \ldots, g_k](x) \neq 0$ for all $x \in I$, where*

$$W_k = W[g_0, \ldots, g_k](x) \doteq \det \left(g_j^{(i)}(x) \right)_{0 \le i,j \le k}$$

is the Wronskian of $(g_0, \ldots, g_k)$ at $x \in I$.

For the sake of completeness, and following [12, 17], we include the main steps of the proof of the following proposition, that uses the averaging method to obtain the Melnikov function associated to (1).

Proposition 5. *Let C_n be the separation curve tangent to the x–axis at the origin and locally convex given in system* (1). *Then its first order Melnikov function M_{C_n} is*

$$M_{C_n}(r) = \frac{1}{r} \sum_{k=0}^{5} \gamma_k g_k(r), \tag{5}$$

where γ_k, $k=0,\dots,5$, are arbitrary real parameters and $\sqrt{x^2+y^2}=r>0$ parameterizes the closed orbits of the unperturbed system ($\varepsilon=0$). Moreover,

$$g_0(r)=1,\qquad g_1(r)=\theta_1(r)-\theta_2(r),\qquad g_2(r)=\frac{1}{r}\big(f_1(r)-f_2(r)\big),$$
$$g_3(r)=f_1^2(r)-f_2^2(r),\qquad g_4(r)=\frac{1}{r}\Big(\sqrt{1-f_1^2(r)}-\sqrt{1-f_2^2(r)}\Big),$$
$$g_5(r)=f_1(r)\sqrt{1-f_1^2(r)}-f_2(r)\sqrt{1-f_2^2(r)},$$

and $f_j(r)=\cos(\theta_j(r))$, where the functions $\theta_j(r)$, $j=1,2$, are given by the cuts of C_n and $x^2+y^2=r^2$ shown in Figure 3. The function M_{C_n} is defined when both functions $\theta_j(r)$ are well-defined.

Proof. In order to apply the averaging method we consider polar coordinates, $(x,y)=(r\cos\theta, r\sin\theta)$, and system (1) writes as

$$\frac{dr}{d\theta}=\frac{\varepsilon A^{\pm}(r,\theta)}{-1+\varepsilon B^{\pm}(r,\theta)/r}=-A^{\pm}(r,\theta)\varepsilon+O(\varepsilon^2),\quad \text{for } \pm G_n(r,\theta)\geq 0,$$

where $G_n(r,\theta)$ is the expression of the separation curve $F_n(x,y)$ in polar coordinates. Notice that $G_n\geq 0$ and $G_n\leq 0$ corresponds to Ω_n^+ and Ω_n^-, respectively, and

$$-A^{\pm}(r,\theta)=\alpha_0^{\pm}r+\alpha_1^{\pm}\cos\theta+\alpha_2^{\pm}\sin\theta+\alpha_3^{\pm}r\cos(2\theta)+\alpha_4^{\pm}r\sin(2\theta),$$

with $\alpha_0^{\pm}=-(a_1^{\pm}+b_2^{\pm})/2$, $\alpha_1^{\pm}=-a_0^{\pm}$, $\alpha_2^{\pm}=-b_0^{\pm}$, $\alpha_3^{\pm}=-(a_1^{\pm}-b_2^{\pm})/2$ and $\alpha_4^{\pm}=(a_2^{\pm}+b_1^{\pm})/2$.

Observe also that the parameters $\alpha_i^{\pm}$ can be chosen arbitrarily, because of the arbitrariness of $a_i^{\pm}$ and $b_j^{\pm}$. The above non-autonomous differential equation is well defined on any interval $r\in[r_0,\beta]$ where $r_0>0$ is a fixed and small enough number and β depends on each specific F_n.

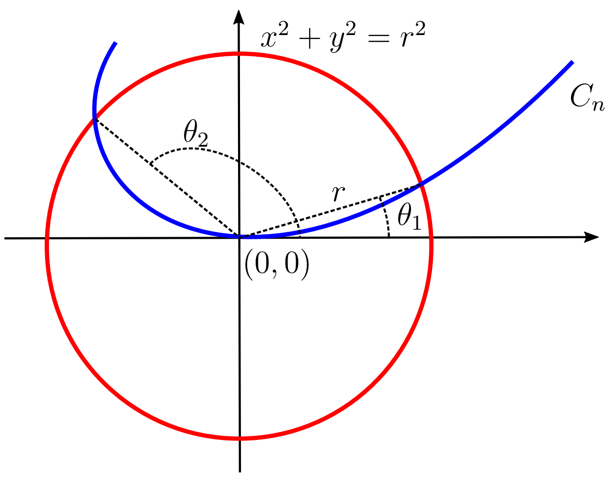

FIGURE 3. Definition of functions $\theta_1(r)$ and $\theta_2(r)$ in Proposition 5.

The first order averaged function (or Melnikov function) is (see [12])

$$\begin{aligned}
M_{C_n}(r) &= -\int_{\theta_1(r)}^{\theta_2(r)} A^+(r,s)\,ds - \int_{\theta_2(r)}^{\theta_1(r)+2\pi} A^-(r,s)\,ds \\
&= 2\alpha_0^-\pi r - (\alpha_0^+ + \alpha_0^-)\big(\theta_1(r)-\theta_2(r)\big)\,r - (\alpha_1^+ - \alpha_1^-)\big(\sin(\theta_1(r))-\sin(\theta_2(r))\big) \\
&+ (\alpha_2^+ - \alpha_2^-)\big(\cos(\theta_1(r))-\cos(\theta_2(r))\big) - \frac{\alpha_3^+ - \alpha_3^-}{2}\big(\sin(2\theta_1(r))-\sin(2\theta_2(r))\big)\,r \\
&+ \frac{\alpha_4^+ - \alpha_4^-}{2}\big(\cos(2\theta_1(r))-\cos(2\theta_2(r))\big)r.
\end{aligned}$$

Writing the above expression in terms of $f_j(r) = \cos(\theta_j(r))$ we arrive to (5). □

Lemma 6. (a) *When C_n is a symmetric curve with respect to the y–axis, that is when $F_n(-x,y) = F_n(x,y)$, the function M_{C_n} reduces to*

$$M_{C_n}(r) = \frac{1}{r}\sum_{k=0}^{3}\beta_k h_k(r), \tag{6}$$

$$h_0(r) = 1,\ h_1(r) = \theta_1(r),\ h_2(r) = \frac{f_1(r)}{r},\ h_3(r) = f_1(r)\sqrt{1-f_1^2(r)},$$

where β_k, $k = 0,\ldots,3$, are arbitrary real constants.

(b) *When $n = 2$ one of the functions g_2, g_3, g_4, and g_5 can be removed in the expression (5) of M_{C_n}. In particular, when the coefficient of xy in F_2 is not zero then the function g_5 can be always removed.*

Proof. (a) In this situation, $f_2 = -f_1$ and $\theta_2 = \pi - \theta_1$. By plugging these identities in (5) we obtain (6).

(b) The two cuts near $(0,0)$ between $F_2(x,y) = 0$ and $x^2+y^2 = r^2$ are the points $\big(rf_k(r), r\sqrt{1-f_k^2(r)}\big)$ for $k = 1,2$. Writing $F_2(x,y) = \sum_{i+j=0}^{2} d_{i,j}x^iy^j$ and imposing that

$$\frac{1}{r^2}\Big(F_2\big(rf_1(r), r\sqrt{1-f_k^2(r)}\big) - F_2\big(rf_2(r), r\sqrt{1-f_2^2(r)}\big)\Big) \equiv 0$$

we obtain

$$d_{1,0}g_2(r) + d_{0,1}g_4(r) + (d_{2,0}-d_{0,2})g_3(r) + d_{1,1}g_5(r) \equiv 0.$$

Since $d_{1,1} \neq 0$, the function g_5 can be expressed in terms of g_2, g_3 and g_4. Thus item (b) follows. □

Proof of Theorem 3 (a). For $n = 2$, by Lemma 6 we have to prove that the 4 functions that define (6) form an ECT-system in a suitable interval. Let us prove first that, under our hypotheses it is not restrictive to write the quadratic polynomial F_2 that defines C_2, as

$$F_2(x,y) = x^2 + Ky^2 - y = 0. \tag{7}$$

In general, $F_2(x,y) = \sum_{i+j=0}^{2} d_{i,j}x^iy^j$. Since it passes through the origin, $d_{0,0} = 0$. Moreover, the symmetry condition $F_2(-x,y) = F_2(x,y)$ implies that $d_{1,0} = d_{1,1} = 0$. Additionally $d_{0,1} \neq 0$, because otherwise F_2 would be a homogeneous conic. Multiplying F_2 by a constant, if necessary, we can assume that $d_{0,1} = -1$, and the convexity condition forces that $d_{2,0}/d_{0,1} < 0$. In short, $F_2(x,y) = L^2x^2 + d_{0,2}y^2 - y$. Finally, changing the scale, that is taking $x_1 = \lambda x$ and $y_1 = \lambda y$ for a suitable λ, we arrive again to a piecewise linear system as (1), but with F_2 as in (7).

To study M_{C_2}, when $F_2(x,y) = x^2+Ky^2-y$, it is convenient to consider separately three cases: $K < 1$, $K = 1$, and $K > 1$. For the first case, we write $K = 1 - d^2$, for $d > 0$. Then we choose C_2 to be a branch of the quadratic curve

$$F_2(x,y) = x^2 + (1-d^2)y^2 - y = 0,$$

which is either a parabola if $d = 1$, or an ellipse if $0 < d < 1$, or a hyperbola if $d > 1$. Notice that C_2 at the origin satisfies $y'(0) = 0$, $y''(0) > 0$, and it is locally convex.

For this curve C_2, we have that $\theta_1(r) = \arccos(f_1(r))$ and we can compute $f_1(r)$ as follows. Using that $(x,y) = \big(rf_1(r), r\sqrt{1-f_1^2(r)}\big)$, we get

$$F_2\big(rf_1(r), r\sqrt{1-f_1^2(r)}\,\big) = 0.$$

Thus, $r\big((1-d^2)r+d^2rf_1^2(r)-\sqrt{1-f_1^2(r)}\big) = 0$. Solving this last expression we obtain

$$f_1(r) = \frac{\sqrt{2d^4r^2 - 2d^2r^2 - 1 + \sqrt{4d^2r^2+1}}}{\sqrt{2}d^2r}.$$

In order to simplify the computations, we can use a new variable u instead of r,

$$r = \Phi(u) \doteq \frac{1-u^2}{4du},$$

to avoid the inner square root in f_1, because $4d^2r^2 + 1 = (1+u^2)^2/(4u^2)$. Now, $0 < u < 1$ when C_2 is a parabola ($d = 1$) or a branch of hyperbola ($d > 1$), and $u \in (u_d,\, 1)$ if it is an ellipse ($0 < d < 1$), with $u_d = (1-d)/(1+d)$. With this change, we get

$$F_1(u) \doteq f_1\left(\Phi(u)\right) = \frac{\Delta(u,d)}{d(u+1)}$$

with

$$\Delta(u,d) = \sqrt{((d+1)u+d-1)((d-1)u+d+1)}.$$

Notice that both factors inside the square root are positive for all values of d, but restricted to the respective intervals of definition. When $d = 1$ this can be easily seen by direct substitution. Otherwise, this becomes apparent by writing

$$((d+1)u+d-1)((d-1)u+d+1) = (d^2-1)\left(u - \frac{1-d}{1+d}\right)\left(u - \frac{1+d}{1-d}\right).$$

In this new variable, the Melnikov function $\widetilde{M}_{C_2}(u) \doteq M_{C_2}(\Phi(u))$ writes as

$$\widetilde{M}_{C_2}(u) = \frac{1}{\Phi(u)}\sum_{k=0}^{3}\beta_k H_k(u),$$

with

$$H_0(u) = 1, \qquad H_1(u) = \arccos\left(\frac{\Delta(u,d)}{d(1+u)}\right),$$

$$H_2(u) = \frac{4u\Delta(u,d)}{(1-u)(1+u)^2}, \qquad H_3(u) = \frac{(1-u)\Delta(u,d)}{d^2(1+u)^2}.$$

From these functions we can compute the ordered list of Wronskians,

$$\begin{aligned}
W[H_0](u) &= 1,\\
W[H_0,H_1](u) &= -\frac{2}{(1+u)\Delta(u,d)},\\
W[H_0,H_1,H_2](u) &= \frac{16(d^2(u+1)^4+2(u-1)^4)}{(u+1)^5(u-1)^3\Delta^2(u,d)},\\
W[H_0,H_1,H_2,H_3](u) &= \frac{2048}{(1+u)^6(u-1)^3\Delta^3(u,d)}.
\end{aligned}$$

Thus, the proof in this case ($K = 1-d^2$) finishes because these Wronskians are non-vanishing and, by Proposition 4, the above 4 functions form an ECT-system in the respective intervals of definition.

For the third case, $K > 1$, we write $K = 1+d^2$, for $d > 0$. Then we choose C_2 to be a piece of the quadratic curve

$$F_2(x,y) = x^2 + (1+d^2)y^2 - y = 0,$$

which is always an ellipse. In this situation

$$f_1(r) = \frac{\sqrt{2d^4r^2 + 2d^2r^2 - 1 + \sqrt{-4d^2r^2+1}}}{\sqrt{2}d^2r},$$

and here, to simplify the computations, it is convenient to introduce the new variable v as,

$$r = \Psi(v) \doteq \frac{2v}{v^2+4d^2}, \quad 0 < v < 2d,$$

also to avoid the inner square root in f_1. Doing similar computations, that we omit for the safe of brevity, we also prove that the corresponding set with 4 functions is an ECT in $(0,2d)$.

Finally, the case $K = 1$ is easier because $f_1(r) = \sqrt{1-r^2}$, and it is straightforward to prove that the functions $h_k(r), k = 0,1,2,3$, given in (6) form an ECT-system in $(0,2d)$.

Hence, in particular, for all $K \in \mathbb{R}$ it is possible to find a Melnikov function $M_{C_2}(r)$ of the form (6) with 3 simple zeros in any given interval in the domain of definition of the function. These zeros give rise to the 3 limit cycles stated in the theorem and prove that $L_2 \geq 3$. This lower bound will be improved in the proof of item (b). $\square$

Proof of Theorem 3 (b) *and* (c). We will study in terms of C_n, and for $\delta > 0$ small enough, whether the first 5 or all the 6 functions that define (5) form an ECT-system for $r \in (0,\delta)$.

We will choose C_3 to be a piece of the cubic curve

$$(x-y)^2+(1-d^2)y^2+ax^3-y=0, \tag{8}$$

that passes trough the origin. Notice that when $a=0$, it is either a parabola if $d=1$, or an ellipse if $0<d<1$, or a hyperbola if $d>1$. When $a\neq 0$ it is a proper cubic. This curve always has at the origin a horizontal tangent point, it is not symmetric with respect to the y–axis and it is locally convex.

As in the proof of item (a) of the theorem, the functions $f_k(r), k=1,2$, that appear in the Melnikov function given in Proposition 5, can be obtained by substituting $(x,y)=(rf, r\sqrt{1-f^2})$ in (8). Then, they are the two solutions, $f=f_k(r)$ for $k=1,2$, of equation

$$r^2(d^2-1)f^2-r(2rf+1)\sqrt{1-f^2}+r^2(2-d^2)+ar^3f^3=0,$$

that satisfy $f_1(0)=1$ and $f_2(0)=-1$, respectively. The series expansion of f_1 writes as

$$\begin{aligned} f_1(r)=&1-\frac{1}{2}r^2+(2-a)r^3-\frac{1}{8}(4a^2-8d^2-32a+57)r^4\\ &+(3ad^2+2a^2-8d^2-14a+24)r^5\\ &+\frac{1}{16}(48a^2d^2-40d^4-384ad^2-84a^2+728d^2+704a-1249)r^6+O\left(r^7\right), \end{aligned}$$

and it can be seen that $f_2(r)=-f_1(-r)$.

Then we can obtain the corresponding series expansions of the six functions defined in (5),

$$\begin{aligned} g_0(r)=&1,\\ g_1(r)=&-\pi+2r-\left(2d^2+4a-\frac{31}{3}\right)r^3+\left(4d^4-2a^2d^2+24ad^2\right.\\ &\left.-57d^2-3a^2-22a+\frac{1543}{20}\right)r^5+O\big(r^7\big),\\ g_2(r)=&\frac{2}{r}-r+\left(2d^2-\frac{57}{4}\right)r^3-\left(5d^4-91d^2+\frac{1249}{8}\right)r^5+O\big(r^7\big),\\ g_3(r)=&-4r+\left(12d^2-26\right)r^3-\left(40d^4-230d^2+\frac{347}{2}\right)r^5+O\big(r^7\big),\\ g_4(r)=&8r^3-\left(32d^2-92\right)r^5+O\big(r^7\big),\\ g_5(r)=&2r-\left(2d^2+4a-9\right)r^3+\left(4d^4-2a^2d^2+24ad^2-53d^2\right.\\ &\left.-7a^2+2a+\frac{163}{4}\right)r^5+O\big(r^7\big). \end{aligned}$$

Finally, we compute all the ordered Wronskians,

$$\begin{aligned}
W_0(r) &= 1,\\
W_1(r) &= 2-\left(6d^2+12a-31\right)r^2+O\left(r^4\right),\\
W_2(r) &= 8r^{-3}-\left(48d^2-248\right)r^{-1}+O\left(r\right),\\
W_3(r) &= \left(1536d^2-1024\right)r^{-3}+\left(-30720d^4+111360d^2-18432\right)r^{-1}\\
&\quad +O\left(r\right),\\
W_4(r) &= -\left(983040d^2+15532032\right)r^{-2}\\
&\quad +33030144d^4+509214720d^2-1666842624+O\left(r^2\right),\\
W_5(r) &= 37748736a\Big(70d^4a^5+105a^3d^6+35ad^8+420a^6d^2-1050d^4a^4\\
&\quad -1120d^6a^2-140d^8-4970a^5d^2+5145d^4a^3+3640d^6a\\
&\quad +630a^6+25116d^2a^4-7406a^2d^4-3500d^6-8820a^5\\
&\quad -73160a^3d^2-10349d^4a+50582a^4+119172a^2d^2\\
&\quad +24036d^4-141180a^3-67274ad^2+204636a^2-27252d^2\\
&\quad -173848a+91336\Big)+O(r^2).
\end{aligned}$$

Notice that since the coefficients of r^{-1} and r^{-3} in W_3 have no common roots as polynomials in d, the Wronskian W_3 is clearly non-vanishing in a neighborhood of the origin. Then, when $a=0$, for all d, the first 5 Wronskians do not vanish in an open neighborhood of the origin. Moreover, by item (b) of Lemma 6, since F_2 has the monomial xy, we already knew that g_5 is a linear combination of the previous 4 functions and 5 is the maximum number of Wronskians that can be non-zero.

On the other hand, when $a\neq 0$ and for most values of d (it suffices that the huge polynomial that gives the first order terms of $W_5(r)$ does not vanish) we know that the six functions g_k, $k=0,1,\ldots,5$, form an ECT-system in a small enough neighborhood of $r=0$, as we wanted to prove.

In particular, when $a=0$ (respectively, $a\neq 0$) it is possible for all d (respectively, for most d) to find a Melnikov function $M_{C_n}(r)$ of the form (5) with 4 (respectively, 5) simple zeros in any small enough given interval $(0,\rho)$. Moreover, the functions g_i and W_i, $i=0,\ldots,5$ have a convergent series in a small enough neighborhood of the origin and the size of such interval is completely independent of the interval $[r_0,\beta]$ in Proposition 5. Hence, we can choose adequately the value of r_0 in such a way that the desired zeros of $M_{C_n}(r)$ are in (r_0,ρ). Consequently, the simple zeros of $M_{C_n}(r)$ give rise to the limit cycles stated in the theorem. □

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