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Approximating Mills ratio

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ABSTRACT

Consider the Mills ratio $f(x) = (1 - \Phi(x))/\phi(x)$, $x \geq 0$, where ϕ is the density function of the standard Gaussian law and Φ its cumulative distribution. We introduce a general procedure to approximate f on the whole $[0, \infty)$ which allows to prove interesting properties where f is involved. As applications we present a new proof that $1/f$ is strictly convex, and we give new sharp bounds of f involving rational functions, functions with square roots or exponential terms. Also Chernoff type bounds for the Gaussian Q-function are studied.

1. Introduction

Recall that the Mills ratio (Mills [22]) is the function

$$f(x) = \frac{1 - \Phi(x)}{\phi(x)} = e^{\frac{x^2}{2}} \int_x^\infty e^{-\frac{t^2}{2}}\, dt, \quad x \geq 0, \tag{1}$$

where $\phi(x) = e^{-x^2/2}/\sqrt{2\pi}$ is the density function of a standard Gaussian law and $\Phi(x) = \int_{-\infty}^x \phi(t)\, dt$ its cumulative distribution function. The study of this function is much older than Mills [22], and through its relation with the function

$$F(x) = e^{x^2} \int_x^\infty e^{-t^2}\, dt \tag{2}$$

given by

$$f(x) = \sqrt{2} F(x/\sqrt{2}\,),$$

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