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On the nonexistence, existence and uniqueness of limit cycles

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Abstract. We present two new criteria for studying the nonexistence, existence and uniqueness of limit cycles of planar vector fields. We apply these criteria to some families of quadratic and cubic polynomial vector fields, and to compute an explicit formula for the number of limit cycles which bifurcate out of the linear centre $\dot{x} = -y$, $\dot{y} = x$, when we deal with the system $\dot{x} = -y + \varepsilon \sum_{i+j=1}^{n} a_{ij} x^i y^j$, $\dot{y} = x + \varepsilon \sum_{i+j=1}^{n} b_{ij} x^i y^j$. Moreover, by using the second criterion we present a method to derive the shape of the bifurcated limit cycles from a centre.

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1. Introduction and statement of the main results

In the qualitative theory of differential equations, research on limit cycles is an interesting and difficult part. Limit cycles of planar vector fields were defined by Poincaré [16]. At the end of the 1920s van der Pol [17], Liénard [14] and Andronov [1] proved that a closed trajectory of a self-sustained oscillation occurring in a vacuum tube circuit was a limit cycle as considered by Poincaré. After this observation, the nonexistence, existence, uniqueness and other properties of limit cycles were studied extensively by mathematicians and physicists, and more recently also by chemists, biologists and economists.

The more well-known method for proving the nonexistence of limit cycles in a simply connected region is the Bendixson–Dulac method and some variations of it. The method of Dulac functions also gives upper bounds for the number of closed trajectories in a multiply connected region. See [23] for more details.

The Poincaré–Bendixson theorem allows us to show the existence of limit cycles under convenient assumptions. The problem of uniqueness of a limit cycle for a given system is in general more difficult than the problem of existence. Some criteria for the uniqueness of limit cycles can be given by using the stability or instability of the limit cycles. There are methods for the uniqueness developed by Poincaré, Andronov, Cherkas, Levinson, Leontovich, Liénard, Massera, Sansone, Zhang Zhifen and many others (see [23]). But in general the sufficient conditions of the previous methods are very restrictive. Of course one of the best methods for studying the nonexistence, existence, and uniqueness of limit cycles is analysing the Poincaré return map defined in a transversal section to the planar flow, but in general such analysis is not easy.

More difficult problems appear when a planar system has more than one limit cycle, and when we try to understand their distribution on the plane. In fact the most famous problem

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