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PERIODS FOR MAPS OF THE FIGURE-EIGHT SPACE

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Let Per(f) denote the set of periods of all periodic points of a map f from a topological space into itself. Let **8** be the figure-eight space. We extend to the **8** the following theorem from the circle due to Block [1981]. Let $\mathbb{S}^1$ be the circle. For every map $f : \mathbb{S}^1 \to \mathbb{S}^1$ with $\mathrm{Per}(f) \cap \{1, 2, \ldots, n\} = \{1, n\}$ and $n > 2$ we have $\mathrm{Per}(f) = \{1, n, n+1, n+2, \ldots\}$. Conversely, for every $n \in \mathbb{N}$ with $n > 2$ there exists a map $f : \mathbb{S}^1 \to \mathbb{S}^1$ such that $\mathrm{Per}(f) = \{1, n, n+1, n+2, \ldots\}$.

For the space **8** we prove the following. Let $f : \mathbf{8} \to \mathbf{8}$ be a continuous map having the branching point fixed and such that $\mathrm{Per}(f) \cap \{1, 2, \ldots, n\} = \{1, n\}$ with $n > 4$. Then Per(f) is either $\{1, n, n+1, n+2, \ldots\}$, or $\{1, n, n+2, n+4, \ldots\}$ with n even, or $\{1, n, n+2, n+4, \ldots\} \cup \{2n+2, 2n+4, 2n+6, \ldots\}$ with n odd. Conversely, for every $n \in \mathbb{N}$ with $n > 4$, if $\mathrm{A}(n)$ is one of the above three subsets of $\mathbb{N}$, then there is a continuous map $f : \mathbf{8} \to \mathbf{8}$ having the branching point fixed and such that $\mathrm{Per}(f) = \mathrm{A}(n)$.

1. Introduction and Statement of the Results

Let $f : X \to X$ be a map on the topological space X. Here a *map* always means a *continuous map*. A point x of X will be called *periodic for* f (or just *periodic*, if f is obvious from the context) if $f^n(x) = x$ for some integer $n > 0$, where f^n is f composed with itself n times. The least n satisfying the above equality is called the *period* of x. The *orbit* of x is the set $\{f^n(x) : n \geq 0\}$, where f^0 is the identity map. We denote by Per(f) the set $\{n : f \text{ has a point of period } n\}$. Clearly $\mathrm{Per}(f) \subset \mathbb{N}$, where $\mathbb{N}$ denotes the set of natural numbers.

In the 1960s Sharkovskiĭ [1964] proved a remarkable theorem about the interrelationships of periodic points of maps on a closed interval. Let $\leq_s$ (the *Sharkovskiĭ ordering*) be the following total ordering of $\mathbb{N}$:

$$1 \leq_s 2 \leq_s 2^2 \leq_s 2^3 \leq_s \cdots$$
$$\leq_s 7 \cdot 2^2 \leq_s 5 \cdot 2^2 \leq_s 3 \cdot 2^2 \leq_s \cdots$$
$$\leq_s 7 \cdot 2 \leq_s 5 \cdot 2 \leq_s 3 \cdot 2 \leq_s \cdots \leq_s 7 \leq_s 5 \leq_s 3\,.$$

We denote by $S(n)$ the initial segment of the Sharkovskiĭ ordering $\leq_s$ ending at $n \in \mathbb{N}$, i.e.,

$$S(n) = \{m \in \mathbb{N} : m \leq_s n\}\,;$$

we also define

$$S(2^\infty) = \{1,\, 2,\, 2^2,\, 2^3,\, 2^4, \ldots\}\,.$$

Interval Theorem [Sharkovskiĭ, 1964]. *Let I be a closed interval.*