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Analytic integrability and characterization of centers for nilpotent singular points

Jaume Giné

Abstract. A method which provides necessary conditions to obtain a local analytic first integral in a neighborhood of a nilpotent singular point is developed. As an application we provide sufficient conditions in order that systems of the form $\dot{x} = y + P_n(x, y)$, $\dot{y} = Q_n(x, y)$ where P_n and Q_n are homogeneous polynomials of degree $n = 2, 3, 4, 5$ have a local analytic first integral of the form $H = y^2 + F(x, y)$, where F starts with terms of order higher than 2. We remark that, in general, the existence of such integral is only guaranteed when the singular point is a nilpotent center and the system has a formal first integral, see [6]. Therefore, we characterize the nilpotent centers of systems which have a local analytic first integral.

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1. Introduction and statement of the main results

One of the classical problems in the qualitative theory of planar analytic differential systems is to characterize the local phase portrait at an isolated singular point. This problem has been solved except if the singular point is of focus–center type, see [2, 1, 8]. Recall that a singular point is said to be of focus–center type if it is either a focus or a center. The problem of distinguishing between a center or a focus is called the *center problem.* Of course, if the linear part of the singular point in nondegenerate (i.e. its determinant does not vanish) the characterization is well known. The problem has also been solved when the linear part is degenerate but not identically zero, see [4], but the resolution is difficult to implement. Another problem is to know if there exists or not a local analytic first integral defined in a neighborhood of a singular point.

If an analytic system has a center at the origin, then after a linear change of variables and a rescalling of the time variable it can be written into one of the following three forms:

$$\begin{array}{rcl} \dot{x} & = & -y + X(x, y), \\ \dot{y} & = & x + Y(x, y), \end{array} \tag{1}$$