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A note on Liouvillian first integrals and invariant algebraic curves

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ABSTRACT

In this work we study the existence and non-existence of finite invariant algebraic curves for a complex planar polynomial differential system having a Liouvillian first integral.

1. Introduction and statement of the main results

In this work we consider a complex planar polynomial differential system

$$\frac{dx}{dt} = \dot{x} = P(x, y), \qquad \frac{dy}{dt} = \dot{y} = Q(x, y), \tag{1}$$

where the dependent variables x and y are complex, and the independent one (the *time*) t can be real or complex, and $P, Q \in \mathbb{C}[x, y]$, where $\mathbb{C}[x, y]$ is the ring of all polynomials in the variables x and y with coefficients in $\mathbb{C}$. We denote by $m = \max\{\deg P, \deg Q\}$ the *degree* of the polynomial system.

Let $f = f(x, y) = 0$ be an algebraic curve in $\mathbb{C}^2$. We say that it is a *finite invariant algebraic curve* for the polynomial differential system (1) if

$$P\frac{\partial f}{\partial x} + Q\frac{\partial f}{\partial y} = kf, \tag{2}$$

for some polynomial $k = k(x, y) \in \mathbb{C}[x, y]$, called the *cofactor* of the algebraic curve $f = 0$. From (2) it is easy to see that the degree of the polynomial k is at most $m - 1$ and that the algebraic curve $f = 0$ is formed by trajectories of the polynomial differential system (1).

Let $h, g \in \mathbb{C}[x, y]$ and assume that h and g are relatively prime in the ring $\mathbb{C}[x, y]$. Then the function $\exp(g/h)$ is called an *exponential factor* of the polynomial differential system (1) if for some polynomial $k \in \mathbb{C}[x, y]$ of degree at most $m - 1$ it satisfies the equation

$$P\frac{\partial \exp(g/h)}{\partial x} + Q\frac{\partial \exp(g/h)}{\partial y} = k\exp(g/h). \tag{3}$$

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