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POLYNOMIAL AND RATIONAL FIRST INTEGRALS FOR PLANAR QUASI–HOMOGENEOUS POLYNOMIAL DIFFERENTIAL SYSTEMS

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Abstract. In this paper we find necessary and sufficient conditions in order that a planar quasi–homogeneous polynomial differential system has a polynomial or a rational first integral. We also prove that any planar quasi–homogeneous polynomial differential system can be transformed into a differential system of the form $\dot{u} = uf(v)$, $\dot{v} = g(v)$ with $f(v)$ and $g(v)$ polynomials, and vice versa.

1. **Introduction .** The characterization of polynomial or rational integrability of a differential system goes back to Poincaré, see [20, 21, 22] and has attracted the attention of many authors, see for instance [2, 3, 8, 14, 17, 18, 19, 23, 24] and references therein. For quasi–homogeneous polynomial differential systems if we control the polynomial first integrals we are controlling all analytical first integrals of the system, see [15, 17].

We assume that there exists an analytic first integral H for an analytic differential system of the form $\dot{x} = P(x, y)$, $\dot{y} = Q(x, y)$. The analytic functions H, P and Q can be decomposed in sum of quasi–homogeneous polynomials of the same weight degree, i.e. $H = H_m + H_{m+1} + \ldots$, $P = P_r + P_{r+1} + \ldots$ and $Q = Q_r + Q_{r+1} + \ldots$. Then, the quasi–homogeneous polynomial of the lowest weight degree H_m must be a first integral of the quasi–homogeneous differential system $\dot{x} = P_r(x, y)$, $\dot{y} = Q_r(x, y)$, see [12, 16]. So the study of the integrability of the quasi–homogeneous polynomial differential systems is a good first step for studying the integrability of more general differential systems, see for instance [1, 16].

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