POLYNOMIAL AND RATIONAL FIRST INTEGRALS FOR PLANAR HOMOGENEOUS POLYNOMIAL DIFFERENTIAL SYSTEMS

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ABSTRACT. In this paper we find necessary and sufficient conditions in order that a planar homogeneous polynomial differential system has a polynomial or rational first integral. We apply these conditions to linear and quadratic homogeneous polynomial differential systems.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

One of the main problems in the qualitative theory of differential systems in $\mathbb{C}^2$ is to determine whether they have a global first integral, i.e. first integrals defined in a dense and open subset of $\mathbb{C}^2$. This problem goes back to Poincaré. In fact, Poincaré, in 1891, started a series of three papers [24, 25, 26] in which he tried to answer the following question: Is it possible to decide if an algebraic differential equation in two variables is algebraically integrable? (in the sense that it has a rational first integral).

For an arbitrary polynomial differential system in $\mathbb{C}^2$ the existence of a rational first integral does not imply the existence of an analytic equation on the coefficients, and neither the degree of the integral nor the genus of the phase curve is bounded by a function of the degree of the differential system, see [16].

The characterization of polynomial or rational integrability for different particular differential systems has attracted the attention of many authors, see for instance [1, 14, 19, 20, 21] and references therein. In the present paper we give the characterization of polynomial or rational integrability for homogeneous polynomial differential systems. Moreover, for such systems when we control the polynomial first integrals, in fact, we control all analytical first integrals, see [15, 19]. Indeed in

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