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# Averaging theory at any order for computing periodic orbits

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ABSTRACT

We provide a recurrence formula for the coefficients of the powers of $\varepsilon$ in the series expansion of the solutions around $\varepsilon = 0$ of the perturbed first-order differential equations. Using it, we give an averaging theory at any order in $\varepsilon$ for the following two kinds of analytic differential equation:

$$\frac{dx}{d\theta} = \sum_{k\geq 1} \varepsilon^k F_k(\theta, x), \qquad \frac{dx}{d\theta} = \sum_{k\geq 0} \varepsilon^k F_k(\theta, x).$$

A planar polynomial differential system with a singular point at the origin can be transformed, using polar coordinates, to an equation of the previous form. Thus, we apply our results for studying the limit cycles of a planar polynomial differential systems.



## 1. Introduction and statement of the main results

In this work, first we deal with the analytic differential equation

$$\frac{dx}{d\theta} = \sum_{k\geq 1} \varepsilon^k F_k(\theta, x), \tag{1}$$

where $x \in \mathbb{R}$, $\theta \in \mathbb{S}^1$ and $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ with $\varepsilon_0$ a small positive real value, and the functions $F_k(\theta, x)$ are $2\pi$-periodic in the variable $\theta$. So, this differential equation is defined in the cylinder $\mathbb{S}^1 \times \mathbb{R}$. We are interested in the *limit cycles* of differential equation (1), i.e. in the isolated periodic orbits with respect to the set of all periodic orbits of Eq. (1).

We denote by $x_\varepsilon(\theta, z)$ the solution of Eq. (1) with initial condition $x_\varepsilon(0, z) = z$. Due to the analyticity of differential equation (1) and the fact that when $\varepsilon = 0$ we have the trivial equation $dx/d\theta = 0$, the solution can be written as

$$x_\varepsilon(\theta, z) = z + \sum_{j\geq 1} x_j(\theta, z)\varepsilon^j, \tag{2}$$

where $x_j(\theta, z)$ are real analytic functions such that $x_j(0, z) = 0$.

We remark that, if the solution $x_\varepsilon(\theta, z)$ is defined for all $\theta \in \mathbb{S}^1$, then we can consider the Poincaré displacement map associated to differential equation (1) given by $P_\varepsilon(z) = x_\varepsilon(2\pi, z) - z$. We note that, by the theorem on continuous dependence of the solution on initial value and parameters, the solution $x_\varepsilon(\theta, z)$ is defined for all $\theta \in \mathbb{S}^1$ if $z$ and $\varepsilon$ are both close enough to 0. We observe that a limit cycle of Eq. (1) corresponds to an isolated zero of the Poincaré map. We are interested in the limit cycles which bifurcate from the periodic orbits of the unperturbed equation, that is from Eq. (1) with $\varepsilon = 0$. We say that a limit cycle bifurcates from the periodic orbit at level $z^* \in \mathbb{R}$ if there exists an analytic function $\zeta(\varepsilon)$ defined in a neighborhood of $\varepsilon = 0$ such that $P_\varepsilon(\zeta(\varepsilon)) = 0$ for all $\varepsilon$ in this neighborhood, this zero is isolated for each fixed value of $\varepsilon \neq 0$, and $\zeta(0) = z^*$.

A limit cycle is said to be of *multiplicity* $m$, with $m \geq 1$ an integer, if this is the multiplicity of $\zeta(\varepsilon)$ as a zero of $P_\varepsilon(z)$ in a punctured neighborhood of $\varepsilon = 0$. We note that the expansion of the Poincaré map in powers of $\varepsilon$ is $P_\varepsilon(z) = \sum_{j\geq 1} x_j(2\pi, z)\varepsilon^j$. We consider $s \geq 1$ the lowest index such that $x_s(2\pi, z)$ is not identically zero. If there exists a limit cycle of Eq. (1) bifurcating from $z^*$, then $x_s(2\pi, z^*) = 0$. Indeed, if there are $m$ limit cycles bifurcating from $z^*$ counted with their multiplicity, then $z^*$ is an isolated zero of $x_s(2\pi, z)$ of multiplicity at least $m$. On the other hand, for each simple (that is, of multiplicity 1) zero $z^*$ of $x_s(2\pi, z)$, there exists a unique limit cycle of Eq. (1) bifurcating from $z^*$. For details about the previous statement, see, for instance, [1].

The following result provides the explicit expression of the function $x_n(\theta, z)$ for any value of $n$.

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