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Universal centres and composition conditions

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Abstract

In this paper, we characterize the universal centres of the ordinary differential equations $d\rho/d\theta = \sum_{i=1}^{\infty} a_i(\theta)\rho^{i+1}$, where $a_i(\theta)$ are trigonometric polynomials, in terms of the composition conditions. These centres are closely related with the classical Poincaré centre problem for planar analytic differential systems.

Additionally, we show that the notion of universal centre is not invariant under changes of variables, and we also provide different families of universal centres. Finally, we characterize all the universal centres for the quadratic polynomial differential systems.

1. *Introduction and statement of the main results*

We consider the analytic ordinary differential equation

$$\frac{d\rho}{d\theta} = \sum_{i=1}^{\infty} a_i(\theta)\rho^{i+1}, \tag{1}$$

on the cylinder $(\rho,\theta) \in \mathbb{R} \times \mathbb{S}^1$ in a neighbourhood of $\rho = 0$ and where $a_i(\theta)$ are trigonometric polynomials in θ. We shall denote the derivative of ρ with respect to θ by $d\rho/d\theta$ or ρ'. By the standard uniqueness theorem, there is a unique solution of equation (1) with the prescribed initial value $\rho(0) = \rho_0$, where $|\rho_0|$ is small enough. The differential equation (1) is determined by its coefficients $a = (a_1(\theta), a_2(\theta), \ldots)$.

We say that equation (1) determines a *centre* if for any sufficiently small initial value $\rho(0)$ the solution of (1) satisfies $\rho(0) = \rho(2\pi)$. The *centre problem* for equation (1) is to find conditions on the coefficients a_i under which the equation has a centre.

The centre problem and an explicit expression for the first return map of the differential equation (1) have been studied by Brudnyi in [**13**, **15**] for a more general class of equations. The expression of the first return map is given in terms of the following iterated integrals of order k :

$$I_{i_1\ldots i_k}(a) := \int\cdots\int_{0\leqslant s_1\leqslant\cdots\leqslant s_k\leqslant 2\pi} a_{i_k}(s_k)\cdots a_{i_1}(s_1)ds_k\cdots ds_1, \tag{2}$$

where, by convention, we assume that for $k = 0$ this expression is equal to 1. Iterated integrals appear in a similar context in the study of several differential equations, see for instance [**3**, **19**, **20**, **22**, **24**]. Let $\rho(\theta;\rho_0;a)$ with $\theta \in [0, 2\pi]$ be the solution of equation (1) corresponding to a with initial value $\rho(0;\rho_0;a) = \rho_0$. Then $P(a)(\rho_0) := \rho(2\pi;\rho_0;a)$ is the first return map of this equation; the following is proved in [**13**, **15**].

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