LIMIT CYCLES BIFURCATING FROM PLANAR POLYNOMIAL QUASI–HOMOGENEOUS CENTERS

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Abstract. In this paper we find an upper bound for the maximum number of limit cycles bifurcating from the periodic orbits of any planar polynomial quasi-homogeneous center, which can be obtained using first order averaging method. This result improves the upper bounds given in [7].

1. Introduction

In this work we deal with polynomial differential systems of the form

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y), \tag{1}$$

where $P(x, y), Q(x, y) \in \mathbb{R}[x, y]$. The dot denotes derivative with respect to an independent variable t real. We say that the degree of the system is $n = \max\{\deg P, \deg Q\}$.

Let $\mathbb{N}$ denote the set of positive integers. The polynomial differential system (1) is *quasi–homogeneous* if there exist $p, q, m \in \mathbb{N}$ such that for arbitrary $\alpha \in \mathbb{R}$,

$$P(\alpha^p x, \alpha^q y) = \alpha^{p+m-1} P(x, y), \; Q(\alpha^p x, \alpha^q y) = \alpha^{q+m-1} Q(x, y), \tag{2}$$

where p and q are called the *weight exponents* of system (1), and m the *weight degree* with respect to the weight exponents p and q. We say that system (1) satisfying conditions (2) is a quasi–homogeneous system of weight (p, q, m). We remark that for the particular case $p = q = 1$, system (1) is the classical homogeneous polynomial differential system of degree m. We note that conditions (2) imply that the origin of coordinates is a singular point of system (1). We will first characterize when the origin of system (1) is a center (that is, it has a neighborhood filled with periodic orbits with the exception of the origin). Lemma 4 statement (ii) (see below) are characterized all the centers of the quasi–homogeneous systems. This characterization is well-known see for instance [7]. Moreover in [1] the authors provide an algorithm for

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obtaining the quasi–homogeneous systems with a given degree which is a combinatorial problem.

When the origin of system (1) is a center, we consider the one-parametric family of systems

$$\dot{x} = P(x,y) + \varepsilon \bar{P}(x,y), \quad \dot{y} = Q(x,y) + \varepsilon \bar{Q}(x,y), \tag{3}$$

where $\varepsilon \in \mathbb{R}$ is the perturbation parameter and $\bar{P}$ and $\bar{Q} \in \mathbb{R}$ are arbitrary polynomials of degree n. Our goal is to give the maximum number of limit cycles which can bifurcate from the periodic orbits of the center localized at the origin of system (1) with $\varepsilon = 0$, inside the family (3) for $\varepsilon \neq 0$ sufficiently small. We can give this maximum number in terms of p, q and n.

Moreover our result is a generalization of the one given in Theorem A of [7] because we do not need the hypothesis that the polynomials P and Q are coprime. Indeed, we will use the averaging method at first order of ε instead of the Abelian integral which is used in [7]. In our approach we use the classical trigonometric functions $\sin\theta$ and $\cos\theta$ instead of the generalized trigonometric functions $Cs\,\theta$ and $Sn\,\theta$ which are related to quasi-homogeneous functions. In our work the integral that we find, see (5), instead of the Abelian integral is an integral of elementary functions.

Our main result is the following one.

Theorem 1. *We consider any quasi–homogeneous polynomial differential system of weight* (p,q,m) *of the form* (1) *having a center at the origin. We denote by* $(p^*,q^*) = (p/M, q/M)$ *where* $M = \gcd(p,q)$. *We perturb system* (1) *inside the class of all polynomial differential systems of degree* n, *that is, we consider systems of the form* (3) *where* $\bar{P}$ *and* $\bar{Q}$ *are arbitrary polynomials of degree* n. *We assume that* $n \geq p^* \geq q^* \geq 1$. *Then, the maximum number of zeros* $r_0 > 0$ *taking into account their multiplicity of the first averaging function* (see (5)) *is at most*

(a) $(2(n+1)p^* - p^{*2} + (3 + (-1)^{n+1})p^* - 7)/4$ *if* p^* *and* q^* *are odd;*
(b) $(2(n+1)p^* - p^{*2} + 2p^* - 4)/4$ *if* p^* *is even and* q^* *is odd; and*
(c) $(2(n+1)p^* - p^{*2} + (-1)^n + 2p^* - 4)/4$ *if* p^* *is odd and* q^* *is even.*

Statements (b) and (c) of Theorem 1 are an improvement of the ones given in [7] because the expression $2p^* - 4$ there appears as $4p^* - 8$.

The upper bounds provided in Theorem 1 cannot always be reached, as the following result shows.

Proposition 2. *We consider the system*

$$\dot{x} = -y^3(x^2+y^4) + \varepsilon \bar{P}(x,y), \quad \dot{y} = x(x^2+y^4) + \varepsilon \bar{Q}(x,y), \tag{4}$$

where $\varepsilon \in \mathbb{R}$ *and* $\bar{P}(x,y)$, $\bar{Q}(x,y)$ *are real polynomials of degree* $n = 7$. *The maximum number of zeros* $r_0 > 0$ *taking into account their multiplicity of the first averaging function* (see (5)) *is at most* 5 *and there are polynomials* $\bar{P}(x,y)$, $\bar{Q}(x,y)$ *for which this upper bound is reached.*

We remark that the bound 5 is lower than the bound given in Theorem 1. In this example, proved in section 4, we can compute the first averaging function and we can improve the bound given by Theorem 1 to get a sharp upper bound.

Another contribution of this paper is that we give the explicit function whose simple zeros provide the periodic solutions of the quasi–homogeneous center in (1) which for ε sufficiently small persist as limit cycles for system (3). This function is

$$\bar{\psi}_1(z) = \sum_{k\in\tilde{\mathcal{S}}_n} \left(\int_0^{2\pi} \varphi_k(\theta)d\theta\right) z^k, \tag{5}$$

where we define the set of indexes

$$\tilde{\mathcal{S}}_n = \{ip + jq \ : \ i,j \geq 0,\ 0 < i+j \leq n+1\},$$

the functions

$$\begin{aligned}\varphi_k(\theta) = \ & \frac{(p\cos^2\theta + q\sin^2\theta)}{g(\theta)^2}\Big[Q(\cos\theta,\sin\theta)\bar{P}_{k-q}(\cos\theta,\sin\theta) \\ & -P(\cos\theta,\sin\theta)\bar{Q}_{k-p}(\cos\theta,\sin\theta)\Big]u(\theta)^{k-m-p-q+1},\end{aligned}$$

where $\bar{P}_\ell$ and $\bar{Q}_\ell$ are the quasi–homogeneous terms of weight (p,q,ℓ) in $\bar{P}$ and $\bar{Q}$, respectively, the trigonometric polynomials

$$\begin{aligned} f(\theta) &= P(\cos\theta,\sin\theta)\cos\theta + Q(\cos\theta,\sin\theta)\sin\theta, \\ g(\theta) &= p\,Q(\cos\theta,\sin\theta)\cos\theta - q\,P(\cos\theta,\sin\theta)\sin\theta. \end{aligned}$$

and the 2π-periodic function, see Lemma 4 statement (ii),

$$u(\theta) := \exp\left(\int_0^\theta \frac{f(s)}{g(s)}\,ds\right).$$

In fact for homogeneous cubic systems perturbed inside the class of all cubic polynomial systems these explicit formulas were given [6]. Some upper bounds for the number of limit cycles which bifurcate from the period annulus of a quasi–homogeneous polynomial differential system with a center have been given in [2]; see also the references therein.

The rest of the paper is organized as follows. In section 2 we present some lemmas which will allow to prove Theorem 1 in section 3. Section 4 contains an application.

2. Preliminary results

First we present the following technical result.

Lemma 3. *Given a quasi–homogeneous system* (1) *of weight* (p, q, m)*, we can suppose without restriction that* p *and* q *are coprime.*

The proof of this lemma is an extension "mutatis-mutandi" of the proof of Lemma 2.1 of [3] for systems with P and Q not coprime, see also [1]. We give here its proof for completeness.

Proof. Let M be the greatest common divisor of p and q. Then $p = Mp^*$ and $q = Mq^*$ with p^* and q^* coprime. If $P(x, y)$ is not zero, let $x^{i_p}y^{j_p}$ be a monomial with nonzero coefficient of $P(x, y)$. Since

$$P(\alpha^p x, \alpha^q y) = \alpha^{p+m-1}P(x, y) \quad \text{for all} \quad \alpha \in \mathbb{R}.$$

we have that $(\alpha^p x)^{i_p}(\alpha^q y)^{j_p} = \alpha^{pi_p}\alpha^{qj_p}x^{i_p}y^{j_p} = \alpha^{p+m-1}x^{i_p}y^{j_p}$ which implies that $pi_p+qj_p = p+m-1$, or equivalently $p(i_p-1)+qj_p = m-1$. Consequently $m - 1$ is divisible by M except if $(i_p, j_p) = (1, 0)$. If $P(x, y) = x$ or $P(x, y) \equiv 0$, and $Q(x, y)$ is not zero then we consider the monomial $x^{i_q}y^{j_q}$ with nonzero coefficient of $Q(x, y)$. Taking into account that

$$Q(\alpha^p x, \alpha^q y) = \alpha^{q+m-1}Q(x, y) \quad \text{for all} \quad \alpha \in \mathbb{R},$$

we obtain $p\, i_q + q(j_q - 1) = m - 1$. From here we deduce that $m - 1$ is divisible by M except if $(i_q, j_q) = (0, 1)$.

Therefore, either $m - 1$ is divisible by M or system (1) writes as one of the following four cases

$$\begin{array}{llll} \dot{x} = x, & \dot{x} = x, & \dot{x} = 0, & \dot{x} = 0, \\ \dot{y} = y, & \dot{y} = 0, & \dot{y} = y, & \dot{y} = 0, \end{array}$$

which are homogenous systems of degree 1 with $(p, q, m) = (1, 1, 1)$ and therefore with p and q coprime.

In any other case we can write $m - 1 = M(m^* - 1)$. In this case we claim that system (1) is (p^*, q^*, m^*) quasi–homogeneous with p^* and q^* coprime. Indeed, we have that any monomial $x^{i_p}y^{j_p}$ of $P(x, y)$ must verify that $p(i_p - 1) + q\, j_p = m - 1$ which can be divided by M to give $p^*(i_p - 1) + q^* j_p = m^* - 1$. In a similar way for any monomial $x^{i_q}y^{j_q}$ of $Q(x, y)$ we obtain $p^*\, i_q + q^*(j_q - 1) = m^* - 1$. Hence we obtain a quasi–homogeneous system of weight (p^*, q^*, m^*). □

In the following we assume that system (1) is a quasi–homogeneous system of weight (p, q, m) with p and q coprime.

Now we take the weighted blow–up $x = r^p \cos\theta$, $y = r^q \sin\theta$ and we apply it to system (1) which becomes

$$\dot{r} = \frac{r^m f(\theta)}{p\cos^2\theta + q\sin^2\theta}, \quad \dot{\theta} = \frac{r^{m-1} g(\theta)}{p\cos^2\theta + q\sin^2\theta}, \tag{6}$$

with

$$f(\theta) = P(\cos\theta, \sin\theta)\cos\theta + Q(\cos\theta, \sin\theta)\sin\theta,$$
$$g(\theta) = p\,Q(\cos\theta, \sin\theta)\cos\theta - q\,P(\cos\theta, \sin\theta)\sin\theta.$$

We note that $p\cos^2\theta + q\sin^2\theta > 0$ for all $\theta \in \mathbb{R}$ because $p,\ q > 0$. The next well-known result characterizes when system (1) has a center at the origin of coordinates in terms of trigonometric polynomials $f(\theta)$ and $g(\theta)$. For completeness, we prove it here.

Lemma 4. *Consider a polynomial system of the form* (1).

(i) *If system* (1) *has a singular point which is a center then this singular point is at the origin of coordinates.*
(ii) *System* (1) *has a center (at the origin of coordinates) if and only if* $g(\theta)$ *has no real roots and* $\int_0^{2\pi} \frac{f(\theta)}{g(\theta)}\, d\theta = 0$.
(iii) *If system* (1) *has a center (at the origin of coordinates) its period annulus is* $\mathbb{R}^2 \setminus \{(0,0)\}$ *(the whole plane).*

Proof. (i) Let (x_0, y_0) be a point different from the origin of coordinates which is a singular point of system (1). By the conditions (2) we see that the whole algebraic curve $L = \{\alpha^p x_0, \alpha^q y_0), \alpha \in \mathbb{R}\}$ is full of singular points of system (1) because

$$P(\alpha^p x_0, \alpha^q y_0) = \alpha^{p+m-1} P(x_0, y_0) = 0,$$
$$Q(\alpha^p x_0, \alpha^q y_0) = \alpha^{q+m-1} Q(x_0, y_0) = 0.$$

Therefore in any punctured neighborhood of (x_0, y_0) there are singular points which implies that there cannot be any neighborhood filled with periodic orbits. Hence (x_0, y_0) cannot be a center for system (1).

(ii) We consider system (6) which is a blow–up system of (1). If $g(\theta^*) = 0$ for $\theta^* \in \mathbb{R}$, we have that the real algebraic curve $L = \{r^p \cos\theta^*, r^q \sin\theta^*), r \in \mathbb{R}\}$ is invariant for system (1). In fact we have $L = \{(x,y) \in \mathbb{R}^2 \ : \ (\cos\theta^*)^q y^p - (\sin\theta^*)^p x^q = 0\}$. And if we define $\varphi(x,y) = (\cos\theta^*)^q y^p - (\sin\theta^*)^p x^q$ we have that $(P,Q) \cdot \nabla\varphi_{|_L} = 0$. Moreover we see that $\varphi(0,0) = 0$ and that in any neighborhood of the origin of coordinates in $\mathbb{R}^2$ there are points of the curve $\varphi(x,y) = 0$, because we are assuming that p and q are coprime and therefore at least one of them is odd. If p is odd, for instance, we see that the function $\varphi(0,y) = (\cos\theta^*)^q y^p$ changes sign in a neighborhood of $y = 0$

(unless $\cos\theta^* = 0$), which implies that $\varphi(x,y) = 0$ has a real branch passing through $(0,0)$. If $\cos\theta^* = 0$, since $g(\theta^*) = 0$ we have that $\sin\theta^* P(0, \sin\theta^*) = 0$ from here we get $P(0, \sin\theta^*) = 0$. This implies that x divides $P(x,y)$ and, therefore, there is an invariant algebraic curve with real branch passing through $(0,0)$. Hence the origin cannot be a center.

When $g(\theta)$ has no real roots, we can consider the differential equation

$$\frac{dr}{d\theta} = r\,\frac{f(\theta)}{g(\theta)}, \tag{7}$$

corresponding to system (6). We remark that system (1) has a center at the origin if and only if all the orbits in a neighborhood of $r = 0$ in system (6) are periodic of period 2π. Indeed, this is the case if and only if all the orbits in a neighborhood of $r = 0$ in equation (7) are periodic of period 2π. We denote by $r(\theta; r_0)$ the solution of (7) with initial condition $r(0; r_0) = r_0$. We remark that

$$r(\theta; r_0) = r_0 \exp\left(\int_0^\theta \frac{f(s)}{g(s)}\, ds\right).$$

We see that these orbits are periodic of period 2π if and only if

$$\int_0^{2\pi} \frac{f(s)}{g(s)}\, ds = 0. \tag{8}$$

(iii) If system (1) has a center then we have that $g(\theta)$ has no real roots. Therefore, system (6) has no singular points in the domain $r > 0$. We note that since $p, q > 0$ then $p\cos^2\theta + q\sin^2\theta > 0$ for all θ. Therefore, system (1) has no singular points except the origin of coordinates and all the orbits rotates in counterclockwise or clockwise sense, because $g(\theta) > 0$ for all θ or $g(\theta) < 0$ for all θ, respectively. If the origin of (1) is a center, then condition (8) is verified, this implies that all the solutions of equation (7) are 2π–periodic in θ, that is, $r(2\pi; r_0) = r_0$ for all $r_0 \in \mathbb{R}$. Hence, any orbit of system (1) which is not the origin of coordinates is periodic. We conclude that the period annulus of the center at the origin is $\mathbb{R}^2 \setminus \{0, 0\}$. $\square$

We consider the perturbed system (3) and the weighted blow–up $x = r^p \cos\theta$, $y = r^q \sin\theta$ and we get

$$
\begin{aligned}
\dot{r} &= \frac{r^m f(\theta)}{p\cos^2\theta + q\sin^2\theta} \\
&+ \varepsilon \frac{r^{1-p}\cos\theta \bar{P}(r^p\cos\theta, r^q\sin\theta) + r^{1-q}\sin\theta \bar{Q}(r^p\cos\theta, r^q\sin\theta)}{p\cos^2\theta + q\sin^2\theta}, \\
\dot{\theta} &= \frac{r^{m-1} g(\theta)}{p\cos^2\theta + q\sin^2\theta} \\
&+ \varepsilon \frac{r^{-q} p\cos\theta \bar{Q}(r^p\cos\theta, r^q\sin\theta) - r^{-p} q\sin\theta \bar{P}(r^p\cos\theta, r^q\sin\theta)}{p\cos^2\theta + q\sin^2\theta}.
\end{aligned}
$$

We assume that system (1) is quasi–homogeneous of weight (p, q, m) with p and q coprime (using Lemma 3) and that it has a center at the origin of coordinates. By Lemma 4, this implies that $g(\theta)$ has no real roots. Therefore, we can consider the ordinary differential equation associated to the above differential system which is

$$
\begin{aligned}
\frac{dr}{d\theta} &= r\,\frac{f(\theta)}{g(\theta)} + \varepsilon\,\frac{(p\cos^2\theta + q\sin^2\theta)}{r^{m+p+q-2} g(\theta)^2} \\
&\quad \Big[r^q Q(\cos\theta, \sin\theta) \bar{P}(r^p\cos\theta, r^q\sin\theta) \\
&\quad - r^p P(\cos\theta, \sin\theta) \bar{Q}(r^p\cos\theta, r^q\sin\theta) \Big] + \mathcal{O}(\varepsilon^2) \\
&= G_0(r,\theta) + \varepsilon G_1(r,\theta) + \mathcal{O}(\varepsilon^2).
\end{aligned}
\tag{9}
$$

We will apply the averaging theory of first order in ε to the equation (9) for studying the limit cycles which bifurcate from the period annulus $\mathcal{P}$ surrounding the origin of the unperturbed system (1). We see that, by hypothesis, an open interval with boundary the origin of coordinates and belonging to $\{(x, y) \in \mathbb{R}^2 \ : \ x > 0, \ y = 0\}$ is a transversal section for system (1) on the whole period annulus $\mathcal{P}$. Therefore there exists an ε sufficiently small such that this interval is also a transversal section for system (3). We consider the Poincaré return map associated to this transversal section and if we denote by $\varphi_\varepsilon(\theta; r_0)$ the solution of equation (9) with initial condition $\varphi_\varepsilon(0; r_0) = r_0$, the Poincaré return map is $\varphi_\varepsilon(2\pi; r_0)$. We remark that when $\varepsilon = 0$ we have that $\varphi_0(2\pi; r_0) = r_0$ for all $r_0 \geq 0$, because all the orbits are periodic with the exception of the origin. By the analytic dependence of the solutions of an ordinary analytic differential equation with respect to parameters and initial conditions we have that there exists an analytic function $\psi_1(r_0)$ such that

$$\varphi_\varepsilon(2\pi; r_0) = r_0 + \varepsilon\psi_1(r_0) + \mathcal{O}(\varepsilon^2).$$

We see that a limit cycle of (3) which bifurcates from a periodic orbit of $\mathcal{P}$ corresponds to a value r^* such that $\varphi_\varepsilon(2\pi; r^*) = r^*$. Therefore, if $\psi_1(r_0)$ is not identically zero, for each simple zero r^* of $\psi_1(r_0)$ there exists a periodic solution of (9) whose initial condition tends to r^* when $\varepsilon \to 0$, see Corollary 5 (a) in [4]. In this case we say that a limit cycle bifurcates from the periodic solution of (1) with initial condition at r^* at first order in ε. In the aforementioned paper we give the formula for $\psi_1(r_0)$ and we present it below. We have that the explicit solution for equation (9) with $\varepsilon = 0$ is

$$\varphi_0(\theta; r_0) = r_0 \exp\left(\int_0^\theta \frac{f(s)}{g(s)}\, ds \right) = r_0 u(\theta).$$

Following Theorem 4 of [4] we have that

$$\psi_1(r_0) = \int_0^{2\pi} \frac{1}{u(\theta)}\, G_1(\theta, r_0 u(\theta)) d\theta. \tag{10}$$

3. Proof of Theorem 1

Our proof is based on the one given in [7] in many aspects.

By Lemma 3, we assume that p and q are coprime. We first observe that the number of zeros $r_0 > 0$ of $\psi_1(r_0)$ coincides with the number of zeros $z > 0$ of $\bar{\psi}_1(z) = z^{m+p+q-2}\psi_1(z)$. By (10) we have that

$$\begin{aligned} \bar{\psi}_1(z) = \; & \int_0^{2\pi} \frac{1}{u(\theta)^{m+p+q-1}} \frac{(p\cos^2\theta + q\sin^2\theta)}{g(\theta)^2} \\ & \Big[z^q u(\theta)^q Q(\cos\theta, \sin\theta) \bar{P}(z^p u(\theta)^p \cos\theta, z^q u(\theta)^q \sin\theta) \\ & - z^p u(\theta)^p P(\cos\theta, \sin\theta) \bar{Q}(z^p u(\theta)^p \cos\theta, z^q u(\theta)^q \sin\theta) \Big] \, d\theta. \end{aligned} \tag{11}$$

We can write the polynomials $\bar{P}$ and $\bar{Q}$ as the sum of their quasi-homogeneous parts.

$$\begin{aligned} \bar{P}(x,y) = \; & \bar{P}_0 + \bar{P}_p(x,y) + \cdots + \bar{P}_{nq}(x,y), \\ \bar{Q}(x,y) = \; & \bar{Q}_0 + \bar{P}_p(x,y) + \cdots + \bar{P}_{nq}(x,y), \end{aligned}$$

with $\bar{P}_k$ and $\bar{Q}_k$ polynomials which are quasi–homogeneous of weight (p,q,k). We note that k takes values in the following set

$$\begin{aligned} \mathcal{S}_n \; & = \{0, p, q, 2p, p+q, 2q, \ldots, np, (n-1)p + q, \ldots, p + (n-1)q, nq\} \\ & = \{ip + jq \;:\; i,j \geq 0,\; 0 \leq i+j \leq n\}. \end{aligned}$$

All polynomial vector fields can be decomposed into quasi–homogeneous components with respect to a given weight (p,q), see for instance [5, page 46].

Hence, from (11) we have

$$
\begin{aligned}
\bar{\psi}_1(z) = \quad & \int_0^{2\pi} \frac{(p\cos^2\theta + q\sin^2\theta)}{u(\theta)^{m+p+q-1}g(\theta)^2} \\
& \Big[z^q u(\theta)^q Q(\cos\theta,\sin\theta) \sum_{k\in\mathcal{S}_n} \bar{P}_k(\cos\theta,\sin\theta) z^k u(\theta)^k \\
& -z^p u(\theta)^p P(\cos\theta,\sin\theta) \sum_{k\in\mathcal{S}_n} \bar{Q}_k(\cos\theta,\sin\theta) z^k u(\theta)^k \Big] \, d\theta.
\end{aligned}
\tag{12}
$$

We define

$$
\begin{aligned}
\varphi_k(\theta) = \quad & \frac{(p\cos^2\theta + q\sin^2\theta)}{g(\theta)^2} \Big[Q(\cos\theta,\sin\theta)\bar{P}_{k-q}(\cos\theta,\sin\theta) \\
& -P(\cos\theta,\sin\theta)\bar{Q}_{k-p}(\cos\theta,\sin\theta)\Big] u(\theta)^{k-m-p-q+1},
\end{aligned}
\tag{13}
$$

where $\bar{P}_{p-q}(\cos\theta,\sin\theta) \equiv \bar{Q}_{q-p}(\cos\theta,\sin\theta) \equiv 0$. By reparameterizing the index k in (12) we get that

$$
\bar{\psi}_1(z) = \sum_{k\in\tilde{\mathcal{S}}_n} \int_0^{2\pi} \varphi_k(\theta) z^k d\theta = \sum_{k\in\tilde{\mathcal{S}}_n} \left(\int_0^{2\pi} \varphi_k(\theta) d\theta \right) z^k, \tag{14}
$$

where

$$
\begin{aligned}
\tilde{\mathcal{S}}_n & = \{p, q, 2p, p+q, 2q, \ldots, (n+1)p, np+q, \ldots, p+nq, (n+1)q\} \\
& = \{ip + jq \ : \ i, j \geq 0, \ 0 < i+j \leq n+1\}.
\end{aligned}
$$

The following lemma corresponds to Proposition 3 of [7].

Lemma 5. *We consider a quasi-homogeneous system* (1) *of weight* (p, q, m) *with a center at the origin of coordinates. Then the following statements hold.*

(a) *If p and q are odd, then m is odd and the periodic orbits of the center are symmetric with respect to the origin of coordinates.*
(b) *If p and q are even, then m is odd.*
(c) *If p is odd and q is even, then m is even and the periodic orbits of the center are symmetric with respect to the y-axis.*
(d) *If p is even and q is odd, then m is even and the periodic orbits of the center are symmetric with respect to the x-axis.*

We remark that since we can assume; without loss of generality, that p and q are coprime, statement (b) of Lemma 5 can be discarded. The described symmetries in statements (a), (c) and (d) mean that

If p and q are odd, by statement (a) of Lemma 5 m is odd, and from (2), we have that

$$P(-x,-y) = -P(x,y),\ Q(-x,-y) = -Q(x,y); \tag{15}$$

if p is even and q is odd, by statement (d) of Lemma 5 m is even, and from (2), it follows that

$$P(x,-y) = -P(x,y), \qquad Q(x,-y) = Q(x,y); \tag{16}$$

if p is odd and q is even, by statement (c) of Lemma 5 m is even, and from (2), we obtain that

$$P(-x,y) = P(x,y), \qquad Q(-x,y) = -Q(x,y); \tag{17}$$

for all $(x,y) \in \mathbb{R}^2$.

Lemma 6. *Let P and Q be polynomials which define the quasi-homogeneous system* (1) *of weight (p,q,m). Assume that $g(\theta)$ has no real roots and that $u(2\pi) = 1$ (that is, system* (1) *has a center at the origin). Then*

(i) *If p and q are odd, $u(\theta+\pi) = u(\theta)$ and $g(\theta+\pi) = g(\theta)$ for all θ.*
(ii) *If p is even and q is odd, $u(-\theta) = u(\theta)$ and $g(-\theta) = g(\theta)$ for all θ.*
(iii) *If p is odd and q is even, $u(\pi-\theta) = u(\theta)$ and $g(\pi-\theta) = g(\theta)$ for all θ.*

Proof. (i) If p and q are odd, we need to show that

$$\int_0^{\theta+\pi} \frac{f(s)}{g(s)}\,ds - \int_0^{\theta} \frac{f(s)}{g(s)}\,ds = 0, \tag{18}$$

for all θ, this implies that $u(\theta+\pi) = u(\theta)$ for all θ. We have that

$$\int_0^{2\pi} \frac{f(s)}{g(s)}\,ds = \int_0^{\pi} \frac{f(s)}{g(s)}\,ds + \int_\pi^{2\pi} \frac{f(s)}{g(s)}\,ds.$$

In the second integral we do the change $s = \tau + \pi$ and we have

$$\int_\pi^{2\pi} \frac{f(s)}{g(s)}\,ds = \int_0^{\pi} \frac{f(\tau+\pi)}{g(\tau+\pi)}\,d\tau.$$

We recall that $\cos(\tau+\pi) = -\cos\tau$ and $\sin(\tau+\pi) = -\sin\tau$. Therefore using (15) we have

$$\begin{aligned} f(\tau+\pi) &= -P(-\cos\tau,-\sin\tau)\cos\tau - Q(-\cos\tau,-\sin\tau)\sin\tau \\ &= P(\cos\tau,\sin\tau)\cos\tau + Q(\cos\tau,\sin\tau)\sin\tau = f(\tau). \end{aligned}$$

and $g(\tau+\pi) = g(\tau)$ analogously. Therefore,

$$\int_0^{\pi} \frac{f(\tau+\pi)}{g(\tau+\pi)}\,d\tau = \int_0^{\pi} \frac{f(\tau)}{g(\tau)}\,d\tau.$$

Thus
$$\int_0^{2\pi} \frac{f(s)}{g(s)}\, ds = 2\int_0^{\pi} \frac{f(s)}{g(s)}\, ds = 0,$$
because by assumption the first integral is zero. Now we consider the left-hand side of (18)
$$\int_0^{\theta+\pi} \frac{f(s)}{g(s)}\, ds - \int_0^{\theta} \frac{f(s)}{g(s)}\, ds = \int_\theta^{\theta+\pi} \frac{f(s)}{g(s)}\, ds = \int_0^{\pi} \frac{f(s)}{g(s)}\, ds = 0,$$
because we have shown in the previous paragraph that $f(s)/g(s)$ is a π-periodic function.

(ii) If p is even and q is odd, we need to show that
$$\int_0^{-\theta} \frac{f(s)}{g(s)}\, ds - \int_0^{\theta} \frac{f(s)}{g(s)}\, ds = 0, \tag{19}$$
for all θ, this implies that $u(-\theta) = u(\theta)$ for all θ. We recall that $\cos(-s) = \cos s$ and $\sin(-s) = -\sin s$. Therefore using (16) we have
$$\begin{aligned} f(-s) &= P(\cos s, -\sin s)\cos s - Q(\cos s, -\sin s)\sin s \\ &= -P(\cos s, \sin s)\cos s - Q(\cos s, \sin s)\sin s = -f(s). \end{aligned}$$
and $g(-s) = g(s)$ analogously. Doing the change $s = -\tau$, we see that
$$\int_0^{-\theta} \frac{f(s)}{g(s)}\, ds = \int_0^{\theta} \frac{f(-\tau)}{g(-\tau)}(-d\tau) = \int_0^{\theta} \frac{f(\tau)}{g(\tau)} d\tau.$$
Therefore (19) is satisfied.

(iii) If p is odd and q is even, we want to show that
$$\int_0^{\pi-\theta} \frac{f(s)}{g(s)}\, ds - \int_0^{\theta} \frac{f(s)}{g(s)}\, ds = 0, \tag{20}$$
for all θ, this implies that $u(\pi-\theta) = u(\theta)$ for all θ. We recall that $\cos(\pi - s) = -\cos s$ and $\sin(\pi - s) = \sin s$. Thus using (17) we have
$$\begin{aligned} f(\pi - s) &= P(-\cos s, \sin s)\cos s + Q(-\cos s, \sin s)\sin s \\ &= -P(\cos s, \sin s)\cos s - Q(\cos s, \sin s)\sin s = -f(s). \end{aligned}$$
and $g(\pi - s) = g(s)$ analogously. Doing the change $s = \pi - \tau$, we have
$$\int_0^{\pi-\theta} \frac{f(s)}{g(s)}\, ds = \int_\pi^{\theta} \frac{f(\pi-\tau)}{g(\pi-\tau)}(-d\tau) = \int_\pi^{\theta} \frac{f(\tau)}{g(\tau)} d\tau.$$
The left-hand side of (17) using the previous equality writes as
$$\int_0^{\pi-\theta} \frac{f(s)}{g(s)}\, ds - \int_0^{\theta} \frac{f(s)}{g(s)}\, ds = \int_\pi^{\theta} \frac{f(s)}{g(s)}\, ds + \int_\theta^{0} \frac{f(s)}{g(s)}\, ds = -\int_0^{\pi} \frac{f(s)}{g(s)}\, ds.$$

On the other hand,doing $s = \pi - \tau$, we have that

$$\int_0^{\pi} \frac{f(s)}{g(s)}\, ds = \int_{\pi}^{0} \frac{f(\tau)}{g(\tau)} d\tau = -\int_0^{\pi} \frac{f(\tau)}{g(\tau)} d\tau.$$

Hence $\displaystyle\int_0^{\pi} \frac{f(\tau)}{g(\tau)} d\tau = 0$ and (20) is satisfied. □

We consider the functions $\varphi_k(\theta)$ defined in (13). We can establish the following result.

Lemma 7. *The following statements hold.*

(i) *If p and q are odd, then* $\displaystyle\int_0^{2\pi} \varphi_k(\theta)\, d\theta = 0$ *when k is odd.*

(ii) *If p or q is even, then* $\displaystyle\int_0^{2\pi} \varphi_k(\theta)\, d\theta = 0$ *when k is even.*

Proof. (i) If p and q are odd by Lemma 6 we have that $u(\theta + \pi) = u(\theta)$ and $g(\theta + \pi) = g(\theta)$. We recall that $\cos(\theta + \pi) = -\cos\theta$ and $\sin(\theta + \pi) = -\sin\theta$. Hence we have

$$\begin{aligned} P(-\cos\theta, -\sin\theta) &= -P(\cos\theta, \sin\theta), \\ Q(-\cos\theta, -\sin\theta) &= -Q(\cos\theta, \sin\theta), \end{aligned}$$

which is (15). Now we consider a $\bar{P}_{k-q}(x, y)$ which is a quasi-homogeneous polynomial of weight $(p, q, k - q)$. Then

$$\bar{P}_{k-q}((-1)^p x, (-1)^q y) = (-1)^{k-q} \bar{P}_{k-q}(x, y).$$

Therefore $\bar{P}_{k-q}(-x, -y) = \bar{P}_{k-q}(x, y)$ because p, q and k are odd. In the same way

$$\bar{Q}_{k-p}((-1)^p x, (-1)^q y) = (-1)^{k-p} \bar{Q}_{k-p}(x, y).$$

So $\bar{Q}_{k-p}(-x, -y) = \bar{Q}_{k-p}(x, y)$. Hence

$$\begin{aligned} \varphi_k(\theta + \pi) &= \frac{(p\cos^2\theta + q\sin^2\theta)}{g(\theta)^2} \Big[-Q(\cos\theta, \sin\theta)\bar{P}_{k-q}(\cos\theta, \sin\theta) \\ &\quad + P(\cos\theta, \sin\theta)\bar{Q}_{k-p}(\cos\theta, \sin\theta) \Big] u(\theta)^{k-m-p-q+1} = -\varphi_k(\theta). \end{aligned}$$

Consequently

$$\int_0^{2\pi} \varphi_k(\theta)\, d\theta = \int_0^{\pi} \varphi_k(\theta)\, d\theta + \int_{\pi}^{2\pi} \varphi_k(\theta)\, d\theta.$$

Now doing the change $\theta = s + \pi$ in the second integral we obtain

$$\int_0^{\pi} \varphi_k(\theta)\, d\theta + \int_0^{\pi} \varphi_k(s + \pi)\, ds = \int_0^{\pi} \varphi_k(\theta)\, d\theta - \int_0^{\pi} \varphi_k(s)\, ds = 0.$$

(ii) If p is even and q is odd by Lemma 6 we have that $u(-\theta) = u(\theta)$ and $g(-\theta) = g(\theta)$. We recall that $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$. Hence we have

$$\begin{aligned} P(\cos\theta, -\sin\theta) &= -P(\cos\theta, \sin\theta), \\ Q(\cos\theta, -\sin\theta) &= Q(\cos\theta, \sin\theta), \end{aligned}$$

which is (16). Now we consider a $\bar{P}_{k-q}(x, y)$ which is a quasi-homogeneous polynomial of weight $(p, q, k-q)$. Then

$$\bar{P}_{k-q}((-1)^p x, (-1)^q y) = (-1)^{k-q}\bar{P}_{k-q}(x, y).$$

Therefore $\bar{P}_{k-q}(x, -y) = -\bar{P}_{k-q}(x, y)$ because q is odd and p and k are even. In the same way

$$\bar{Q}_{k-p}((-1)^p x, (-1)^q y) = (-1)^{k-p}\bar{Q}_{k-p}(x, y).$$

So $\bar{Q}_{k-p}(x, -y) = \bar{Q}_{k-p}(x, y)$. Hence

$$\begin{aligned} \varphi_k(-\theta) &= \frac{(p\cos^2\theta + q\sin^2\theta)}{g(\theta)^2}\Big[Q(\cos\theta, \sin\theta)(-\bar{P}_{k-q}(\cos\theta, \sin\theta)) \\ &\quad + P(\cos\theta, \sin\theta)\bar{Q}_{k-p}(\cos\theta, \sin\theta)\Big] u(\theta)^{k-m-p-q+1} = -\varphi_k(\theta). \end{aligned}$$

Since $\varphi_k(\theta)$ is a 2π-periodic function we have

$$\int_0^{2\pi} \varphi_k(\theta)\, d\theta = \int_{-\pi}^{\pi} \varphi_k(\theta)\, d\theta = \int_{-\pi}^{0} \varphi_k(\theta)\, d\theta + \int_0^{\pi} \varphi_k(\theta)\, d\theta. \tag{21}$$

Now doing the change $\theta = -s$ in the first integral we obtain

$$\int_\pi^0 \varphi_k(-s)(-ds) + \int_0^\pi \varphi_k(\theta)\, d\theta = -\int_0^\pi \varphi_k(s)\, ds + \int_0^\pi \varphi_k(\theta)\, d\theta = 0.$$

If p is odd and q is even by Lemma 6 we have that $u(\pi - \theta) = u(\theta)$ and $g(\pi - \theta) = g(\theta)$. We recall that $\cos(\pi - \theta) = -\cos\theta$ and $\sin(\pi - \theta) = \sin\theta$. Hence we have

$$\begin{aligned} P(-\cos\theta, \sin\theta) &= P(\cos\theta, \sin\theta), \\ Q(-\cos\theta, \sin\theta) &= -Q(\cos\theta, \sin\theta), \end{aligned}$$

that is, we obtain (17). Now we consider a $\bar{P}_{k-q}(x, y)$ which is a quasi-homogeneous polynomial of weight $(p, q, k-q)$. Then

$$\bar{P}_{k-q}((-1)^p x, (-1)^q y) = (-1)^{k-q}\bar{P}_{k-q}(x, y).$$

Therefore $\bar{P}_{k-q}(-x, y) = \bar{P}_{k-q}(x, y)$ because p is odd and q and k are even. In the same way

$$\bar{Q}_{k-p}((-1)^p x, (-1)^q y) = (-1)^{k-p}\bar{Q}_{k-p}(x, y).$$

So $\bar{Q}_{k-p}(x,-y) = -\bar{Q}_{k-p}(x,y)$. Hence

$$\varphi_k(\pi-\theta) = \frac{(p\cos^2\theta + q\sin^2\theta)}{g(\theta)^2}\Big[-Q(\cos\theta,\sin\theta)\bar{P}_{k-q}(\cos\theta,\sin\theta) \\ -P(\cos\theta,\sin\theta)(-\bar{Q}_{k-p}(\cos\theta,\sin\theta))\Big]u(\theta)^{k-m-p-q+1} = -\varphi_k(\theta).$$

Consequently we get

$$\int_0^{2\pi}\varphi_k(\theta)\,d\theta = \int_0^{\pi}\varphi_k(\theta)\,d\theta + \int_{\pi}^{2\pi}\varphi_k(\theta)\,d\theta.$$

Now doing the change $\theta = \pi - s$ in the second integral we have

$$\int_0^{\pi}\varphi_k(\theta)\,d\theta + \int_0^{-\pi}\varphi_k(\pi-s)(-ds) = \int_0^{\pi}\varphi_k(\theta)\,d\theta - \int_{-\pi}^{0}\varphi_k(\theta)\,d\theta.$$

Since $\varphi_k(\theta)$ is a 2π-periodic function it satisfies equality (21). Therefore

$$\int_{-\pi}^{0}\varphi_k(\theta)\,d\theta + \int_0^{\pi}\varphi_k(\theta)\,d\theta = \int_0^{\pi}\varphi_k(\theta)\,d\theta - \int_{-\pi}^{0}\varphi_k(\theta)\,d\theta,$$

which implies

$$\int_{-\pi}^{0}\varphi_k(\theta)\,d\theta = 0.$$

Moreover doing the change $\theta = \pi - s$ we have that

$$\int_0^{\pi}\varphi_k(\theta)\,d\theta = \int_{\pi}^{0}\varphi_k(\pi-s)(-ds) = \int_0^{\pi}\varphi_k(\pi-s)\,ds = -\int_0^{\pi}\varphi_k(s)\,ds.$$

Therefore

$$\int_0^{\pi}\varphi_k(\theta)\,d\theta = 0.$$

Finally

$$\int_0^{2\pi}\varphi_k(\theta)\,d\theta = \int_{-\pi}^{0}\varphi_k(\theta)\,d\theta + \int_0^{\pi}\varphi_k(\theta)\,d\theta = 0.$$

$\square$

The difference between the bounds in Theorem 1 and the ones in [7] comes from the fact that in [7] it was erroneously stated that if p or q is even, then $\int_0^{2\pi}\varphi_k(\theta)\,d\theta = 0$ when k is odd, instead of the correct statement (ii) of Lemma 7.

By (14) and Lemma 7 we have that

$$\bar{\psi}_1(z) = \sum_{k\in\tilde{\mathcal{S}}_n}\left(\int_0^{2\pi}\varphi_k(\theta)d\theta\right)z^k = \sum_{k\in\tilde{\mathcal{S}}_n^*}\left(\int_0^{2\pi}\varphi_k(\theta)d\theta\right)z^k, \tag{22}$$

where $\tilde{\mathcal{S}}_n = \{ip + jq \,:\, i,j \geq 0,\ 0 < i+j \leq n+1\} = \mathcal{S}_{n+1} \setminus \{0\}$ and we define

$$\tilde{\mathcal{S}}_n^* = \begin{cases} \{k = ip + jq \in \tilde{\mathcal{S}}_n \,:\, k \text{ even}\} & \text{if } p \text{ and } q \text{ odd}, \\ \{k = ip + jq \in \tilde{\mathcal{S}}_n \,:\, k \text{ odd}\} & \text{if } p \text{ or } q \text{ even}. \end{cases} \tag{23}$$

Recall that p and q are assumed to be coprime by Lemma 3. We define the following set of indexes from which we will determine its cardinal:

(24)

$$B(p,q,n) =$$

$$= \left\{ k = ip + jq \,:\, i,j \geq 0, 0 \leq i+j \leq n, \begin{array}{l} k \text{ even if } p \text{ and } q \text{ odd} \\ k \text{ odd if } p \text{ or } q \text{ even} \end{array} \right\}$$

$$= \left\{ ip + jq \,:\, i,j \geq 0, 0 \leq i+j \leq n, \begin{array}{l} i+j \text{ even if } p \text{ and } q \text{ odd} \\ j \text{ odd if } p \text{ even and } q \text{ odd} \\ i \text{ odd if } p \text{ odd and } q \text{ even} \end{array} \right\}$$

We remark that if p and q are odd, the cardinal of the set $\tilde{\mathcal{S}}_n^*$ is the cardinal of the set $B(p,q,n+1)$ minus 1. This "minus 1" comes from the fact that the value $i+j=0$ is not considered in $\tilde{\mathcal{S}}_n^*$. If p or q is even, the cardinal of the set $\tilde{\mathcal{S}}_n^*$ coincides the cardinal of the set $B(p,q,n+1)$.

Next lemma provides the cardinal of the set $B(p,q,n)$.

Lemma 8. *Let p, q and n be positive integers with p and q coprime and such that $n \geq p \geq q$.*

(a) *If p and q are odd, then*

$$|B(p,q,n)| = \frac{2np - p^2 + (3 + (-1)^n)p + 1}{4}.$$

(b) *If p is even and q is odd, then*

$$|B(p,q,n)| = \frac{2np - p^2 + 2p}{4}.$$

(c) *If p is odd and q is even, then*

$$|B(p,q,n)| = \frac{2np - p^2 + 2p - (-1)^n}{4}.$$

Proof. We first consider the particular case that $p = q = 1$. Then we have that $\mathcal{S}_n = \{i \,:\, 0 \leq i \leq n+1\}$, $\tilde{\mathcal{S}}_n = \mathcal{S}_n \setminus \{0\}$ and

$$\tilde{\mathcal{S}}_n^* = \left\{ k \in \tilde{\mathcal{S}}_n \,:\, k \text{ even} \right\}.$$

We see that

$$B(1,1,n) = \{k : 0 \leq k \leq n, k \text{ even}\}$$

and its cardinal is $(n+1)/2$ if n is odd and $(n+2)/2$ if n is even. These values coincide with statement (a) of the lemma substituting p by 1. For the rest of the proof, we assume that $p > q$.

We first consider the set $\{ip + jq : i, j \geq 0, 0 \leq i + j \leq n\}$. We depict its values in the following way: we take the first quadrant of coordinate axes with i in the horizontal axes and j in the vertical axes. The values of these coordinates belong to the triangle $i, j \geq 0, 0 \leq i+j \leq n$. Next to each point (i, j) belonging to this triangle, we write the value $ip+jq$. Figure 1 corresponds to this triangle for the case $n = 12$, $p = 3$ and $q = 2$.

We define the trapezoid $\mathcal{T}_{p,q,n} = \{(i, j) : i, j \geq 0, 0 \leq i + j \leq n, j < p\}$. Figure 2 shows this trapezoid for the case $n = 12$, $p = 3$ and $q = 2$.

If (i, j) and (i', j') are two different points on the triangle $i, j \geq 0, 0 \leq i + j \leq n$, for which $ip + jq = i'p + j'q$, then $i \neq i'$ and $j \neq j'$. We take, for instance, the case $i > i'$. Then, we have $(i - i')p = (j' - j)q$. Since $p > q \geq 1$ and p and q are coprime, we deduce that the value $j' - j$ is a multiple of p, say $j' - j = mp$ for a positive integer m, and $i - i' = mq$. Geometrically, one sees that the relation $ip + jq = i'p + j'q$ can only be satisfied if the points (i', j') and (i, j) are the vertices north-west and south-east of a rectangle of height mp and width mq. As a consequence of this fact, we claim that all the values of the function $ip + jq$ are different for different points on the trapezoid $\mathcal{T}_{p,q,n}$; and that each of all the values of the set $\{ip + jq : i, j \geq 0, 0 \leq i + j \leq n\}$ is taken exactly once by one point on the trapezoid $\mathcal{T}_{p,q,n}$. The first claim comes from the fact that no rectangle of height a multiple of p is contained in the trapezoid $\mathcal{T}_{p,q,n}$ and the second claim from the fact that from any point of the triangle $i, j \geq 0, 0 \leq i + j \leq n$ one can draw a rectangle of height mp and width mq whose south-east vertex reaches the trapezoid $\mathcal{T}_{p,q,n}$. For the rest of the proof, we will only consider the points in the trapezoid $\mathcal{T}_{p,q,n}$. There are three cases p and q odd; p even and q odd; and p odd and q even.

(a) Assume that p and q are odd. We have the set

$$B(p,q,n) = \{ip + jq : i, j \geq 0, 0 \leq i + j \leq n, i + j \text{ even}\}.$$

The elements of the set $B(p,q,n)$ are the points of the trapezoid $\mathcal{T}_{p,q,n}$ which belong to segments of slope -1 with one vertex in the point $(i, 0)$ for i even and the other vertex in the point $(i - p + 1, p - 1)$ if $i - p + 1 > 0$ or $(0, i)$ otherwise. Figure 3 shows the values of the set $B(5,1,8)$ encircled, and Figure 4 the values of the set $B(5,1,9)$.

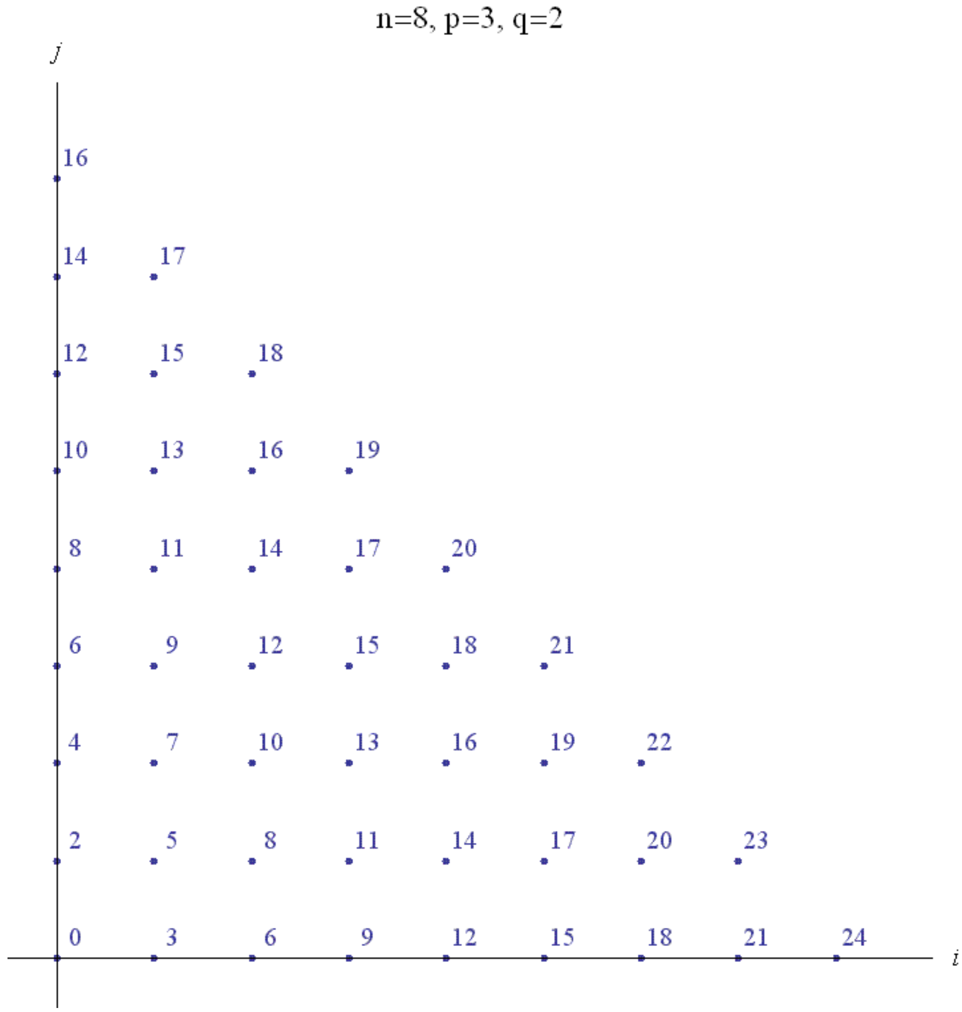

FIGURE 1. The triangle $i, j \geq 0, 0 \leq i + j \leq 8$ with values $3i + 2j$.

We first assume that n is even and we compute the cardinal of the set $B(p, q, n)$. We see that in the row corresponding to the line $j = 0$, there are $n/2 + 1$ points encircled. In each of the rows corresponding to the lines $j = 1$ and $j = 2$, there are $n/2$ points. In each of the rows corresponding to the lines $j = 3$ and $j = 4$, there are $n/2 - 1$ points. Each time that we take a row twice higher, we lose one point. Each of the last two rows, corresponding to the lines $j = p - 2$ and $j = p - 1$ (recall that p is odd by assumption), contain $(n - p + 3)/2$

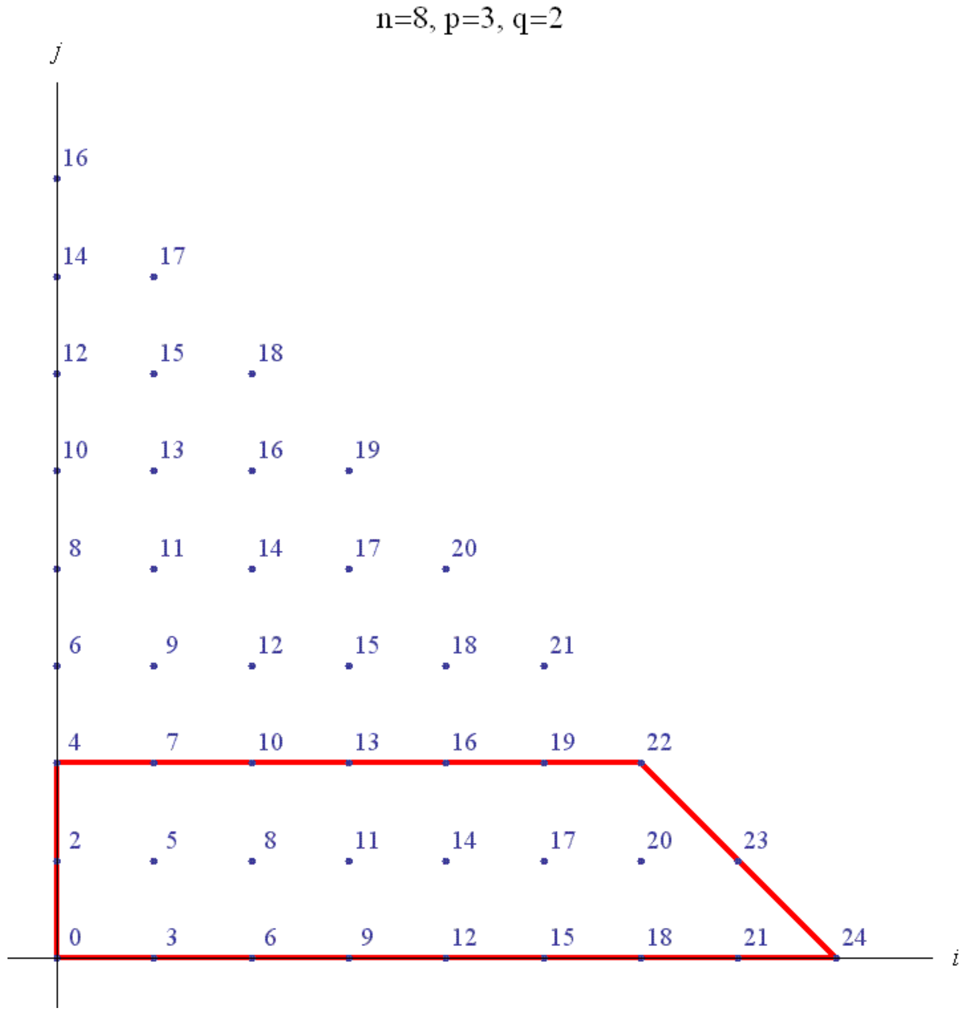

FIGURE 2. The trapezoid $\mathcal{T}_{3,2,8}$.

points. Therefore, the cardinal of the set $B(p,q,n)$ is the sum

$$\frac{n}{2}+1+2\sum_{\ell=0}^{(p-3)/2}\frac{n-\ell}{2}.$$

By adding up this arithmetic progression, we get that

$$|B(p,q,n)| = \frac{2np-p^2+4p+1}{4}.$$

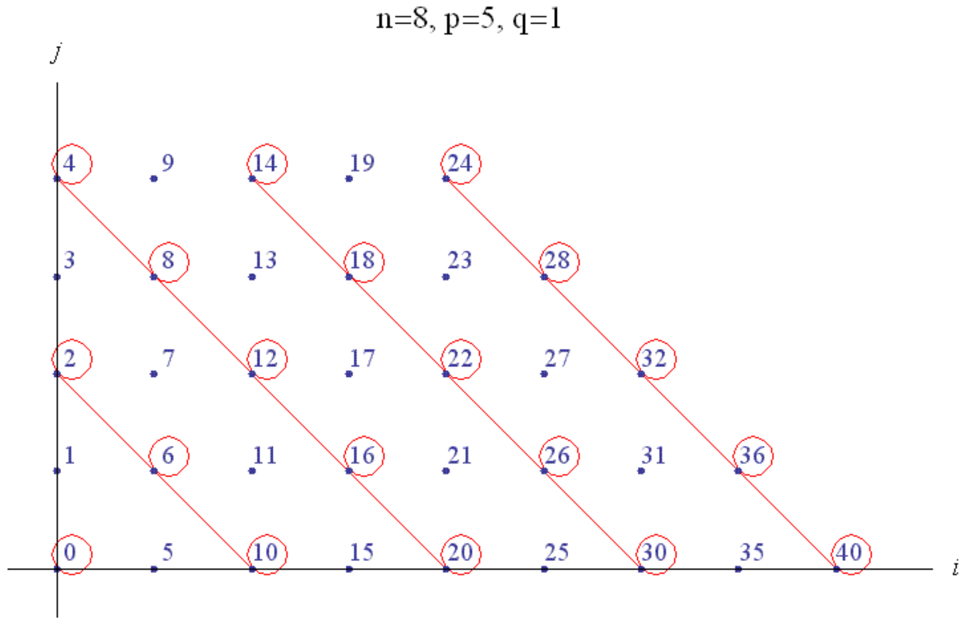

FIGURE 3. The values of set $B(5,1,8)$ are encircled.

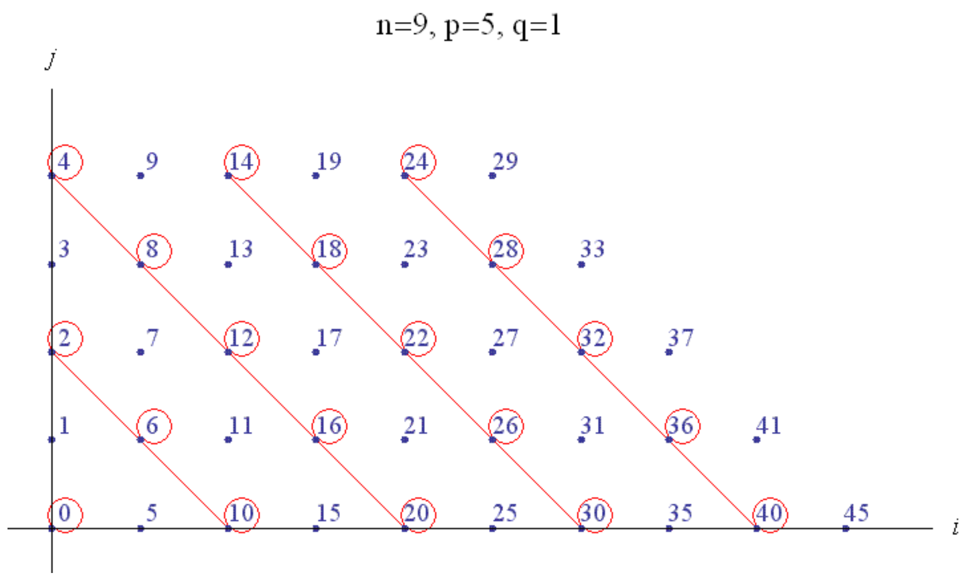

FIGURE 4. The values of set $B(5,1,9)$ are encircled.

If n is odd, we see that the values of the set $B(p,q,n)$ coincide with the values of the set $B(p,q,n-1)$, see Figures 3 and 4. Therefore

$$|B(p,q,n)| = \frac{2(n-1)p - p^2 + 4p + 1}{4} = \frac{2np - p^2 + 2p + 1}{4}.$$

The latter expressions coincide with the ones given in statement (a) of Lemma 8.

(b) Assume that p is even and q is odd. We have the set

$$B(p,q,n) = \{ip + jq \, : \, i,j \geq 0, 0 \leq i+j \leq n, j \text{ odd}\}.$$

The elements of this set are the points of the trapezoid $\mathcal{T}_{p,q,n}$ which belong to horizontal lines with one vertex in the point $(0,j)$ for j odd and the other vertex in the point $(n-j,j)$. Figure 5 shows the values of the set $B(4,1,7)$ encircled.

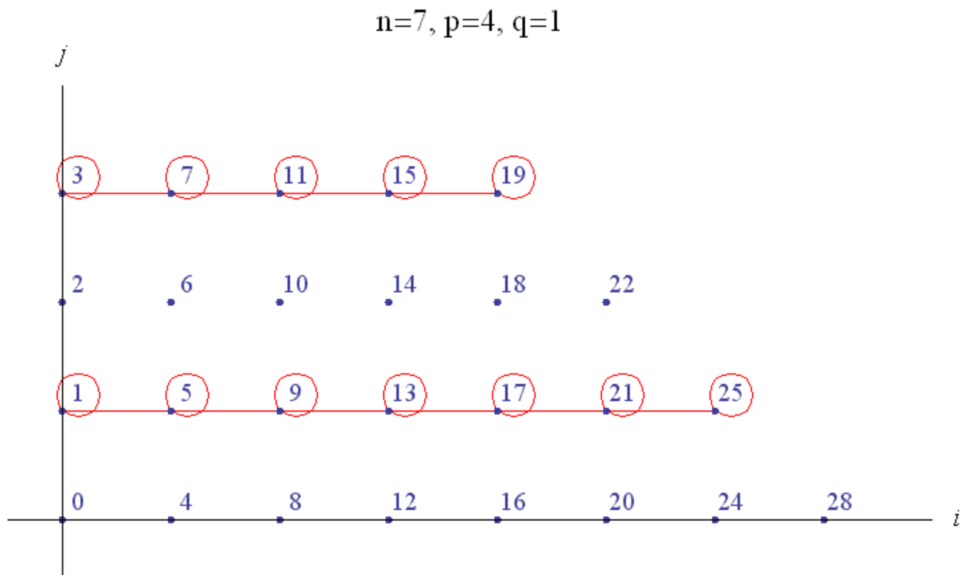

Figure 5. The values of set $B(4,1,7)$ are encircled.

We see that the row corresponding to $j=1$ contains n points encircled. The row corresponding to level $j=3$ contains $n-2$ points. Each time that we take a higher row (with j odd) we lose 2 points. The last row, which corresponds to $j=p-1$ (recall that p is even by assumption), contains $n-p+2$ points. Therefore, the cardinal of the set $B(p,q,n)$ is the sum of an arithmetic progression:

$$|B(p,q,n)| = \sum_{\ell=0}^{p/2-1} n - 2\ell = \frac{2np - p^2 + 2p}{4}.$$

(c) Assume that p is odd and q is even. We have the set

$$B(p,q,n) = \{ip + jq \, : \, i,j \geq 0, 0 \leq i+j \leq n, i \text{ odd}\}.$$

The elements of this set are the points of the trapezoid $\mathcal{T}_{p,q,n}$ which belong to vertical lines with one vertex in the point $(i,0)$ for i odd and the other vertex in the point $(i,p-1)$ if $i \le p-1$, or $(i,n-i)$ otherwise. Figure 6 shows the values of the set $B(5,2,7)$ encircled and Figure 7 the values of the set $B(5,2,8)$.

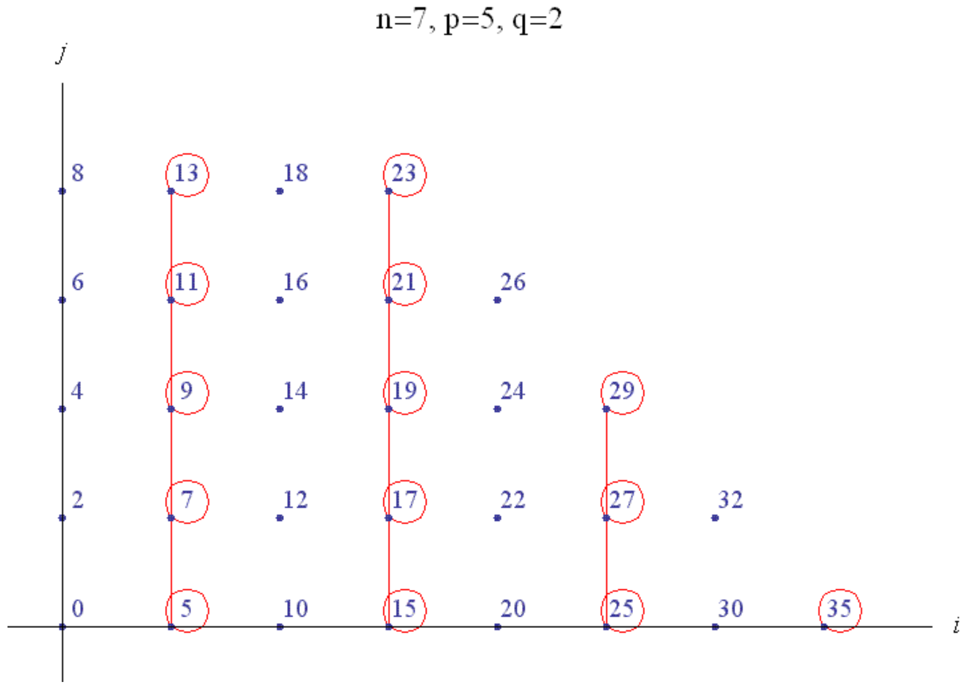

FIGURE 6. The values of set $B(5,2,7)$ are encircled.

Assume first that n is odd. See Figure 6 for an example. In the first $(n-p+2)/2$ columns (recall that p is odd by assumption), corresponding to the lines $i = 1,3,5,\ldots,n-p+1$, we have p points encircled. Then we lose two points each time we move one line to the right (that is, from the line of abscissa i to the line of abscissa $i+2$). The vertical line $i = n-p+3$ contains $p-2$ points of the trapezoid $\mathcal{T}_{p,q,n}$, the vertical line $i = n-p+5$ contains $p-4$ points, and the last "line" (with abscissa $i = n$) contains just 1 point. Hence the cardinal of the set $B(p,q,n)$ is the product of the p points in each of the first $(n-p+2)/2$ columns plus the sum of an arithmetic progression:

$$|B(p,q,n)| \;=\; p\,\frac{n-p+2}{2} \;+\; \sum_{\ell=0}^{(p-3)/2} 2\ell+1 \;=\; \frac{2np-p^2+2p+1}{4}.$$

Assume now that n is even. See Figure 7 for an example. As in the previous case, in the first $(n-p+1)/2$ columns (recall that p is odd

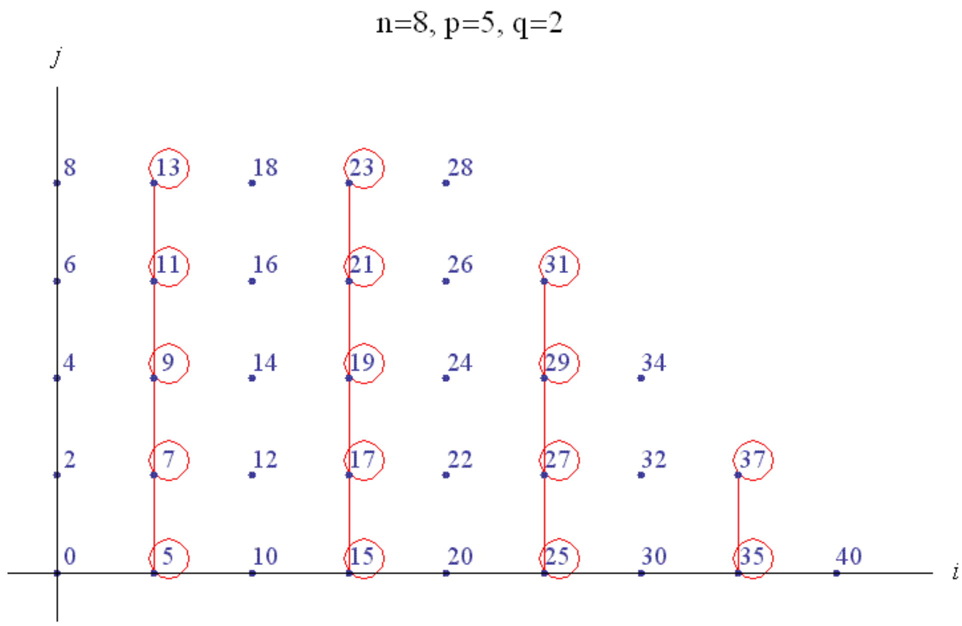

FIGURE 7. The values of set $B(5,2,8)$ are encircled.

by assumption), corresponding to the lines $i = 1, 3, 5, \ldots, n-p$, we have p points encircled. The next column at the right, corresponding to $i = n-p+2$ contains $p-1$ points. Then we lose two points each time we move one line to the right (that is, from the line of abscissa i to the line of abscissa $i+2$). The vertical line $i = n-p+4$ contains $p-3$ points of the trapezoid $\mathcal{T}_{p,q,n}$, and the last column (with abscissa $i = n-1$) contains 2 points. Hence the cardinal of the set $B(p,q,n)$ is the product of the p points in each of the first $(n-p+1)/2$ columns plus the sum of an arithmetic progression:

$$|B(p,q,n)| = p\,\frac{n-p+1}{2} + \sum_{\ell=1}^{(p-1)/2} 2\ell = \frac{2np-p^2+2p-1}{4}.$$

The latter expressions coincide with the ones given in statement (c) of Lemma 8. $\square$

By Lemma 3, we can assume that p and q are coprime and, thus, the bounds given in Theorem 1 in terms of p^* and q^* can be rewritten using p and q. The first averaging function is defined in (10) and from its expression given in (22), we have that the maximum number of its zeros $r_0 > 0$ is the number of monomials in $\tilde{\mathcal{S}}_n^*$ minus 1. Indeed, as we have already stated, if p and q are odd, then $|\tilde{\mathcal{S}}_n^*| = |B(p,q,n+1)|-1$; and if p or q is even, then $|\tilde{\mathcal{S}}_n^*| = |B(p,q,n+1)|$; see (23) and (24).

Therefore, the maximum number of zeros $r_0 > 0$ of the first averaging function is $|B(p,q,n+1)|-2$ if p and q are odd and $|B(p,q,n+1)|-1$ if p or q is even. The bounds provided in Theorem 1 are these values, once taking Lemma 8 into account.

4. Example

We consider the following planar polynomial differential system

$$\dot{x} \;=\; -y^3(x^2+y^4), \quad \dot{y} \;=\; x(x^2+y^4),$$

which is quasi-homogeneous of weight $(2,1,6)$. We note that the polynomials $P(x,y)$ and $Q(x,y)$ which define this system are not coprime. The function $H(x,y) = 2x^2+y^4$ is a first integral of this system and, therefore, the origin is a center. We perturb this center with polynomials of degree 7 and we get Proposition 2.

Proof of Proposition 2. We take

$$\bar{P}(x,y) \;=\; \sum_{i=0}^{7}\sum_{j=0}^{7-i} p_{ij}x^iy^j, \quad \bar{Q}(x,y) \;=\; \sum_{i=0}^{7}\sum_{j=0}^{7-i} q_{ij}x^iy^j,$$

where p_{ij} and q_{ij} are real constants. The weighted blow-up $x = r^2\cos\theta$, $y = r\sin\theta$, transforms the unperturbed system (4) with $\varepsilon = 0$ into

$$\dot{r} \;=\; \frac{r^6\, f(\theta)}{2\cos^2\theta+\sin^2\theta}, \quad \dot{\theta} \;=\; \frac{r^5\, g(\theta)}{2\cos^2\theta+\sin^2\theta},$$

with $f(\theta) = \cos^3\theta\sin\theta(\cos^2\theta+\sin^4\theta)$ and $g(\theta) = (2\cos^2\theta+\sin^4\theta)$ $(\cos^2\theta+\sin^4\theta)$. The first averaging function is defined in (10), where the function $G_1(r,\theta)$ is given in (9) and the function $u(\theta)$ is

$$\begin{aligned} u(\theta) \;&=\; \exp\left(\int_0^\theta \frac{f(s)}{g(s)}\,ds\right) \;=\; \exp\left(\int_0^\theta \frac{\cos^3 s\sin s}{2\cos^2 s+\sin^4 s}\,ds\right) \\ &=\; \frac{2^{1/4}}{\left(2\cos^2\theta+\sin^4\theta\right)^{1/4}}. \end{aligned}$$

We consider the functions $\varphi_k(\theta)$ defined in (13) and the expression of the first averaging function (14) as the sum of monomials of z whose coefficients are the integrals of the functions $\varphi_k(\theta)$ over a period. As a consequence of the proof of Theorem 1 (see the proof of Lemma 8), we only need to take the values of k which are odd and belong to the trapezoid $\mathcal{T}_{2,1,7}$. The values on the trapezoid are depicted in Figure 8 and the values of k to be considered are encircled.

Therefore we only need to consider the functions φ_1, φ_3, φ_5, φ_7, φ_9, φ_{11}, φ_{13} because the integral over a period of $\varphi_k(\theta)$ for k even is null,

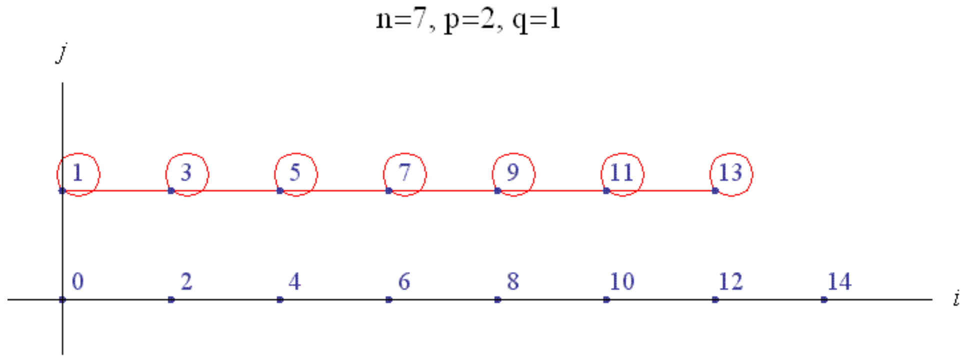

FIGURE 8. The values k to be considered are encircled.

see Lemma 7. We have 7 functions but we will show that the integral of φ_1 over a period is null and that the integrals over a period of the other functions φ_k, $k = 3, 5, 7, 9, 11, 13$, can be taken to be not null. From (13) we have that

$$
\begin{aligned}
\varphi_1(\theta) &= \frac{(2\cos^2\theta+\sin^2\theta)}{g(\theta)^2}\Big[\cos\theta(\cos^2\theta+\sin^4\theta)p_{00}\Big]u(\theta)^{-7};\\
\varphi_3(\theta) &= \frac{(2\cos^2\theta+\sin^2\theta)}{g(\theta)^2}\Big[\cos\theta(\cos^2\theta+\sin^4\theta)(p_{10}\cos\theta+p_{02}\sin^2\theta)\\
&\quad +\sin^3\theta(\cos^2\theta+\sin^4\theta)q_{01}\sin\theta\Big]u(\theta)^{-5};\\
\varphi_5(\theta) &= \frac{(2\cos^2\theta+\sin^2\theta)}{g(\theta)^2}\Big[\cos\theta(\cos^2\theta+\sin^4\theta)(p_{20}\cos^2\theta\\
&\quad +p_{12}\cos\theta\sin^2\theta+p_{04}\sin^4\theta)+\sin^3\theta(\cos^2\theta+\sin^4\theta)\\
&\quad (q_{11}\cos\theta\sin\theta+q_{03}\sin^3\theta)\Big]u(\theta)^{-3};\\
\varphi_7(\theta) &= \frac{(2\cos^2\theta+\sin^2\theta)}{g(\theta)^2}\Big[\cos\theta(\cos^2\theta+\sin^4\theta)(p_{30}\cos^3\theta\\
&\quad +p_{22}\cos^2\theta\sin^2\theta+p_{14}\cos\theta\sin^4\theta+p_{06}\sin^6\theta)\\
&\quad +\sin^3\theta(\cos^2\theta+\sin^4\theta)(q_{21}\cos^2\theta\sin\theta+q_{13}\cos\theta\sin^3\theta\\
&\quad +q_{05}\sin^5\theta)\Big]u(\theta)^{-1};
\end{aligned}
$$

$$
\begin{aligned}
\varphi_9(\theta) \;=\;& \frac{(2\cos^2\theta+\sin^2\theta)}{g(\theta)^2}\Big[\cos\theta(\cos^2\theta+\sin^4\theta)(p_{40}\cos^4\theta \\
& +p_{32}\cos^3\theta\sin^2\theta+p_{24}\cos^2\theta\sin^4\theta+p_{16}\cos\theta\sin^6\theta) \\
& +\sin^3\theta(\cos^2\theta+\sin^4\theta)(q_{31}\cos^3\theta\sin\theta+q_{23}\cos^2\theta\sin^3\theta \\
& +q_{15}\cos\theta\sin^5\theta+q_{07}\sin^7\theta)\Big]u(\theta); \\
\varphi_{11}(\theta) \;=\;& \frac{(2\cos^2\theta+\sin^2\theta)}{g(\theta)^2}\Big[\cos\theta(\cos^2\theta+\sin^4\theta)(p_{50}\cos^5\theta \\
& +p_{42}\cos^4\theta\sin^2\theta+p_{34}\cos^3\theta\sin^4\theta)+\sin^3\theta(\cos^2\theta+\sin^4\theta) \\
& (q_{41}\cos^4\theta\sin\theta+q_{33}\cos^3\theta\sin^3\theta+q_{25}\cos^2\theta\sin^5\theta)\Big]u(\theta)^3; \\
\varphi_{13}(\theta) \;=\;& \frac{(2\cos^2\theta+\sin^2\theta)}{g(\theta)^2}\Big[\cos\theta(\cos^2\theta+\sin^4\theta)(p_{60}\cos^6\theta \\
& +p_{52}\cos^5\theta\sin^2\theta)+\sin^3\theta(\cos^2\theta+\sin^4\theta)(q_{51}\cos^5\theta\sin\theta \\
& +q_{43}\cos^4\theta\sin^3\theta)\Big]u(\theta)^5.
\end{aligned}
$$

We observe that if a periodic function $\vartheta(\theta)$ of period 2π is such that $\vartheta(\pi-\theta) = -\vartheta(\theta)$, then the value of the integral of this function over a period is null. This observation implies that

$$\int_0^{2\pi} \varphi_1(\theta)\, d\theta \;=\; 0.$$

Analogously, if a periodic function $\vartheta(\theta)$ of period 2π is such that $\vartheta(-\theta) = -\vartheta(\theta)$, then the value of the integral of this function over a period is null. Using these observations, we can get rid of some terms and we can define the functions $\tilde{\varphi}_k(\theta)$, for $k = 3, 5, 7, 9, 11, 13$, such that

$$\int_0^{2\pi} \tilde{\varphi}_k(\theta)\, d\theta \;=\; \int_0^{2\pi} \varphi_k(\theta)\, d\theta.$$

We have

$$
\begin{aligned}
\tilde{\varphi}_3(\theta) &= \frac{(2\cos^2\theta+\sin^2\theta)}{g(\theta)^2}\Big[\cos\theta(\cos^2\theta+\sin^4\theta)p_{10}\cos\theta \\
&\quad +\sin^3\theta(\cos^2\theta+\sin^4\theta)q_{01}\sin\theta\Big]u(\theta)^{-5}; \\
\tilde{\varphi}_5(\theta) &= \frac{(2\cos^2\theta+\sin^2\theta)}{g(\theta)^2}\Big[\cos\theta(\cos^2\theta+\sin^4\theta)p_{12}\cos\theta\sin^2\theta \\
&\quad +\sin^3\theta(\cos^2\theta+\sin^4\theta)q_{03}\sin^3\theta\Big]u(\theta)^{-3}; \\
\tilde{\varphi}_7(\theta) &= \frac{(2\cos^2\theta+\sin^2\theta)}{g(\theta)^2}\Big[\cos\theta(\cos^2\theta+\sin^4\theta)(p_{30}\cos^3\theta \\
&\quad +p_{14}\cos\theta\sin^4\theta)+\sin^3\theta(\cos^2\theta+\sin^4\theta)(q_{21}\cos^2\theta\sin\theta \\
&\quad +q_{05}\sin^5\theta)\Big]u(\theta)^{-1}; \\
\tilde{\varphi}_9(\theta) &= \frac{(2\cos^2\theta+\sin^2\theta)}{g(\theta)^2}\Big[\cos\theta(\cos^2\theta+\sin^4\theta)(p_{32}\cos^3\theta\sin^2\theta \\
&\quad +p_{16}\cos\theta\sin^6\theta)+\sin^3\theta(\cos^2\theta+\sin^4\theta)(q_{23}\cos^2\theta\sin^3\theta \\
&\quad +q_{07}\sin^7\theta)\Big]u(\theta); \\
\tilde{\varphi}_{11}(\theta) &= \frac{(2\cos^2\theta+\sin^2\theta)}{g(\theta)^2}\Big[\cos\theta(\cos^2\theta+\sin^4\theta)(p_{50}\cos^5\theta \\
&\quad +p_{34}\cos^3\theta\sin^4\theta)+\sin^3\theta(\cos^2\theta+\sin^4\theta)(q_{41}\cos^4\theta\sin\theta \\
&\quad +q_{25}\cos^2\theta\sin^5\theta)\Big]u(\theta)^3; \\
\tilde{\varphi}_{13}(\theta) &= \frac{(2\cos^2\theta+\sin^2\theta)}{g(\theta)^2}\Big[\cos\theta(\cos^2\theta+\sin^4\theta)p_{52}\cos^5\theta\sin^2\theta \\
&\quad +\sin^3\theta(\cos^2\theta+\sin^4\theta)q_{43}\cos^4\theta\sin^3\theta\Big]u(\theta)^5.
\end{aligned}
$$

We note that each of the periodic functions appearing in $\tilde{\varphi}_{2\ell+1}$ with $\ell=1,2,\ldots,6$ whose coefficients are p_{10}, q_{01}, p_{12}, q_{03}, p_{30}, p_{14}, q_{21}, q_{05}, p_{32}, p_{16}, q_{23}, q_{07}, p_{50}, p_{34}, q_{41}, q_{25}, p_{52}, q_{23} are strictly positive. This implies that given any point $(c_1,c_2,c_3,c_4,c_5,c_6)\in\mathbb{R}^6$, there is a choice of the polynomials $\bar{P}(x,y)$ and $\bar{Q}(x,y)$ for which

$$\int_0^{2\pi}\tilde{\varphi}_{2\ell+1}(\theta)\,d\theta \;=\; c_\ell, \qquad \ell=1,2,\ldots,6.$$

We have, as a consequence of (14), that the first averaging function becomes

$$\bar{\psi}_1(z) = \sum_{\ell=1}^{6} \left(\int_0^{2\pi} \varphi_{2\ell+1}(\theta) d\theta \right) z^{2\ell+1} = \sum_{\ell=1}^{6} c_\ell \, z^{2\ell+1}. \tag{25}$$

Therefore the polynomial $\bar{\psi}_1(z)$ has at most 5 positive roots and there are values of c_ℓ for which it has exactly 5 positive and simple roots. □

We remark that for any given system (3), we can remove useless terms from the integrand of (5) in an analogous way as we did in the previous example. Indeed, only integrands which are strictly positive will appear, and consequently the sharp upper bound of simple zeros of the function $\bar{\psi}_1(z)$ can always be computed as in the proof of Proposition 2.

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