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On the integrable rational Abel differential equations

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Abstract. In Cheb-Terrab and Roche (Comput Phys Commun 130(1–2):204–231, 2000) a classification of the Abel equations known as solvable in the literature was presented. In this paper, we show that all the integrable rational Abel differential equations that appear in Cheb-Terrab and Roche (Comput Phys Commun 130(1–2):204–231, 2000) and consequently in Cheb-Terrab and Roche (Eur J Appl Math 14(2):217–229, 2003) can be reduced to a Riccati differential equation or to a first-order linear differential equation through a change with a rational map. The change is given explicitly for each class. Moreover, we have found a unified way to find the rational map from the knowledge of the explicitly first integral.

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1. Introduction and statement of the results

In this work, we study the class of integrable *rational Abel differential equations* of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f_0(x) + f_1(x)y + f_2(x)y^2 + f_3(x)y^3, \tag{1}$$

where $f_i(x)$ are rational functions of x. Here *integrable* means that the Abel differential equation has an explicit first integral $H(x, y)$ defined in all $\mathbb{R}^2$ except in a Lebesgue set of zero measure.

Abel equations appear in the reduction of order of many second- and high-order families, and hence are frequently found in the modelling of real problems in varied areas. In [5] it is given a classification according to invariant theory of the integrable Abel differential equations, i.e. of the Abel equations known as solvable in the literature.

The classification of the known integrable cases is derived from the analysis of the works of Abel [1], Apell [2], Liouville [14–16], and Kamke's textbook [13]. During these last 130 years all the integrable rational Abel differential equations that have been found have been reduced by Cheb-Terrab and Roche [5] to four classes depending on one parameter and seven classes formed by a unique equation. All these classes are summarized in Appendix. We remark that the Class 1 of Appendix can be written into the form (1) doing the change $\{X = y,\ Y = 1/(-x - 3y + 3y^2)\}$.

Our main result is the following.

Theorem 1. *Any integrable rational Abel differential equation given in Appendix can be transformed to a Riccati differential equation or to a first-order linear differential equation doing a change with a rational map.*

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