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Weierstrass integrability in Liénard differential systems ☆

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ABSTRACT

In this work we study the Liénard differential systems that admit a Weierstrass first integral or a Weierstrass inverse integrating factor.

1. Introduction

We consider the Liénard differential equations of the form

$$\ddot{x} + f(x)\dot{x} + g(x) = 0, \tag{1}$$

where $f, g : \mathbb{R} \to \mathbb{R}$ are real C^k-functions and the dot denotes, as usual, derivative with respect to the independent variable t. Here k runs over $1, 2, \ldots, \infty, \omega$. Of course C^ω denotes the class of analytic functions. In what follows we denote by F and G the two functions such that $F'(x) = f(x)$, $G'(x) = g(x)$, $F(0) = 0$ and $G(0) = 0$. We can write the differential equation (1), taking coordinates x and $Y = \dot{x}$, as the differential system

$$\dot{x} = Y, \qquad \dot{Y} = -f(x)Y - g(x), \tag{2}$$

and in the coordinates x and $y = \dot{x} + F(x)$, it becomes

$$\dot{x} = y - F(x), \qquad \dot{y} = -g(x). \tag{3}$$

The application of these differential equations to the natural sciences and technology is enormous, and well justifies their continued study. Moreover as we will see many other systems have transformations which bring them to the form (1) or to the related systems (2) and (3). The most important recent lines of research in the Liénard differential systems are the study of the center problem and of their limit cycles, see [6,7,12,20,21] and references therein. In the present paper we study the integrability problem for such systems.

For the moment a universal definition of integrability for a dynamical system seems elusive. There exist several definitions of integrability and it is still an open problem to clarify completely the relationships between them, see [16]. For

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