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On the planar integrable differential systems

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Abstract. In this paper, we prove that the C^1 planar differential systems that are integrable and non-Hamiltonian roughly speaking are C^1 equivalent to the linear differential systems $\dot{u} = u$, $\dot{v} = v$. Additionally, we show that these systems have always a Lie symmetry. These results are improved for the class of polynomial differential systems defined in $\mathbb{R}^2$ or $\mathbb{C}^2$.

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1. Introduction and statement of the results

We deal with the *planar differential systems*

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y), \tag{1}$$

where $P, Q : U \to \mathbb{R}$ are C^k functions in the variables x and y, U is an open subset of $\mathbb{R}^2$, and the dot denotes derivative with respect to the independent variable t. The open set U is called the *domain of definition* of system (1). Here, k runs over $1, 2, \ldots, \infty, \omega$. Of course, C^ω denotes the class of analytic functions.

Let φ be the C^k flow defined by the differential system (1). We denote by Σ the union of all separatrices of φ, for a definition of separatrix, see [1,10]. It is known that Σ is a closed invariant subset of U. A component C_i of $U \setminus \Sigma$ with the restricted flow $\varphi|_{C_i}$ is called a *canonical region* of φ. Then, the local flow $\varphi|_{C_i}$ has a C^k first integral H_i on every canonical region C_i of φ, see [1,9], i.e., there exists a non-constant C^k function $H_i : C_i \to \mathbb{R}$, which is constant on the orbits of the flow contained in C_i, or equivalently $P(H_i)_x + Q(H_i)_y|_{C_i} = 0$.

When the differential system (1) has a non-constant function $H : U \setminus \Sigma \to \mathbb{R}$, which is constant on the orbits of the flow contained in $U \setminus \Sigma$ (or equivalently $PH_x + QH_y = 0$ on $U \setminus \Sigma$), we say that we have a *canonical first integral* of system (1). A planar differential system having a canonical first integral is called *canonical integrable.*

Since any flow on U has a first integral H_i on every canonical region C_i of $U \setminus \Sigma = \cup_{i \in I} C_i$ of the same differentiability than the flow, we have that *for a C^r vector field with $r \neq \omega$ we always can define a C^r first integral H on $U \setminus \Sigma$ taking $H|_{C_i} = H_i$.* So for $r \neq \omega$, any C^r planar flow is integrable on $U \setminus \Sigma$. A difficult problem, non-solved in general, is how compute such a first integral H?

Let $\Sigma' \subset \Sigma$, where as usual Σ is the set of all separatrices of the differential system (1). Then if system (1) has a non-constant function $H : U \setminus \Sigma' \to \mathbb{R}$ which is constant on the orbits of the flow contained in

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