ON LIOUVILLIAN INTEGRABILITY OF THE FIRST–ORDER POLYNOMIAL ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. Recently the authors provided an example of an integrable Liouvillian planar polynomial differential system that has no finite invariant algebraic curves, see [8]. In this note we prove that if a complex differential equation of the form $y' = a_0(x) + a_1(x)y + \cdots + a_n(x)y^n$ with $a_i(x)$ polynomials for $i = 0, 1, \ldots, n$, $a_n(x) \neq 0$ and $n \geq 2$ has a Liouvillian first integral, then it has a finite invariant algebraic curve. So, this result applies to the Riccati and Abel polynomial differential equations. We shall prove that in general this result is not true when $n = 1$, i.e. for linear polynomial differential equations.

1. Introduction and the main results

By definition a *complex planar polynomial differential system* or simply a *polynomial system* is a differential system of the form

$$\frac{dx}{dt} = \dot{x} = P(x, y), \qquad \frac{dy}{dt} = \dot{y} = Q(x, y), \tag{1}$$

where the dependent variables x and y are complex, and the independent one (the *time*) t can be real or complex, and $P, Q \in \mathbb{C}[x, y]$ where $\mathbb{C}[x, y]$ is the ring of all polynomials in the variables x and y with coefficients in $\mathbb{C}$. We denote by $m = \max\{\deg P, \deg Q\}$ the *degree* of the polynomial system.

Let $f = f(x, y) = 0$ be an algebraic curve in $\mathbb{C}^2$. We say that it is *invariant* or that it is a *finite invariant algebraic curve* by the polynomial system (1) if $P\,\partial f/\partial x + Q\,\partial f/\partial y = kf$, for some polynomial $k = k(x, y) \in \mathbb{C}[x, y]$, called the *cofactor* of the algebraic curve $f = 0$. Note that the degree of the polynomial k is at most $m - 1$.

Let $h, g \in \mathbb{C}[x, y]$ and assume that h and g are relatively prime in the ring $\mathbb{C}[x, y]$. Then the function $\exp(g/h)$ is called an *exponential factor* of the polynomial system (1) if for some polynomial $k \in \mathbb{C}[x, y]$ of degree at most $m - 1$ it satisfies equation $P\,\partial \exp(g/h)/\partial x + Q\,\partial \exp(g/h)/\partial y = k \exp(g/h)$. If $\exp(g/h)$ is an exponential factor it is easy to show that $h = 0$ is an invariant algebraic curve.

Let U be an open subset of $\mathbb{C}^2$. We say that a non-constant function $H : U \to \mathbb{C}$ is a *first integral* of the polynomial system (1) in U if H is constant on the trajectories of the polynomial system (1) contained in U.

We say that a non-constant function $R : U \to \mathbb{C}$ is an *integrating factor* of the polynomial system (1) in U if R satisfies that $\partial(RP)/\partial x + \partial(RQ)/\partial y = 0$, in the points $(x, y) \in U$.

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