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A method for characterizing nilpotent centers ☆

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ABSTRACT

To characterize when a nilpotent singular point of an analytic differential system is a center is of particular interest, first for the problem of distinguishing between a focus and a center, and second for studying the bifurcation of limit cycles from it or from its period annulus. We provide necessary conditions for detecting nilpotent centers based on recent developments. Moreover we survey the last results on this problem and illustrate our approach by means of examples.

1. Introduction and statement of the main results

This work deals mainly with the distinction between a center and a focus in the case of a nilpotent singular point, the so-called *center problem* for nilpotent singular points. A related problem is to characterize when there exists an analytic first integral in a neighborhood of a singular point which is a center, see [11]. Let $p \in \mathbb{R}^2$ be a singular point of a differential system in $\mathbb{R}^2$. We recall that p is a *center* if there is a neighborhood U of p such that all the orbits of $U \setminus \{p\}$ are periodic, and p is a *focus* if there is a neighborhood U of p such that all the orbits of $U \setminus \{p\}$ spiral either in forward or in backward time to p.

Assume that p is a center that we can suppose at the origin of coordinates. After a linear change of variables and a scaling of the time variable (if necessary), the system can be written in one of the following three forms:

$$\dot{x} = -y + F_1(x, y), \qquad \dot{y} = x + F_2(x, y); \tag{1}$$

$$\dot{x} = y + F_1(x, y), \qquad \dot{y} = F_2(x, y); \tag{2}$$

$$\dot{x} = F_1(x, y), \qquad \dot{y} = F_2(x, y); \tag{3}$$

where F_1 and F_2 are real analytic functions without constant and linear terms, defined in a neighborhood of the origin, and the dot denotes derivative with respect to the independent variable t, usually called the time. The center is called of *linear type* (also badly called *non-degenerate*), *nilpotent* or *degenerate*, if it can be written after an affine change of variables and a scaling of time as system (1), (2) or (3), respectively.

The characterization of the linear type centers is well-known in terms of the existence of an analytic first integral, see [26,33]. For this type of centers it is also possible to use the Poincaré return map to characterize the existence of

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