ZERO–HOPF BIFURCATION IN THE VOLTERRA–GAUSE SYSTEM OF PREDATOR–PREY TYPE

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ABSTRACT. We prove that the Volterra–Gause system of predator–prey type exhibits two kinds of zero–Hopf bifurcations for convenient values of their parameters. In the first one periodic solution bifurcates from a zero–Hopf equilibrium, and in the second five periodic solutions bifurcate from another zero–Hopf equilibrium. This study is done using the averaging theory of second order.

1. Introduction

The very first mathematical predator-prey models have been originally proposed by Lotka [8] and Volterra [12]. In their seminal works, the interaction between prey x and predator y and between predator y and top-predator z was represented by the functional responses xy and yz respectively. Nevertheless, such a representation did not take into account the satiety of the predator and that of the top-predator, *i.e.* the saturation of the predation rate which is in this case unlimited. In the middle of the thirties Gause [3, 4] decided to make an "experimental verification of the mathematical theory of the struggle for existence". Then, he obtained a reasonable fit to a predation rate curve by taking the square root of x (and of y respectively)[1]. So, he replaced the functional response xy of the predator by $x^{1/2}y$ and that of the top-predator yz can be replaced by $y^{1/2}z$. Thus, the Volterra-Gause model proposed by Ginoux *et al.* [5] is a three-dimensional model including a prey x, a predator y and a top-predator z, which they named the Volterra-Gause model because it combines the original model of Volterra [12] incorporating a logisitic limitation of Verhulst [11] type on growth of the prey x and a limitation of Gause [3, 4] type on the intensity of predation of the predator y on the prey x and of the top-predator z on the predator y. In its original form the Volterra-Gause model has eight parameters but expressing its equations in a dimensionless form makes it possible to reduce this number to three. The dimensionless model is presented below.

This differential system can be written as:

$$
\begin{aligned}
\dot{x} &= a(x(1-x) - \sqrt{x}y),\\
\dot{y} &= -by + \sqrt{x}y - \sqrt{y}z,\\
\dot{z} &= c(\sqrt{y} - d)z,
\end{aligned}
\tag{1}
$$

2010 *Mathematics Subject Classification.* Primary 34C23, 34C25, 37G10.

Key words and phrases. Volterra–Gause system; predator–prey system, zero–Hopf bifurcation; periodic orbits.

[1]Let's note that in the beginning of the seventies, Rosenzweig [9] generalized this procedure by taking x to the g^{th} power with $0 < g \leqslant 1$. Thus, the predator rate reads: $x^g y$ for the predator and $y^g z$ for the top-predator.