INTEGRABILITY CONDITIONS OF A RESONANT SADDLE IN GENERALIZED LIÉNARD-LIKE COMPLEX POLYNOMIAL DIFFERENTIAL SYSTEMS

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Abstract. We consider a complex differential system with a resonant saddle at the origin. We compute the resonant saddle quantities and using Gröbner bases we find the integrability conditions for such systems up to a certain degree. We also establish a conjecture about the integrability conditions for such systems when they have arbitrary degree.

1. Introduction

The center problem for polynomial vector fields in the real plane with an elementary singular point of the form

$$\dot{x} = -y + \cdots, \qquad \dot{y} = x + \cdots,$$

where the dots means higher order terms is a subject of much work during these last decades, see for instance [2, 6]. These type of systems can be embedded by the change of variable $u = x + iy$ and the corresponding conjugate variable $v = x - iy$ into the complex system of the form

$$\dot{u} = u + \cdots, \qquad \dot{v} = -v + \cdots.$$

The next generalization of the above system is to consider the case of a polynomial system in the complex plane of the form

$$\dot{u} = \lambda_1 u + \cdots, \qquad \dot{v} = -\lambda_2 v + \cdots, \tag{1}$$

where $\lambda_1, \lambda_2 \in \mathbb{C}$. However if $k := (\lambda_1, \lambda_2)$ does not satisfies the resonant condition $(\alpha, k) - \lambda_m = 0$, for all $m \in \{1, 2\}$ and for all $\alpha \in \mathbb{N}_0^2$ with $|\alpha| \geq 2$ then system (1) is formally equivalent to its normal form $\dot{u} = \lambda_1 u$, $\dot{v} = -\lambda_2 v$, see [6]. Hence we consider the case with a $p : -q$ resonant elementary singular point

$$\dot{u} = p\, u + \cdots, \qquad \dot{v} = -q\, v + \cdots, \tag{2}$$

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with $p, q \in \mathbb{Z}$. If $p, q > 0$, $(p, q) = 1$ then the linear part has the analytic first integral $H_0 = x^q y^p$ and we can seek the conditions for the existence of an analytic first integral $H = H_0 + \cdots$ for system (2). Hence get the equation $\dot{H} = v_1 H_0^2 + v_3 H_0^3 + \cdots$, and the so-called *$p : -q$ resonant saddle quantities v_i* are polynomials in the coefficients of system (2). If all the v_i are zero we say that we have an *analytic resonant saddle*, see [7] and references therein.

In this work we aim to study analytic differential systems in the complex plane of the form

$$\dot{x} = x + g(x)y, \qquad \dot{y} = -y + f(y)x, \tag{3}$$

where $f(y) = \sum_{j \geq 1} a_j y^j$ and $g(x) = \sum_{j \geq 1} b_j x^j$ are analytic functions without constant terms. In fact system (3) has a $1 : -1$ resonant saddle singular point at the origin. System (3) with $g(x) = 0$ was studied in [4] where the following result was given.

Theorem 1. ([4]) *The complex polynomial differential system* (3) *with $g(x) = 0$ has an integrable saddle at the origin if and only if one of the following two conditions holds:*

(1) $a_1 = a_2 = 0$;
(2) $a_i = 0$ *for* $i \geq 2$.

The case $g(x) \neq 0$ is much harder and we have studied the polynomial case when f and g are polynomials of degree less than or equal to 6, and we have obtained the following result.

Theorem 2. *The complex polynomial differential system* (3) *when f and g are polynomials of degree ≤ 6 has an analytic integrable saddle at the origin, if and only if, the following conditions hold:*

$$a_2 + b_2 = -a_1 a_3 + b_1 b_3 = a_1^2 a_4 + b_1^2 b_4 = -a_1^3 a_5 + b_1^3 b_5 = a_1^4 a_6 + b_1^4 b_6 = 0.$$

The proof of this theorem is given in section 2.

From this result we can establish the following conjecture for the complex polynomial differential system (3) when the degrees of the polynomials f and g are arbitrary.

Conjecture 3. *The complex polynomial differential system* (3) *has an analytic integrable saddle at the origin, if and only if, the following conditions holds:*

$$\begin{aligned} a_2 + b_2 &= 0, \\ -a_1^{i-2} a_i + b_1^{i-2} b_i &= 0, \ \textit{for } i \textit{ odd}, \\ a_1^{i-2} a_i + b_1^{i-2} b_i &= 0, \ \textit{for } i \textit{ even}. \end{aligned}$$

2. Proof of Theorem 2

The sufficiency of Theorem 2 is proved in the following lemma.

Lemma 4. *System* (3) *when f and g are polynomials of degree ≤ 6 with* $a_2 + b_2 = -a_1a_3 + b_1b_3 = a_1^2a_4 + b_1^2b_4 = -a_1^3a_5 + b_1^3b_5 = a_1^4a_6 + b_1^4b_6 = 0$ *has an analytic first integral defined in a neighborhood of the origin.*

Proof. First we vanish the conditions taking $b_2 = -a_2$, $a_3 = k_3b_1$, $b_3 = k_3a_1$, $a_4 = -k_4b_1^2$, $b_4 = k_4a_1^2$, $a_5 = k_5b_1^3$, $b_5 = k_5a_1^3$, $a_6 = -k_6b_1^4$, $b_6 = k_6a_1^4$, where k_3, k_4, k_5 and k_6 are arbitrary constants. System (3) takes the form

$$\begin{aligned} \dot{x} &= x + (a_1x + a_2x^2 + b_1k_3x^3 - b_1^2k_4x^4 + b_1^3k_5x^5 - b_1^4k_6x^6)y, \\ \dot{y} &= -y + (b_1y - a_2y^2 + a_1k_3y^3 + a_1^2k_4y^4 + a_1^3k_5y^5 + a_1^4k_6y^6)x. \end{aligned}$$

Now we do the change of coordinates $X = b_1x$ and $Y = a_1y$ and the scaling of time $dt = d\tau/(b_1a_1)$ and the system is transformed into

$$\begin{aligned} \dot{X} &= X(b_1a_1 + Y(b_1a_1 + a_2X + k_3X^2 - k_4X^3 + k_5X^4 - k_6X^5)), \\ \dot{Y} &= Y(-b_1a_1 + X(b_1a_1 - a_2Y + k_3Y^2 + k_4Y^3 + k_5Y^4 + k_6Y^5)). \end{aligned}$$

Finally we do a rotation of angle $\varphi = \pi/4$ which is given by the linear change

$$u = \frac{X}{\sqrt{2}} - \frac{Y}{\sqrt{2}}, \qquad v = \frac{X}{\sqrt{2}} + \frac{Y}{\sqrt{2}},$$

and the system takes the form

$$\dot{u} = v + vP(u, v^2), \qquad \dot{v} = u + Q(u, v^2). \tag{4}$$

System (4) is invariant by the symmetry $(u, v, t) \to (u, -v, -t)$. hence the system is a time-reversible system following the definition given in [5]. Moreover all the time-reversible systems are inside the Sibirsky subvariety which are inside the Center variety i.e., systems (4) that have an analytic first of the form $H = uv + \cdots$ around the origin. Hence system (4) and the original one have a resonant integrable saddle at the origin. □

The necessity condition of Theorem 2 is proved in the following lemma.

Lemma 5. *If system* (3) *when f and g are of degree ≤ 6 has an integrable saddle at the origin then the following conditions hold:*

$$a_2 + b_2 = -a_1a_3 + b_1b_3 = a_1^2a_4 + b_1^2b_4 = -a_1^3a_5 + b_1^3b_5 = a_1^4a_6 + b_1^4b_6 = 0.$$

Proof. In system (3) with f and g polynomials of degree ≤ 6 we introduce the change of variables $X = x + iy$, $Y = x - iy$ and the scaling of time $t \mapsto -t/i$. With this change the system becomes into the form

$$\begin{aligned} \dot{X} &= -Y + F(X,Y), \\ \dot{Y} &= X + G(X,Y), \end{aligned} \tag{5}$$

where the coefficients of F and G are complex. Next we take polar coordinates, i.e., $X = r\cos\theta$ and $Y = r\sin\theta$ and doing this change of variables system (5) takes the form

$$\dot{r} = \sum_{s=2}^{7} P_s(\theta)\, r^s, \quad \dot{\theta} = 1 + \sum_{s=2}^{7} Q_s(\theta)\, r^{s-1}, \tag{6}$$

where P_s and Q_s are trigonometric polynomials of degree s. To determine the necessary conditions to have a formal first integral in a neighborhood of the origin we propose a Poincaré series of the form $H(r,\theta) = \sum_{m=2}^{\infty} H_m(\theta) r^m$, where $H_2(\theta) = 1/2$ and $H_m(\theta)$ are homogeneous trigonometric polynomials respect to θ of degree m. Imposing that this power series is a formal first integral of system (6) we obtain $\dot{H}(r,\theta) = \sum_{k=2}^{\infty} V_{2k} r^{2k}$, where the V_{2k} are in fact the *saddle quantities* that depend on the parameters of system (3). From the recursive equations that generate V_{2k} we can see that these V_{2k} are polynomials in the parameters of system (3), see [1, 3, 6]. Due to the Hilbert Basis theorem, the ideal $J =< V_4, V_6, ... >$ generated by the saddle quantities is finitely generated, i.e. there exist $v_1, v_2, ..., v_k$ in J such that $J =< v_1, v_2, ..., v_k >$. Such a set of generators is called a basis of J and the conditions $v_j = 0$ for $j = 1, \ldots, k$ provide a finite set of necessary conditions to have a formal first integral around the origin.

In fact we determine a number of saddle quantities thinking that inside these number there is the set of generators.

In our case the necessity is straightforward because the first saddle quantity is $V_4 = a_2 + b_2$. Then we take $b_2 = -a_2$. The next saddle quantity is $V_6 = a_1 a_3 - b_1 b_3$. To vanish this quantity we take $a_3 = k_3 b_1$ and $b_3 = k_3 a_1$ where k_3 is an arbitrary constant. We compute the next saddle quantity and we obtain $V_8 = a_1^2 a_4 + b_1^2 b_4$. Hence we take, as before, $a_4 = k_4 b_1^2$ and $b_4 = -k_4 a_1^2$, where k_4 is an arbitrary constant. The next saddle quantity is $V_{10} = a_1^3 a_5 - b_1^3 b_5$. Hence we take, as before, $a_5 = k_5 b_1^3$ and $b_5 = k_5 a_1^3$, where k_5 is an arbitrary constant. Under these conditions $V_{12} = a_1^4 a_6 + b_1^4 b_6$ and we take $a_6 = k_6 b_1^4$ and $b_6 = -k_6 a_1^4$, where k_6 is an arbitrary constant. The next saddle quantity V_{14} is zero and we assume that the rest are also zero and that we have vanished a set of generators. $\square$

Proof of Theorem 2. The proof of Theorem 2 is an immediate consequence of Lemmas 4 and 5. □

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