ON THE INTEGRABILITY OF LIÉNARD SYSTEMS WITH A STRONG SADDLE

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ABSTRACT. We study the local analytic integrability for real Liénard systems, $\dot{x} = y - F(x)$, $\dot{y} = x$, with $F(0) = 0$ but $F'(0) \neq 0$, which implies that it has a strong saddle at the origin. First we prove that this problem is equivalent to study the local analytic integrability of the $[p : -q]$ resonant saddles. This result implies that the local analytic integrability of a strong saddle is a hard problem and only partial results can be obtained. Nevertheless this equivalence gives a new method to compute the so-called resonant saddle quantities transforming the $[p : -q]$ resonant saddle into a strong saddle.

1. Introduction and main results

Two of the main problems about the planar Liénard differential systems are to know whether they are integrable or not, and to give criteria for controlling their number of limit cycles. In this work we deal with the first problem. For more information on the second mentioned problem, see [1]. Liénard differential systems appeared in electrical circuits with nonlinear elements, but later many other situations have been modeled by these type of differential equations, see [7] and references therein.

Assume that a real planar analytic differential system has a weak focus. It is well-known that this singular point is a center if and only if the equation has an analytic first integral defined in a neighborhood of this point, see for instance [18]. Consequently the center problem for non-degenerate singular points is equivalent to the local analytic integrable problem for such singular points.

The following theorem characterizes the centers for the classical real analytic Liénard systems, see the proof in [3, 5, 6].

Theorem 1 (Cherkas). *Consider the differential system*

$$\dot{x} = y + F(x), \qquad \dot{y} = -x,$$

with $F(x)$ analytic at zero and $F(0) = F'(0) = 0$, then the origin is a center if and only if F is an even function.

Recently have been proved a similar result, but for classical Liénard systems with a weak saddle at the origin, that is, systems of the form

$$\dot{x} = y + F(x), \qquad \dot{y} = x,$$

with F analytic and $F(0) = F'(0) = 0$. In this case the eigenvalues are ± 1. Recall that a saddle point is *weak* if its eigenvalues are $\pm\lambda$ with $0 \neq \lambda \in \mathbb{R}$. The duality between centers and saddles transforms one center into one saddle when we consider the system in $\mathbb{C}^2$, see [7].

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The result obtained in [7] is the following:

Theorem 2. *Consider the analytic differential systems in* $\mathbb{C}^2$ *of the form*

$$\dot{x} = y + F(x), \qquad \dot{y} = ax, \tag{1}$$

with $0 \neq a \in \mathbb{C}$ *and where* $F(x)$ *is an analytic function without linear and constant terms. System* (1) *is locally integrable at the origin if and only if* $F(x)$ *is an even function of* x.

The result covers the classical Liénard case in $\mathbb{R}^2$ with a weak focus at the origin which corresponds to $a = -1$, and the weak saddle case that has $a = 1$, and also extends to $\mathbb{C}^2$ some of the results obtained in [16].

In [7] Theorem 2 is generalized to get the characterization of the real analytic integrable weak saddles for general Liénard systems in $\mathbb{R}^2$, that is, for systems of the form

$$\dot{x} = y + F(x), \qquad \dot{y} = g(x), \tag{2}$$

where F and g are analytic functions of x, $F(x)$ without linear and constant terms and with $g(0) = 0$ and $g'(0) \neq 0$. This generalization use the ideas given in [3, 6], see also [2, 8, 9], and is also recently proved in [16]. The result can be written in the following way.

Corollary 3. *System* (2) *has an integrable resonant weak saddle (resp. weak focus) at the origin if and only if* $g'(0) > 0$ *(resp.* $g'(0) < 0$*) and* $F(x) = \phi(G(x))$*, for some analytic function* ϕ *with* $\phi(0) = 0$ *and where* $G(x) = \int_0^x g(\xi) d\xi$.

Moreover in [7], using Lüroth theorem as in [4], an effective characterization of real integrable polynomial Liénard systems with a weak saddle is obtained. In [10] some other integrable Liénard-like complex systems with a weak saddle are also characterized.

In this work we aim to characterize the analytic integrability of the real differential systems of the form

$$\dot{x} = y + F(x), \qquad \dot{y} = x, \tag{3}$$

where $F(x)$ is an analytic function of x with $F(0) = 0$ but $F'(0) \neq 0$ which implies that system (3) has a strong saddle at the origin. Without loss of generality we can take $F'(0) = 1/c - c$, with $c \in \mathbb{R}^+ \setminus \{1\}$. We avoid the case $c = 1$ which corresponds to the weak saddle case. Then the eigenvalues λ_1 and λ_2 of the linear part are $-c < 0 < 1/c$. It is well-known, see [13, 17], that if $-\lambda_1/\lambda_2 \notin \mathbb{Q}^+$ then the differential equation (3) has no local analytic first integral in a neighborhood of the origin. Consequently a necessary condition for having an analytic first integral at the origin is $c^2 = q/p \in \mathbb{Q}^+$. In fact, if $F(x) = (\sqrt{p/q} - \sqrt{q/p})\, x$ it is not difficult to prove that

$$H(x, y) = \left(\sqrt{p}\, x + \sqrt{q}\, y\right)^q \left(\sqrt{p}\, y - \sqrt{q}\, x\right)^p$$

is a first integral of the linear Liénard system.

Before stating our main result, we introduce some notation and definitions. The center problem for analytic differential systems in $\mathbb{R}^2$ with a nondegenerate singular point, that is, systems of the form $\dot{x} = -y + \cdots$, $\dot{y} = x + \cdots$, where the dots mean higher order terms, can be transformed by the complex change of variable $u = x + iy$, $v = x - iy$ into the complex differential system of the form $\dot{u} = u + \cdots$, $\dot{v} = -v + \cdots$. The natural generalization of the above system is to consider analytic differential system in $\mathbb{C}^2$ of the form

$$\dot{u} = \lambda\, u + \cdots, \qquad \dot{v} = -\mu\, v + \cdots, \tag{4}$$

where $\lambda, \mu \in \mathbb{C} \setminus \{0\}$. As we have said a necessary condition to have analytic integrality at the origin of system (4) is $\lambda/\mu = p/q \in \mathbb{Q}^+$ with $\gcd(p, q) = 1$. This case is called $[p : -q]$

resonant case. In this situation the local analytic integrability is sometimes possible with some more necessary conditions. We describe how to obtain these necessary conditions.

The $[p:-q]$ resonant case can be written as

$$\dot{u} = p\,u + \cdots\,, \qquad \dot{v} = -q\,v + \cdots\,, \tag{5}$$

after a scaling of time if necessary, with $p, q \in \mathbb{Z}^+$. The linear part of system (5) has the analytic first integral $H_0 = u^q v^p$ and the inverse integrating factor $V_0 = u^{1-q}v^{1-p}$. Now we can search conditions for the existence of an analytic first integral $H(u,v) = H_0 + \cdots$ for system (5) and we get the equation

$$\dot{H} = v_1 H_0^2 + v_3 H_0^3 + \cdots\,,$$

where v_i are the so-called $[p:-q]$ *resonant saddle quantities* which are polynomials in the coefficients of system (5). Therefore we have a formal *analytic resonant saddle* if all the v_i are zero, see [12, 19] and references therein. Moreover in this case we have also the existence of a local analytic first integral, see [18].

The first result of this work is the following.

Theorem 4. *System* (3) *with a strong saddle at the origin, that is, with $F(0) = 0$ but $F'(0) = 1/c - c \neq 0$ and with $c^2 = q/p \in \mathbb{Q}^+$ can be transformed into a system with a $[p:-q]$ resonant saddle at the origin.*

Theorem 4 is proved in section 4.

The analytic integrability problem of a $[p:-q]$ resonant saddle is a very hard problem as the several papers dedicated to this problem in the last years have shown, see for instance [12, 14, 15, 19, 20] and references therein. Hence the first consequence of Theorem 4 is that the analytic integrability of a strong saddle is as difficult as the analytic integrability problem of the $[p:-q]$ resonant saddle. The second consequence is that we can derive *a new method to compute the resonant saddle quantities of a system with a $[p:-q]$ resonant saddle* as we show in section 2.

In [7] it was proposed the following open problem: *Are there nonlinear integrable systems inside the family of system* (3)? The following proposition establishes that in a particular case the unique integrable case is the linear one.

Proposition 5. *The Liénard differential system with a strong saddle at the origin*

$$\dot{x} = y + (1/c - c)x + a_2x^2 + a_3x^3, \qquad \dot{y} = x,$$

with $c^2 = q/p = 2/1 \in \mathbb{Q}^+$ has a unique integrable case which corresponds to $a_2 = a_3 = 0$.

Moreover, for several $[p:-q]$ resonant saddles of homogeneous and non homogeneous differential systems that can be transformed into a system of the form (3) it has been checked that the only integrable case is when $F(x) = ax$. Hence we establish the following conjecture.

Conjecture 6. *The unique integrable case of the Liénard system* (3) *is the linear one.*

What is not true is that the unique integrable case of a general strong saddle be the linear one. Indeed, if we consider more general systems of the form

$$\dot{x} = y + F(x,y), \qquad \dot{y} = x, \tag{6}$$

where $F(x,y) = ax + \cdots$ with $a \neq 0$, here the dots mean higher order terms which implies that system (6) has also a strong saddle at the origin, these systems can have analytic integrable saddle different from the linear one as the following example shows.

Proposition 7. *The differential system*

$$\dot{x} = y - \frac{x}{\sqrt{2}} + \frac{bx^2}{\sqrt{2}} + 2bxy + \sqrt{2}by^2, \quad \dot{y} = x, \tag{7}$$

has an analytic integrable strong saddle at the origin because has the analytic first integral

$$H = e^{\frac{1}{2b^3}\left(4\sqrt{2}+6by+\frac{3(\sqrt{2}+2by)}{(1+b(x+\sqrt{2}y))^2}-\frac{12(\sqrt{2}+by}{1+b(x+\sqrt{2}y)}\right)}(1+b(x+\sqrt{2}y))^{-\frac{3\sqrt{2}}{b^3}}.$$

It is easy to check that the function H in Proposition 7 is a first integral of system (7).

2. A new method to compute the resonant saddle quantities

We consider the following analytic differential system with a $[p:-q]$ resonant saddle at the origin

$$\dot{u} = p\,u + \sum_{i=2}^{\infty} a_{ij}u^i v^j, \quad \dot{v} = -q\,v + \sum_{i=2}^{\infty} b_{ij}u^i v^j, \tag{8}$$

where p, $q \in \mathbb{Z}^+$ and a_{ij}, $b_{ij} \in \mathbb{R}$. Now we define $c^2 = q/p$ and we made the scaling of time $t \to cp\,t$ and system (8) takes the form

$$\dot{u} = \frac{u}{c} + \sum_{i=2}^{\infty} \tilde{a}_{ij}u^i v^j, \quad \dot{v} = -c\,v + \sum_{i=2}^{\infty} \tilde{b}_{ij}u^i v^j, \tag{9}$$

where $\tilde{a}_{ij}$, $\tilde{b}_{ij} \in \mathbb{R}$. Next we do the linear change of variables

$$x = -\frac{-u+cv}{1+c^2}, \qquad y = \frac{cu+v}{1+c^2},$$

and system (9) becomes

$$\dot{x} = y + \left(\frac{1}{c} - c\right)x + \sum_{i=2}^{\infty} c_{ij}x^i y^j, \quad \dot{y} = x + \sum_{i=2}^{\infty} d_{ij}x^i y^j. \tag{10}$$

Then we implement the change of variables

$$U = \frac{x}{\sqrt{2}} - \frac{y}{\sqrt{2}}, \qquad V = \frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}},$$

and system (10) becomes

$$\begin{aligned} \dot{U} &= \frac{1-2c-c^2}{2c}U + \frac{1-c^2}{2c}V + \sum_{i=2}^{\infty} \tilde{c}_{ij}U^iV^j, \\ \dot{V} &= \frac{1-c^2}{2c}U + \frac{1+2c-c^2}{2c}V + \sum_{i=2}^{\infty} \tilde{d}_{ij}U^iV^j, \end{aligned} \tag{11}$$

where $\tilde{c}_{ij}$, $\tilde{d}_{ij} \in \mathbb{R}$. Next we introduce the complex change of variables $U = X+iY$, $V = X-iY$ and system (11) can be written as

$$\dot{X} = -Y + i\left(c - \frac{1}{c}\right)X + \sum_{i=2}^{\infty} \tilde{\tilde{c}}_{ij}X^iY^j, \quad \dot{Y} = X + \sum_{i=2}^{\infty} \tilde{\tilde{d}}_{ij}X^iY^j, \tag{12}$$

where here $\tilde{\tilde{c}}_{ij}$, $\tilde{\tilde{d}}_{ij} \in \mathbb{C}$. A first integral of the linear part of system (12) is given by

$$H_0 = \left(\sqrt{\frac{q}{p}}X + iY\right)^p \left(X - i\sqrt{\frac{q}{p}}Y\right)^q,$$

and consequently a first integral of system (12) must be of the form $H = H_0 + \cdots$ where the dots are higher order terms, see [11]. Notice that system (12) has complex parameters and it has the linear part of a center-focus type. Hence the classical method of passing to polar coordinates and to find obstructions to have a formal first integral is now possible.

Therefore in system (12) we take the polar coordinates $X = r\cos\theta$ and $Y = r\sin\theta$ and we propose the Poincaré power series

$$H(r,\theta) = \sum_{m=2}^{\infty} H_m(\theta) r^m, \tag{13}$$

where $H_2(\theta) = 1/2$ and $H_m(\theta)$ are homogeneous trigonometric polynomials respect to θ of degree m. Computing the derivative of H we have

$$\dot H(r,\theta) = \sum_{k=2}^{\infty} V_{2k} r^{2k},$$

where the V_{2k}'s are the obstructions to have H as a formal first integral of system (12). This V_{2k} are the *saddle quantities* which are polynomials in the parameters of system (12), see [18]. However these quantities correspond to the saddle quantities of the original system (8) and consequently are polynomials in the real parameters of system (8).

The equivalence that establishes Theorem 4 also helps to understand why it is more difficult to classify of the integrable systems of a $[p : -q]$ resonance when we increase p or q. What happens is that the first integral of the linear part increases also its degree, and hence the saddle quantities appear for degrees higher in the terms of the proposed power series (13). Consequently the saddle quantities are polynomials of higher degree in the variables of system (8), and it is more difficult to find the zeros of the ideal generated by these saddle quantities.

Proof of Proposition 5. Using the method described in the previous section we find that the first two nonzero saddle quantities are

$$\begin{aligned} V_6 &= \sqrt{2}a_2^2 + 21a_3,\\ V_8 &= 177017\sqrt{2}a_2^6 + 4078901a_2^4a_3 + 3855267\sqrt{2}a_2^2a_3^2 - 33480a_3^3, \end{aligned}$$

modulo the previous one zero, which implies $a_2 = a_3 = 0$. □

3. Proof of Theorem 4

The proof is based on apply several transformations to system (3). First we do a rotation of angle $\varphi = \pi/4$ which is given by the linear change

$$u = \frac{x}{\sqrt{2}} - \frac{y}{\sqrt{2}}, \qquad v = \frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}},$$

and system (3) with $F(0) = 0$ but $F'(0) = 1/c - c \neq 0$ takes the form

$$\dot u = \frac{1}{2}(-2 + \frac{1}{c} - c)\,u + \frac{1-c^2}{2c} v + \cdots, \quad \dot v = \frac{1-c^2}{2c}\,u + \frac{1}{2}(2 + \frac{1}{c} - c)\,v + \cdots. \tag{14}$$

The eigenvalues of the linear part are $\lambda_1 = 1/c$, and $\lambda_2 = -c$ whose eigenvectors are

$$v_1 = \left(\frac{1-c}{1+c}, 1\right), \qquad v_2 = \left(\frac{c+1}{c-1}, 1\right),$$

respectively. Now we propose the linear change of variables

$$X = \frac{1-c}{1+c}u + v, \qquad Y = \frac{c+1}{c-1}u + v,$$

and the scaling of time $t \to ct$, and system (14) takes the form

$$\dot{X} = X + \cdots, \quad \dot{Y} = -c^2 Y + \cdots. \tag{15}$$

Taking into account that $c^2 = q/p \in \mathbb{Q}^+$ doing a second scaling of time we obtain the system

$$\dot{X} = pX + \cdots, \quad \dot{Y} = -q Y + \cdots. \tag{16}$$

which corresponds to a $[p : -q]$ resonant saddle and the proof of the theorem is completed.

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