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Integrability conditions of a resonant saddle in generalized Liénard-like complex polynomial differential systems

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ABSTRACT

We consider a complex differential system with a resonant saddle at the origin. We compute the resonant saddle quantities and using Gröbner bases we find the integrability conditions for such systems up to a certain degree. We also establish a conjecture about the integrability conditions for such systems when they have arbitrary degree.

1. Introduction

The center problem for polynomial vector fields in the real plane with an elementary singular point of the form

$$\dot{x} = -y + \cdots, \qquad \dot{y} = x + \cdots,$$

where the dots means higher order terms is a subject of much work during these last decades, see for instance [2,6]. These type of systems can be embedded by the change of variable $u = x + iy$ and the corresponding conjugate variable $v = x - iy$ into the complex system of the form

$$\dot{u} = u + \cdots, \qquad \dot{v} = -v + \cdots.$$

The next generalization of the above system is to consider the case of a polynomial system in the complex plane of the form

$$\dot{u} = \lambda_1 u + \cdots, \qquad \dot{v} = -\lambda_2 v + \cdots, \tag{1}$$

where $\lambda_1, \lambda_2 \in \mathbb{C}$. However if $k := (\lambda_1, \lambda_2)$ does not satisfies the resonant condition $(\alpha, k) - \lambda_m = 0$, for all $m \in \{1, 2\}$ and for all $\alpha \in \mathbb{N}_0^2$ with $|\alpha| \geq 2$ then system (1) is formally equivalent to its normal form $\dot{u} = \lambda_1 u$, $\dot{v} = -\lambda_2 v$, see [6]. Hence we consider the case with a $p : -q$ resonant elementary singular point

$$\dot{u} = p\,u + \cdots, \qquad \dot{v} = -q\,v + \cdots, \tag{2}$$

with $p, q \in \mathbb{Z}$. If $p,\ q > 0$, $(p, q) = 1$ then the linear part has the analytic first integral $H_0 = x^q y^p$ and we can seek the conditions for the existence of an analytic first integral $H = H_0 + \cdots$ for system (2). Hence get the equation $\dot{H} = v_1 H_0^2 + v_3 H_0^3 + \cdots$, and the so-called *$p : -q$ resonant saddle quantities* v_i are polynomials in the coefficients of system (2). If all the v_i are zero we say that we have an *analytic resonant saddle*, see [7,8] and references therein.

In this work we aim to study analytic differential systems in the complex plane of the form

$$\dot{x} = x + g(x)y, \qquad \dot{y} = -y + f(y)x, \tag{3}$$

where $f(y) = \sum_{j\geq 1} a_j y^j$ and $g(x) = \sum_{j\geq 1} b_j x^j$ are analytic functions without constant terms. In fact system (3) has a $1 : -1$ resonant saddle singular point at the origin. System (3) with $g(x) = 0$ was studied in [4] where the following result was given.

Theorem 1. [4] *The complex polynomial differential system (3) with $g(x) = 0$ has an integrable saddle at the origin if and only if one of the following two conditions holds:*

(1) $a_1 = a_2 = 0$;
(2) $a_i = 0$ *for* $i \geq 2$.

The case $g(x) \neq 0$ is much harder and we have studied the polynomial case when f and g are polynomials of degree less than or equal to 6, and we have obtained the following result.

Theorem 2. *The complex polynomial differential system (3) when f and g are polynomials of degree ≤ 6 has an analytic integrable saddle*

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