Bull. London Math. Soc. 47 (2015) 315–324
doi:10.1112/blms/bdv005

Centers for the Kukles homogeneous systems with odd degree

Jaume Giné, Jaume Llibre and Claudia Valls

Abstract

For the polynomial differential system $\dot{x} = -y$, $\dot{y} = x + Q_n(x, y)$, where $Q_n(x, y)$ is a homogeneous polynomial of degree n there are the following two conjectures raised in 1999. (1) Is it true that the previous system for $n \geqslant 2$ has a center at the origin if and only if its vector field is symmetric about one of the coordinate axes? (2) Is it true that the origin is an isochronous center of the previous system with the exception of the linear center only if the system has even degree? We prove both conjectures for all n odd.

1. *Introduction and statement of the main results*

Kukles [**10**] in 1944 examined the conditions under which the origin for the differential system of the form

$$\dot{x} = -y, \quad \dot{y} = x + a_1x^2 + a_2xy + a_3y^2 + a_4x^3 + a_5x^2y + a_6xy^2 + a_7y^3 \tag{1}$$

is a center. For long time, it had been thought that the conditions given by Kukles were necessary and sufficient conditions, but some new cases have been found, see [**2**, **4**, **9**]. In [**4**], the center problem for the class of system (1) with $a_7 = 0$ (reduced Kukles system) was resolved, moreover it was shown that at most five limit cycles bifurcate from the origin. In [**11**], the center problem for system (1) was solved in the case $a_2 = 0$ and it was shown that at most six limit cycles bifurcate from the origin, see also [**12**]. The first complete solution of the center-focus problem for Kukles' system (1) was obtained in [**12**]. In [**14**], the complete solution was also given using the Cherkas' method of passing to a Liénard equation, see also the works [**13**, **15**].

System (1) is derived from a second-order differential equation and it has been used as a test bed for future studies in the center problem, see [**13**]. In fact, the study of this family exhibits properties and issues which are important in the problem of the full classification of cubic systems with a center.

In this paper, we continue the characterization of the centers for a linear center where the $\dot{y}$ equation is perturbed by a homogeneous polynomial, that is, systems of the form

$$\dot{x} = -y, \quad \dot{y} = x + Q_n(x, y), \tag{2}$$

where $Q_n(x, y)$ is a homogeneous polynomial of degree n, that is,

$$Q_n(x, y) = \sum_{j=0}^{n} c_j x^j y^{n-j}, \quad c_j \in \mathbb{R}. \tag{3}$$

These systems are called *Kukles homogeneous systems*, see [**7**].

Received 22 May 2014; revised 19 November 2014; published online 1 March 2015.

2010 *Mathematics Subject Classification* 34C05 (primary), 37C10 (secondary).

The first author is partially supported by a MINECO/FEDER grant number MTM2011-22877 and an AGAUR (Generalitat de Catalunya) grant number 2014SGR 1204. The second author is partially supported by a MINECO/FEDER grant MTM2008-03437 and MTM2013-40998-P, an AGAUR grant number 2014 SGR568, an ICREA Academia, the grants FP7-PEOPLE-2012-IRSES 318999 and 316338, FEDER-UNAB-10-4E-378. The third author is supported by Portuguese National Funds through FCT – Fundação para a Ciência e a Tecnologia within the project PTDC/MAT/117106/2010 and by CAMGSD.