CHIELLINI HAMILTONIAN LIÉNARD DIFFERENTIAL SYSTEMS

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Abstract. We characterize the centers of the Chiellini Hamiltonian Liénard second-order differential equations $x' = y$, $y' = -f(x)y - g(x)$ where $g(x) = f(x)(k - \alpha(1+\alpha)\int f(x)dx)$ with $\alpha, k \in \mathbb{R}$. Moreover we study the phase portraits in the Poincaré disk of these systems when $f(x)$ is linear.

1. Introduction

In the study of differential equations, a Liénard equation in $\mathbb{R}$ is a second–order differential equation of the form

$$x'' + f(x)x' + g(x) = 0,$$

named in honor of the French physicist Alfred-Marie Liénard [13], who during the development of radio and vacuum tube technology, introduced such equations to model oscillating circuits. These equations have been studied intensively, thus now in MathSciNet are more than 1480 articles which appear the keywords "Liénard" and "equation", some of these recent papers are for instance [2, 6, 7, 10, 12, 14] and the references quoted therein.

Here we deal with Liénard second–order differential equations

$$x'' + f(x)x' + g(x) = 0,$$

where f and g are polynomials, or equivalently with first–order differential system of equations

$$\dot{x} = y, \qquad \dot{y} = -f(x)y - g(x), \tag{1}$$

in $\mathbb{R}^2$. In 1931 Chiellini [3] proved that system (1) is integrable if the functions $f(x)$ and $g(x)$ satisfy the condition

$$\frac{d}{dx}\left(\frac{g(x)}{f(x)}\right) = sf(x), \tag{2}$$

where s is a constant. This condition is now known as the Chiellini condition.

Recently Ghose Choudhury and Guha in [9] studied when the Liénard systems (1) satisfying Chiellini condition admit a Hamiltonian formulation

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and they proved that this holds if in (2) we have $s = -\alpha(1+\alpha)$ with $\alpha \in \mathbb{R}$. More precisely, if

$$g(x) = f(x)\left(k - \alpha(1+\alpha)\int f(x)dx\right), \tag{3}$$

where $k \in \mathbb{R}$, then the Liénard systems (1) satisfying (3) admit an integrating factor $R(x,y)$ such that the differential system

$$\dot{x} = R(x,y)y = -\frac{\partial H}{\partial y}, \qquad \dot{y} = R(x,y)(-f(x)y - g(x)) = \frac{\partial H}{\partial x},$$

is Hamiltonian with the Hamiltonian function

$$H = \begin{cases} \left(\dfrac{\alpha(y(1+\alpha)f(x)+g(x))}{(1+2\alpha)f(x)}\right)^{\alpha}\left(y - \dfrac{g(x)}{\alpha f(x)}\right)^{1+\alpha} & \text{if } \alpha \notin \{-1,-1/2,0\}, \\ e^{y+F(x)}(k+y)^{-k} & \text{if } \alpha \in \{-1,0\}, \\ e^{\frac{4(F(x)+4k)}{F(x)+4k+2y}}(F(x)+4k+2y)^4 & \text{if } \alpha = -1/2, \end{cases}$$

where $F(x) = \displaystyle\int f(x)dx$. Note that H is a Darboux first integral, for more details on Darboux integrability see Chapter 8 of [5]. We remark that the first integrals H are not given in [9].

First we are interested in the center problem for the Liénard systems (1) satisfying (3) and we get the next result.

Theorem 1. *The singular point $(x_0, 0)$ of system (1) is a center if it satisfies the conditions (3), $g(x_0) = 0$ and $g'(x_0) > 0$.*

Theorem 1 is proved in section 2

Now we consider system (1) satisfying condition (3) in the case in which $f(x)$ is linear that is, $f(x) = ax + b$ with $a, b \in \mathbb{R}$, $a \neq 0$. The second main result is concerned with the phase portraits in the Poincaré disk of these systems, i.e.

$$\dot{x} = y, \qquad \dot{y} = -\frac{1}{2}(b+ax)(2k+2y-2\alpha(1+\alpha)bx - \alpha(1+\alpha)ax^2), \tag{4}$$

with $a \neq 0$ and $\alpha^2(\alpha+1)^2 + k^2 \neq 0$ (otherwise $g(x) \equiv 0$). See Chapter 5 in [5] for the definition and results concerning the Poincaré disk and the Poincaré compactification and see Chapter 1 in [5] for the definition of topologically equivalence.

The second main result in the paper is the following.

Theorem 2. *The phase portraits of system (4) in the Poincaré disk are topologically equivalent to the following:*

Figure 1 if $\alpha(\alpha+1) = 0$ and $ak > 0$;
Figure 2 if $\alpha(\alpha+1) = 0$ and $ak < 0$;

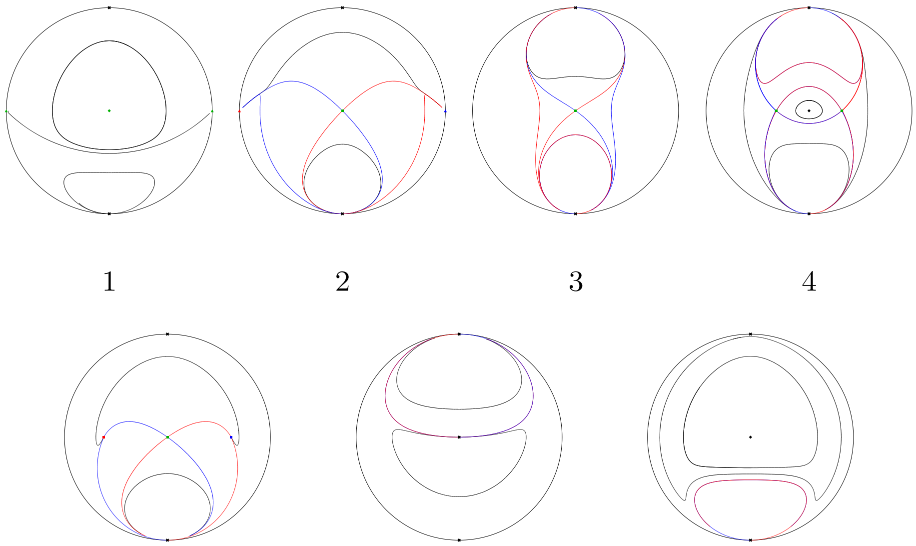

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Figure 3 if $\alpha(\alpha+1) > 0$ *and* $\alpha(\alpha+1)(\alpha(\alpha+1)b^2 + 2ak) \leq 0$*;*
Figure 4 if $\alpha(\alpha+1) > 0$ *and* $\alpha(\alpha+1)(\alpha(\alpha+1)b^2 + 2ak) > 0$*;*
Figure 5 if $\alpha(\alpha+1) < 0$ *and* $\alpha(\alpha+1)(\alpha(\alpha+1)b^2 + 2ak) < 0$*;*
Figure 6 if $0 > \alpha(\alpha+1) \geq -1/4$ *and* $\alpha(\alpha+1)(\alpha(\alpha+1)b^2 + 2ak) = 0$*;*
Figure 7 if $\alpha(\alpha+1) < 0$ *and* $\alpha(\alpha+1)(\alpha(\alpha+1)b^2 + 2ak) > 0$.

Theorem 3 is proved in section 3.

2. Proof of Theorem 1

The singular point $(x_0, 0)$ of system (1) with the translation $x = X + x_0$ pass to the origin of coordinates. Indeed, system (1) becomes

$$\dot{X} = y, \qquad \dot{y} = -f(X + x_0)y - g(X + x_0) = -\bar{f}(X)y - \bar{g}(X), \tag{5}$$

with $\bar{g}(0) = 0$ and $\bar{g}'(0) > 0$.

In order to prove Theorem 1 for system (5) we first recall that this system can be transformed by the change of variables $y = Y - \bar{F}(x)$ into

$$\dot{x} = Y - \bar{F}(x), \qquad \dot{Y} = -\bar{g}(x), \tag{6}$$

where $\bar{F}(x) = \displaystyle\int \bar{f}(x)dx$.

We recall the following result proved in [8], see also [4, 11].

Theorem 3. *Consider the differential system*

$$\dot{x} = \varphi(y) - F(x), \qquad \dot{y} = -g(x). \tag{7}$$

where all the involved functions are analytic and satisfy

$$\varphi(y) = y^{2m-1} + O(y^{2m}), \qquad F(x) = a_j x^j + O(x^{j+1}), \text{ and}$$

$$G(x) = \int g(x)dx = \frac{x^{2\ell}}{2\ell} + O(x^{2\ell+1}),$$

with m, j and ℓ nonzero positive integers. Consider $F(\xi(u)) = f_j u^j + O(u^{j+1})$, where $\xi(u)$ is defined as the inverse function of

$$u = x\sqrt[2\ell]{2\ell\frac{G(x)}{x^{2\ell}}},$$

and assume that $j > \ell(2m-1)/m$. Then it has a center at the origin if there exist analytic functions β, γ, and h satisfying $\alpha(0) = 0$, $\gamma(x) = cx^\ell + O(x^{\ell+1})$ with $c \neq 0$, and such that

$$F(x) = \beta(h(x)), \qquad G(x) = \gamma(h(x)).$$

Now we are going to apply Theorem 3 to our system (6). In this case $m = \ell = 1$ and $j = 2$, because

$$0 < \bar{g}'(0) = \bar{f}'(0)(k - \alpha(1+\alpha)\bar{F}(0)),$$

and so $\bar{f}'(0) \neq 0$. Now

$$\begin{aligned} G(x) &= \int \bar{g}(x)dx = \int \Big[\bar{f}(x)\Big(k - \alpha(1+\alpha)\int \bar{f}(x)dx\Big)\Big]dx \\ &= k\int \bar{f}(x)dx - \alpha(1+\alpha)\int \bar{F}(x)\bar{f}(x)dx = k\bar{F}(x) - \alpha(1+\alpha)\frac{1}{2}\bar{F}^2(x). \end{aligned}$$

Hence applying Theorem 3 with $\beta(x) = x$, $h(x) = F(x)$ and $\gamma(x) = kx - \alpha(1+\alpha)x^2/2$, we have that Theorem 1 is proved.

3. Proof of Theorem 3

The finite singular points of system (4) are

$$p_1 = (-b/a, 0), \quad p_{2,3} = \Big(-\frac{(\alpha(1+\alpha)b \pm \sqrt{\alpha(1+\alpha)(\alpha(1+\alpha)b^2 + 2ak)}}{\alpha(1+\alpha)a}, 0\Big).$$

Note that $p_{2,3}$ only exist when $\alpha(1+\alpha) \neq 0$ and $\alpha(1+\alpha)(\alpha(1+\alpha)b^2 + 2ak) > 0$, because when $\alpha(1+\alpha) \neq 0$ and $\alpha(1+\alpha)(\alpha(1+\alpha)b^2 + 2ak) = 0$, then $p_1 = p_2 = p_3$.

First we study the finite singular point p_1. The eigenvalues of the Jacobian matrix at p_1 are

$$\pm\frac{\sqrt{-\alpha(1+\alpha)b^2 - 2ak}}{\sqrt{2}}.$$

If $\alpha(1+\alpha) = 0$ then p_1 is a saddle if $ak < 0$ and a center if $ak > 0$. Now we assume that $\alpha(1+\alpha) \neq 0$. So, p_1 is a center if $\alpha(1+\alpha)b^2 + 2ak > 0$, a saddle if $\alpha(1+\alpha)b^2 + 2ak < 0$ and it is nilpotent if $\alpha(1+\alpha)b^2 + 2ak = 0$ (note that in this last case, that is, when $k = -\alpha(1+\alpha)b^2/(2a))$ the Jacobian matrix at p_1 is

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Now we take $k = -\alpha(1+\alpha)b^2/(2a)$ and we introduce the change of variables $X = x + b/a, Y = y$. With this change of variables system (4) becomes

$$X' = Y, \quad Y' = -aXY + \frac{1}{2}\alpha(1+\alpha)a^2X^3.$$

Applying Theorem 3.5 in [5] and taking into account that $\alpha(1+\alpha) \geq -1/4$ (because $\alpha(1+\alpha)$ has a unique minimum at $\alpha = -1/2$ with value $-1/4$) we get that p_1 is a saddle if $\alpha(1+\alpha) > 0$ and the union of one hyperbolic and one elliptic sector if $\alpha(1+\alpha) \in [-1/4, 0)$.

Now we study the finite singular points $p_{2,3}$. In order that they exist we must have $\alpha(1+\alpha) \neq 0$ and $\alpha(1+\alpha)(\alpha(1+\alpha)b^2 + 2ak) > 0$. The eigenvalues of the Jacobian matrix at p_2 are

$$-\frac{\sqrt{\alpha}}{\sqrt{1+\alpha}}\sqrt{\alpha(1+\alpha)b^2 + 2ak}, \quad \frac{\sqrt{\alpha(1+\alpha)b^2 + 2ak}}{\sqrt{\frac{\alpha}{1+\alpha}}},$$

and the eigenvalues of the Jacobian matrix at p_3 are

$$\frac{\sqrt{\alpha}}{\sqrt{1+\alpha}}\sqrt{\alpha(1+\alpha)b^2 + 2ak}, \quad -\frac{\sqrt{\alpha(1+\alpha)b^2 + 2ak}}{\sqrt{\frac{\alpha}{1+\alpha}}}.$$

Hence, p_2 and p_3 are both saddles if $\alpha(1+\alpha) > 0$ and $\alpha(1+\alpha)b^2+2ak > 0$ and they are both nodes (p_2 being unstable and p_3 being stable) if $\alpha(1+\alpha) < 0$ and $\alpha(1+\alpha)b^2 + 2ak < 0$. In short we have the following finite singular points:

(i) two nodes (p_2 unstable and p_3 stable) and a saddle (p_1) if $\alpha(1+\alpha)b^2 + 2ak < 0$ and $\alpha(1+\alpha) < 0$;
(ii) two saddles (p_2 and p_3) and one center (p_1) if $\alpha(1+\alpha)b^2 + 2ak > 0$ and $\alpha(1+\alpha) > 0$;
(iii) one saddle p_1 if either $\alpha(1+\alpha) = 0$ and $ak < 0$, or $\alpha(1+\alpha) > 0$ and $\alpha(1+\alpha)b^2 + 2ak \leq 0$;
(iv) one center p_1 if either $\alpha(1+\alpha) = 0$ and $ak > 0$, or $\alpha(1+\alpha) < 0$ and $\alpha(1+\alpha)b^2 + 2ak > 0$;
(v) one singular point formed by one hyperbolic and one elliptic sector if $\alpha(1+\alpha) \in [-1/4, 0)$ and $\alpha(1+\alpha)b^2 + 2ak = 0$.

Now we compute the infinite singular points in the local chart U_1 (see again Chapter 5 in [5] for its definition). We have

$$
\begin{aligned}
\dot u &= \frac{1}{2}\alpha(1+\alpha)a^2 + \frac{3}{2}\alpha(1+\alpha)abv - auv + (\alpha(1+\alpha)b^2 - ak)v^2 - buv^2 \\
&\quad - bkv^3 - u^2v^2, \\
\dot v &= -uv^3.
\end{aligned} \tag{8}
$$

Note that there are no infinite singular points on the local chart U_1 if $\alpha(1+\alpha)a^2 \neq 0$.

In the case $\alpha(1+\alpha) = 0$ then system (8), after simplifying by the common factor v, we have

$$\dot u = -(au + akv + buv + bkv^2 + u^2v), \quad \dot v = -uv^2.$$

The unique infinite singular point is $(u, v) = (0, 0)$. In this case the Jacobian matrix at this point is

$$\begin{pmatrix} -a & -ak \\ 0 & 0 \end{pmatrix}$$

and so the point is semi-hyperbolic with eigenvalues $-a, 0$. Applying Theorem 2.19 in [5] we get that it is a saddle if $ak > 0$, and a node if $ak < 0$, stable if $a > 0$ and otherwise it is unstable.

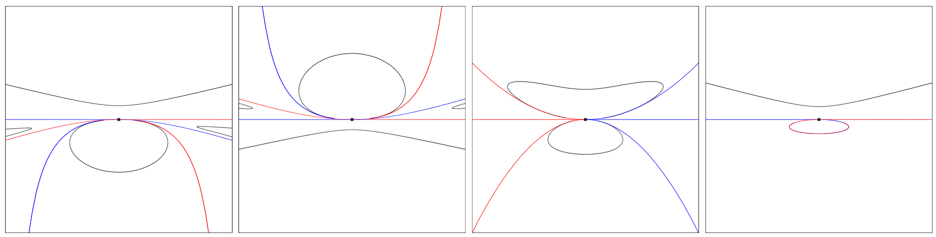
Now we study the origin in the local chart U_2 (see again Chapter 5 in [5] for its definition). We have

$$
\begin{aligned}
\dot u &= v^2 + au^2v + buv^2 - \frac{1}{2}\alpha(1+\alpha)a^2u^4 - \frac{3}{2}\alpha(1+\alpha)abu^3v, \\
&\quad - (\alpha(1+\alpha)b^2 - ak)u^2v^2 + bkuv^3 \\
\dot v &= auv^2 + bv^3 - \frac{1}{2}\alpha(1+\alpha)a^2u^3v - \frac{3}{2}\alpha(1+\alpha)abu^2v^2 \\
&\quad - (\alpha(1+\alpha)b^2 - ak)uv^3 + bkv^4.
\end{aligned}
$$

The origin of the local chart U_2 is a singular point. The Jacobian matrix at the origin is the 2×2 zero matrix and so the origin of U_2 is linearly zero. We need to apply blow-up techniques to study it, see for instance [1]. Doing so, we get the following:

- if $\alpha(1+\alpha) < 0$ the origin of the local chart U_2 is the union of three hyperbolic and one elliptic sectors. Its local phase portrait on the local chart U_2 is topologically equivalent to the one of Figure 8 if $a > 0$, and to the one of Figure 9 if $a < 0$;
- if $\alpha(1+\alpha) > 0$ the origin of the local chart U_2 is the union of two elliptic sectors separated by parabolic sectors. Its local phase portrait on the local chart U_2 is topologically equivalent to the one of Figure 10;

- if $\alpha(1+\alpha) = 0$ the origin of the local chart U_2 is the union of one hyperbolic and one elliptic sectors separated by two parabolic sectors. Its local phase portrait on the local chart U_2 is topologically equivalent to the one of Figure 11.



8 9 10 11

Taking into account the local phase portraits at the finite and infinite singular points we get the following cases:

- if $\alpha(\alpha+1) = 0$ and $ak > 0$, there is a unique finite singular point which is a center, the origin of the local chart U_1 is a saddle and the origin of the local chart U_2 is the union of one hyperbolic and one elliptic sectors separated by two parabolic sectors see Figure 11. Gluing this information together we get the phase portrait of Figure 1.
- if $\alpha(\alpha+1) = 0$ and $ak < 0$, there is a unique finite singular point which is a saddle, the origin of the local chart U_1 is a node and the origin of the local chart U_2 is the union of one hyperbolic and one elliptic sectors separated by two parabolic sector as see Figure 11. Gluing this information together we get the phase portrait of Figure 2.
- if $\alpha(\alpha+1) > 0$ and $\alpha(\alpha+1)(\alpha(\alpha+1)b^2+2ak) \leq 0$, there is a unique finite singular point which is a saddle, no infinite singular points in the local chart U_1 and the origin of U_2 is the union of two elliptic sector separated by two parabolic sectors see Figure 10. Gluing this information together we get the phase portrait of Figure 3.
- if $\alpha(\alpha+1) > 0$ and $\alpha(\alpha+1)(\alpha(\alpha+1)b^2+2ak) > 0$, there are three finite singular points which are two saddles and one center, no infinite singular points in the local chart U_1 and the origin of U_2 is the union of two elliptic sectors separated by two parabolic sectors see Figure 10. Gluing this information together we get the phase portrait of Figure 4.
- if $\alpha(\alpha+1) < 0$ and $\alpha(\alpha+1)(\alpha(\alpha+1)b^2+2ak) < 0$, there are three finite singular points which are two nodes and one saddle, no infinite singular points in the local chart U_1 and the origin of U_2 is the union of three hyperbolic and one elliptic sectors as in Figures

8 and 9. Gluing this information together we get the phase portrait of Figure 5.

- if $0 > \alpha(\alpha+1) \geq -1/4$ and $\alpha(\alpha+1)(\alpha(\alpha+1)b^2+2ak) = 0$, there is a unique finite singular point which is the union of one hyperbolic and one elliptic sectors separated by two parabolic sectors, no infinite singular points in the local chart U_1, and the origin of U_2 is the union of three hyperbolic and one elliptic sectors as in Figures 8 and 9. Gluing this information together we get the phase portrait of Figure 6.
- if $\alpha(\alpha+1) < 0$ and $\alpha(\alpha+1)(\alpha(\alpha+1)b^2+2ak) > 0$, there is a unique finite singular point which is a center, no infinite singular points in the local chart U_1, and the origin of U_2 is the union of three hyperbolic and one elliptic sectors as in Figures 8 and 9. Gluing this information together we get the phase portrait of Figure 7.

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