PERIODS OF PERIODIC HOMEOMORPHISMS OF PINCHED SURFACES WITH ONE OR TWO BRANCHING POINTS

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Abstract. In this paper we characterize all the possible sets of periods of a periodic homeomorphism defined on compact connected pinched surfaces with one or two branching points.

1. Introduction

A *pinched surface* $\mathbb{S}$, here studied, is a compact set formed by one or two *vertices* (points) and *handles*, where here a handle is homeomorphic to an open cylinder, i.e. to the set $(0,1)\times\mathbb{S}^1$, where $(0,1)$ is the open interval of the real line and $\mathbb{S}^1$ is the circle. The boundaries of every handle are vertices. Furthermore, the handles are pairwise disjoint, and the pinched surfaces that we consider here will always be connected.

Let $\mathbb{S}$ be a pinched surface and let $z\in\mathbb{S}$ be a vertex. We consider a small open neighborhood U (in $\mathbb{S}$) of z. The number of connected components of $U\setminus\{z\}$ is called the *valence* of z and is denoted by $\mathrm{Val}(z)$. Observe that this definition is independent of the choice of U if U is sufficiently small. A vertex of valence 1 is called an *endpoint of* $\mathbb{S}$ and a vertex of valence larger than 1 is called a *branching point of* $\mathbb{S}$.

A continuous map $f:\mathbb{S}\to\mathbb{S}$ is called *periodic* if there exists a positive integer n such that the iterate f^n is the identity map, i.e. $f^n(x)=x$ for all $x\in S$, or $f=id$.

Let $f:\mathbb{S}\to\mathbb{S}$ be a continuous map. A point $z\in S$ such that $f(z)=z$ is called a *fixed point*, or a periodic point of period 1. The point $z\in\mathbb{S}$ is *periodic* of *period* $m>1$ if $f^m(z)=z$ and $f^k(z)\neq z$ for $k=1,\ldots,m-1$. Of course, in the whole paper $f^m(x)$ denotes the m–th iterate of the point x by the map f. We denote by $\mathrm{Per}(f)$ the *set of periods of all periodic points of* f.

In this work our aim is to characterize the sets $\mathrm{Per}(f)$ when $f:\mathbb{S}\to\mathbb{S}$ is a periodic homeomorphism of some classes of pinched surface $\mathbb{S}$. The full characterization of the sets $\mathrm{Per}(f)$ for every pinched surface looks as a very hard problem.

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There are a lot of important works in this context. For example, in [13] it was studied the fixed point sets of pointwise almost periodic homeomorphism on the sphere $\mathbb{S}^2$ in the orientation-reversing and orientation-preserving cases, where a homeomorphism is called pointwise almost periodic if the collection of orbit closures forms a decomposition of $\mathbb{S}^2$. With a weaker condition, almost periodic, or a still weaker condition, weakly almost periodic, the fixed point set was studied in [14]. Also the sets of periods of some diffeomorphisms have been studied, see for instance [6, 7] and the references quoted there.

Dealing with homeomorphisms defined on graphs there are works in the same direction. In [5] the authors characterized the sets of periods of homeomorphisms defined on some classes of finite connected compact graphs. Probably the first result on the set of periods of a homeomorphism of a graph is due to Fuller [4]. See also Halpern [8] and Brown [2]. For other results on the set of periods of continuous maps from a graph into itself see for instance [1] or [12], and the references quoted there. For periodic properties of homeomorphisms on dendrites or Sierpinski curve see [9] and [10] respectively.

This paper is organized as follows. In section 2 we characterize the set $\text{Per}(f)$ when f is a periodic homeomorphism defined in a pinched surface with a unique handle. When a pinched surface has more than one handle but a unique branching point we obtain the set $\text{Per}(f)$ in section 3. Sections 4 and 5 are dedicated to the case of pinched surfaces with two branching points, and there we characterize the possible sets $\text{Per}(f)$ for a periodic homeomorphism defined in such surfaces.

2. A pinched surface with a unique handle

We start this section proving that there exist only two possibilities for a pinched surface with a unique handle.

Proposition 1. *Let $\mathbb{S}$ be a pinched surface with a unique handle. Then $\mathbb{S}$ is either a 2-dimensional sphere $\mathbb{S}^2$ (see Figure 1) or a pinched sphere $\mathbb{S}/\{P_N, P_s\}$ which is a sphere $\mathbb{S}^2$ with two points identified (see Figures 2, 3).*

Proof. Let $\mathbb{S}$ be a pinched surface. By definition the handle of $\mathbb{S}$ is homeomorphic to an open cylinder $(0,1)\times\mathbb{S}^1$, which is homeomorphic to the open cylinder $(-1,1)\times\mathbb{S}^1$. Identifying the boundaries $\{-1\}\times\mathbb{S}^1$ and $\{1\}\times\mathbb{S}^1$ to the points $(0,0,-1)$ and $(0,0,1)$ respectively, we obtain that $\mathbb{S}$ is homeomorphic to the sphere $\mathbb{S}^2=\{(x,y,z)\in\mathbb{R}^3 \,:\, ||(x,y,z)||=1\}$, which is a pinched surface with unique handle and two vertices. If we identify the boundaries $\{-1\}\times\mathbb{S}^1$ and $\{1\}\times\mathbb{S}^1$ to the same point, we obtain a pinched surface with a unique handle and only one vertice. In this case the identification can be done in two as we explain below.

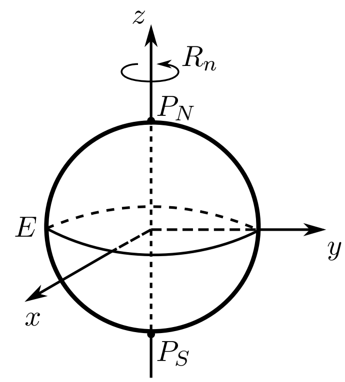

FIGURE 1. A 2–dimensional sphere $\mathbb{S}^2$.

Starting with the sphere $\mathbb{S}^2$ and considering the south pole $P_S = (0, 0, -1)$ and the north pole $P_N = (0, 0, 1)$ as the two vertices of our pinched surface $\mathbb{S}$ with a unique handle we can identify P_S and P_N to the same point in the following two different ways:

(a) by the interior of the sphere $\mathbb{S}^2$: we call this object S_0
(b) by outside of the sphere $\mathbb{S}^2$: we call this object S_1

More precisely, in the case (a) we consider the segment $\overline{P_S P_N}$ linking the points P_S and P_N, that is, $tP_S + (1-t)P_N$, with $0 \le t \le 1$. Then, over this segment we identify P_S and P_N to the same point $J_0 = (0, 0, 0)$ by the interior of the sphere $\mathbb{S}^2$. In the case (b) we consider a circle in the yz-plane centered at a point $(0, d, 0)$ and radius $r = \sqrt{1 + d^2}$, with $d > 1$. Points P_S and P_N belong tho this circle and then, over this circle, we can identify P_S and P_N to the same point $J_1 = (0, d + r, 0)$ by outside the sphere $\mathbb{S}^2$.

In order to prove that there exist only two pinched surface with a unique handle we shall prove that the last two objects obtained by identifying two points of the sphere (S_0 and S_1) are homeomorphic each other. To prove this we take a neighborhood U_0 of J_0 (an open sphere in $\mathbb{R}^3$) and a neighborhood U_1 of J_1 (an open sphere in $\mathbb{R}^3$) and we observe that $S_0 \cap U_0$ and $S_1 \cap U_1$ are homeomorphic to a cone. Moreover, we have that $S_0 \cap (U_0^{\mathsf{c}})$ and $S_1 \cap (U_1^{\mathsf{c}})$ are homeomorphic to an open cylinder. □

From now on we will always refer to a pinched sphere $\mathbb{S}/\{P_N, P_S\}$ as in the Figure 3 for simplicity of understanding the results.

The proof of the next proposition follows immediately from the definition of a homeomorphism.

Proposition 2. *Let $f : \mathbb{S} \to \mathbb{S}$ be a periodic homeomorphism of a pinched surface $\mathbb{S}$. Let z be a vertex of valence $k > 1$. Then $f(z)$ is a vertex of valence k.*

An important concept that we need is the topological equivalence between functions, that is, given continuous functions $f : S \to S$ and $f_1 : S_1 \to S_1$, we

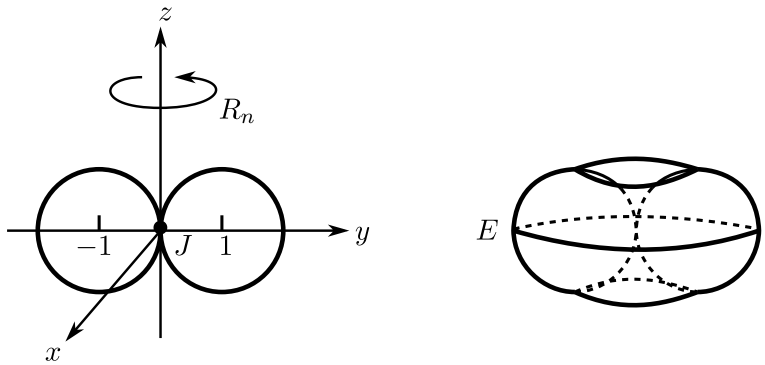

FIGURE 2. Figure eight. Its rotation around the z–axis generates a pinched sphere $\mathbb{S}$.

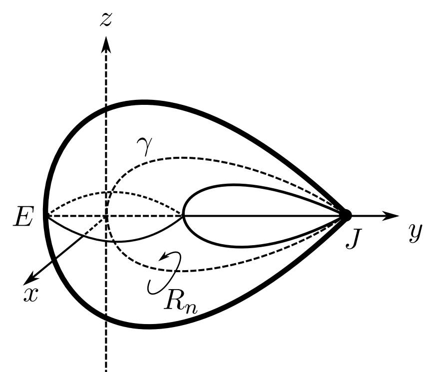

FIGURE 3. A pinched sphere $\mathbb{S}$ topologically is a sphere $\mathbb{S}^2$ with two points identified as in the figure. Also it is a pinched flower surface with a unique handle.

say that f and f_1 are *topologically equivalent* if there exists a homeomorphism $h : S \to S_1$ such that $(f_1 \circ h)(x) = (h \circ f)(x)$, for all $x \in S$.

We say that S is a *topological sphere* if S is homeomorphic to $\mathbb{S}^2$. In [3] and [11] it was proved the following result.

Theorem 3. *Every periodic map f of a topological sphere S into itself is topologically equivalent to a continuous periodic map $f_1 : \mathbb{S}^2 \to \mathbb{S}^2$ such that f_1 is either the identity, or a rotation with respect an axis which pass through the center of the sphere, or a reflection with respect to a plane passing for the center of the sphere, or to a rotation followed by a reflection with the axis of the rotation orthogonal to the plane of reflection.*

In what follows we consider R_n the rotation with respect to the z–axis by an angle $2\pi/n$ of the sphere $\mathbb{S}^2$, and when we consider a reflection it will be a reflection with respect to the plane $z = 0$ of the sphere $\mathbb{S}^2$ (see Figure 1). This will be the case when we consider a sphere $\mathbb{S}^2$ with the points P_S and P_N identified by the interior of the sphere $\mathbb{S}^2$, that is, a pinched sphere

$\mathbb{S}/\{P_N, P_S\}$ with a unique handle and only one vertice (case (a) in the proof of Proposition 1, see Figure 2).

When we are working with a pinched sphere $\mathbb{S}/\{P_N, P_S\}$ with the points P_S and P_N identified by outside of the sphere $\mathbb{S}^2$ (case (b) in the proof of Proposition 1) we consider R_n the rotation by an angle $2\pi/n$ with respect the curve γ given in Figure 3. In order to be more precisely curve γ is a curve in the $yz-$plane obtained from segment $\overline{P_S P_N}$ by identifying P_S and P_N to the same point $J_1 = (0, d+r, 0)$ over the circle centered at the point $(0, d, 0)$ and radius $r = \sqrt{1+d^2}$ with $d > 1$ as in the proof of Proposition 1 . Hence the rotation R_n around γ works as a rotation around the $z-$axis and after that we identify the points P_S and P_N to the point J_1 (see Figure 3). When we consider a reflection it will be a reflection with respect to the plane $z = 0$ of the pinched sphere $\mathbb{S}/\{P_N, P_S\}$.

Our first result establishes a correspondence between periodic homeomorphisms on the sphere and homeomorphisms defined on a pinched sphere. But for this we will consider a identification between $\mathbb{S}^2$ and $\mathbb{S}/\{P_N, P_S\}$ given by a map $D : \mathbb{S}^2 \to \mathbb{S}/\{P_N, P_S\}$ satisfying the following conditions:

(i) D preserve the equator E (see Figures 2 and 3), i.e. $D(E) = E$;
(ii) $D(P_N) = D(P_S) = J$, where $J = J_0$ or $J = J_1$ depending how the points P_S and P_N are identified (by the interior or by outside the sphere $\mathbb{S}^2$);
(iii) $D(\mathbb{S}^2 \cap \{z > 0\} \setminus \{P_N\}) = (\mathbb{S}/\{P_N, P_s\}) \cap \{z > 0\}$ and $D(\mathbb{S}^2 \cap \{z < 0\} \setminus \{P_S\}) = (\mathbb{S}/\{P_N, P_S\}) \cap \{z < 0\}$.

It is not difficult to see that $D|_{\mathbb{S}^2 \setminus \{P_N, P_S\}} : \mathbb{S}^2 \setminus \{P_N, P_S\} \to (\mathbb{S}/\{P_N, P_S\}) \setminus \{J\}$ is a homeomorphism.

Proposition 4. *The following statements hold.*

(a) *Given a periodic homeomorphism $F : \mathbb{S}^2 \to \mathbb{S}^2$ having, either fixed points in P_N and P_S, or a $2-$periodic orbit formed by P_N and P_S, there is a periodic homeomorphism $f : \mathbb{S}/\{P_N, P_S\} \to \mathbb{S}/\{P_N, P_s\}$ such that $F|_{\mathbb{S}^2 \setminus \{P_N, P_S\}} = D^{-1} \circ f \circ D|_{\mathbb{S}^2 \setminus \{P_N, P_S\}}$.*
(b) *Given a periodic homeomorphism $f : \mathbb{S}/\{P_N, P_S\} \to \mathbb{S}/\{P_N, P_S\}$ there is a periodic homeomorphism $F : \mathbb{S}^2 \to \mathbb{S}^2$ having, either fixed points in P_N and P_S, or a $2-$periodic orbit formed by P_N and P_S, such that $f|_{\mathbb{S} \setminus \{J_1\}} = D \circ F \circ D^{-1}|_{\mathbb{S} \setminus \{J_1\}}$.*

Proof. We start proving statement (a). Consider a periodic homeomorphism $F : \mathbb{S}^2 \to \mathbb{S}^2$ having P_N and P_S as fixed points, or as a 2–periodic orbit. Then we define a periodic homeomorphism $f : \mathbb{S}/\{P_N, P_S\} \to \mathbb{S}/\{P_N, P_S\}$ as follows. We identify $\mathbb{S}^2 \setminus \{P_N, P_S\}$ with $\mathbb{S}/\{P_N, P_S\} \setminus \{J\}$ using map D defined above, and in $\mathbb{S}/\{P_N, P_S\} \setminus \{J\}$ we take $f = D \circ F \circ D^{-1}$. We complete the definition of f taking $f(J) = J$. This proves statement (a).

We shall prove statement (b). Let U be a neighborhood of the branching point J. Given a periodic homeomorphism $f : \mathbb{S}/\{P_N, P_S\} \to \mathbb{S}/\{P_N, P_S\}$ we define a periodic homeomorphism $F : \mathbb{S}^2 \to \mathbb{S}^2$ as follows. We identify

$\mathbb{S}/\{P_N, P_S\} \setminus \{J\}$ with $\mathbb{S}^2 \setminus \{P_N, P_S\}$ using map D defined above, and in $\mathbb{S}^2 \setminus \{P_N, P_S\}$ we take $F = D^{-1} \circ f \circ D$. Since $f : \mathbb{S}/\{P_N, P_S\} \to \mathbb{S}/\{P_N, P_S\}$ is a homeomorphism there are only two possibilities for the points in U, i.e. either $f(U \cap \{z > 0\}) \subset U \cap \{z > 0\}$, or $f(U \cap \{z > 0\}) \subset U \cap \{z < 0\}$. In the first case we complete the definition of F taking $F(P_N) = P_N$ and $F(P_S) = P_S$; and in the second case we take $F(P_N) = P_S$ and $F(P_S) = P_N$. This completes the proof of statement (b). $\square$

Theorem 5. *Let $\mathbb{S}$ be a pinched surface with a unique handle and let $f : \mathbb{S} \to \mathbb{S}$ be a periodic homeomorphism. Then, the following statements hold.*

(a) *If $\mathbb{S}$ is $\mathbb{S}^2$, then* $\mathrm{Per}(f)$ *is*
either $\{1\}$ if f is topologically equivalent to the identity,
or $\{1, n\}$ if f is topologically equivalent to a rotation R_n, and $f^n = id$,
or $\{1, 2\}$ if f is topologically equivalent to a reflection, and $f^2 = id$,
or $\{2, n, 2n\}$ when n is odd, and $\{2, n\}$ when n is even, if f is topologically equivalent to a rotation R_n followed by a reflection, and $f^{2n} = id$.

(b) *If $\mathbb{S}$ is a pinched sphere $\mathbb{S}/\{P_N, P_S\}$, then* $\mathrm{Per}(f)$ *is*
either $\{1\}$ if f is topologically equivalent to the identity,
or $\{1, n\}$ if f is topologically equivalent to a rotation R_n, and $f^n = id$,
or $\{1, 2\}$ if f is topologically equivalent to a reflection, and $f^2 = id$,
or $\{1, n, 2n\}$ when n is odd, and $\{1, n\}$ when n is even, if f is topologically equivalent to a rotation R_n followed by a reflection, and $f^{2n} = id$.

Proof. Let $\mathbb{S}$ be a pinched surface with a unique handle and let $f : \mathbb{S} \to \mathbb{S}$ be a periodic homeomorphism. Firstly, we assume that $\mathbb{S}$ is $\mathbb{S}^2$. Hence, using Theorem 3, the proof is quite straightforward. More precisely if f is topologically equivalent to the identity, then $\mathrm{Per}(f) = \{1\}$.

If f is topologically equivalent to a rotation R_n the vertices P_N and P_S remain fixed by f while the other points are rotate by an angle $2\pi/n$ and then these points are fixed by f^n. Therefore, $\mathrm{Per}(f) = \{1, n\}$. and $f^n = id$.

When f is topologically equivalent to a reflection with respect to the plane $z = 0$ we get that all the points of the equator E are fixed by f while the others have period 2. Hence $\mathrm{Per}(f) = \{1, 2\}$, and $f^2 = id$.

If f is topologically equivalent to a rotation R_n followed by a reflection with respect to the plane $z = 0$ then the points P_N and P_S have period 2, all the points in the equator E have period n and the other points have period $2n$ when n is odd or period n when n is even. To see this we just observe that after each reflection these points change hemisphere. Thus, $\mathrm{Per}(f) = \{2, n, 2n\}$ if n is odd or $\mathrm{Per}(f) = \{2, n\}$ if n is even. This completes the proof of statement (a).

For statement (b) we use the same arguments as before. We just to take into account that since S is a pinched sphere $\mathbb{S}/\{P_N, P_S\}$, which can be obtained from $\mathbb{S}^2$ by identifying two points, there exist only two possibilities to do this identification that preserve f as a homeomorphism. More precisely, we can identify two fixed points, or two points of a periodic orbit of period 2.

In the first case we can assume, without loss of generality, that north pole P_N and south pole P_S of sphere $\mathbb{S}^2$ are fixed points of a periodic homeomorphism $F : \mathbb{S}^2 \to \mathbb{S}^2$. Hence, using the identification given by map $D : \mathbb{S}^2 \to \mathbb{S}/\{P_N, P_S\}$ defined above, we conclude that F satisfy the hypotesis of item (a) and then $\mathrm{Per}(F)$ can be all the possibilities listed in (a). But since $f : \mathbb{S}/\{P_N, P_S\} \to \mathbb{S}/\{P_N, P_S\}$ is a periodic homeomorphism the identified point J remains as a fixed point of f and his period is 1 which is the same period of the points P_N and P_S for F. Thus $\mathrm{Per}(f) = \mathrm{Per}(F)$ and we are done.

When we identify two points of a periodic orbit of period 2 we can have two situations, that is, if this identification occurs in the unique periodic orbit of period 2 of a periodic homeomorphism $F : \mathbb{S}^2 \to \mathbb{S}^2$ then, using the identification $D : \mathbb{S}^2 \to \mathbb{S}/\{P_N, P_S\}$, this orbit becomes a fixed point of $f : \mathbb{S}/\{P_N, P_S\} \to \mathbb{S}/\{P_N, P_S\}$ implying that $2 \notin \mathrm{Per}(f)$ and $1 \in \mathrm{Per}(f)$. But if there exist more than one periodic orbit of period 2 of F we get that $1, 2 \in \mathrm{Per}(f)$ and then the set $\mathrm{Per}(f)$ remains the same as in statement (a). So, statement (b) is proved. □

3. Flower surface

A *flower surface* is a surface with a unique branching point z and a finite number of handles called here *petals*. There exist three different types of closed petals (sphere, pinched sphere and pinched torus), consequently we have three different types of flowers having the same kind of petals, that is, a *sphere flower* when all the petals are spheres, a *pinched sphere flower* when all the petals are pinched spheres and a *pinched torus flower* when all the petals are pinched torus. See a pinched torus flower surface with 5 petals in Figure 4 and a sphere flower surface with 4 petals in Figure 5.

Lemma 6. *Let $f : S \to S$ be a periodic homeomorphism of a sphere flower surface with r petals denoted by $\mathbb{S}^2_1, \mathbb{S}^2_2, \dots, \mathbb{S}^2_r$. Suppose that $f(\mathbb{S}^2_i) = \mathbb{S}^2_{i+1}$ for all $i = 1, 2, \dots, r-1$ and $f(\mathbb{S}^2_r) = \mathbb{S}^2_1$. Then $\mathrm{Per}(f)$ is either $\{1, r\}$, or $\{1, r, nr\}$, or $\{1, r, 2r\}$.*

Proof. Considering z the unique branching point of the sphere flower surface S with r petals, by Proposition 2, we have that $f(z) = z$, and then $1 \in \mathrm{Per}(f)$. In particular $f^r(z) = z$.

Furthermore, the hypotheses assure that $f^r|_{\mathbb{S}^2_i} : \mathbb{S}^2_i \to \mathbb{S}^2_i$ is a periodic homeomorphism of the sphere $\mathbb{S}^2_i$. Thus, $\mathrm{Per}(f^r)$ is one of the sets from Theorem 5(a), except the set $\{2, n, 2n\}$ because here we cannot have f^r

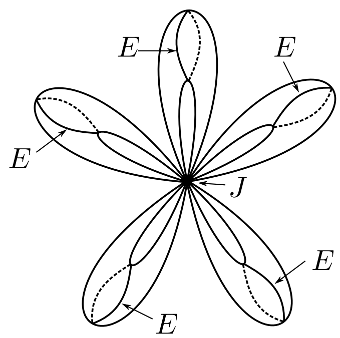

FIGURE 4. A pinched torus flower surface.

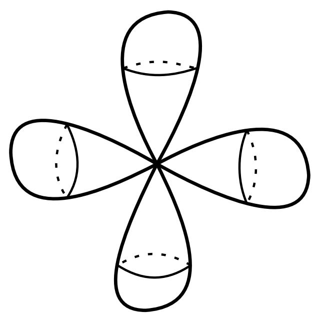

FIGURE 5. A sphere flower surface.

topologically equivalent to a rotation R_n followed by a reflection since if f^r were we would have $f^r(z) \neq z$, which is a contradiction with $f(z) = z$.

When f^r is topologically equivalent to the identity we get that $\mathrm{Per}(f^r) = \{1\}$. Since the branching point z is a fixed point of f and the other points are fixed only by f^r, not of f, we conclude that the points $x \in \mathbb{S}_i^2$ different from z are periodic of period r of f. Then $\mathrm{Per}(f) = \{1, r\}$.

Assuming that f^r is topologically equivalent to a rotation R_n, and $(f^r)^n = id$, we get $\mathrm{Per}(f^r) = \{1, n\}$. There exist two fixed points of f^r. One of these two points is the branching point z and the other is a point $w \in \mathbb{S}_i^2$ different from z. Since z is the unique fixed point of f we conclude that w is a periodic point of period r to f. Furthermore, all the other points in $\mathbb{S}_i^2$ different from z and w are rotated by a rotation R_n, and thus they have period $r \cdot n$ to f. Then $\mathrm{Per}(f) = \{1, r, rn\}$.

In the case that f^r is topologically equivalent to a reflection, and $(f^r)^2 = id$, we get $\mathrm{Per}(f^r) = \{1, 2\}$. Here the reflection is with respect to a topological plane π containing the branching point z. Consequently we have that all the points in the set $\pi \cap \mathbb{S}_i^2$ are fixed by f^r and the other points in $\mathbb{S}_i^2 \setminus \pi \cap \mathbb{S}_i^2$ are periodic of period 2 by f^r. Since the branching point z is the unique fixed point of f we conclude that the points in $\pi \cap \mathbb{S}_i^2$ different from z are periodic of period r for f and the points in $\mathbb{S}_i^2 \setminus \pi \cap \mathbb{S}_i^2$ are periodic of period $2r$ for f. Thus $\mathrm{Per}(f) = \{1, r, 2r\}$, and we are done. □

We have similar results for a pinched sphere flower and for a pinched torus flower.

Lemma 7. *Let $f : S \to S$ be a periodic homeomorphism of a pinched torus flower surface S with r petals denoted by $\mathbb{T}_1, \mathbb{T}_2, \ldots, \mathbb{T}_r$. Suppose that $f(\mathbb{T}_i) = \mathbb{T}_{i+1}$ for all $i = 1, 2, \ldots, r-1$ and $f(\mathbb{T}_r) = \mathbb{T}_1$. Then* $\mathrm{Per}(f)$ *is either $\{1, r\}$, or $\{1, nr\}$, or $\{1, r, 2r\}$, or $\{1, nr, 2nr\}$.*

Proof. Considering z the unique branching point of the r–flower S, by Proposition 2, we have that $f(z) = z$, and then $1 \in \mathrm{Per}(f)$.

The hypotheses assure that $f^r|_{\mathbb{T}_i} : \mathbb{T}_i \to \mathbb{T}_i$ is a periodic homeomorphism of the pinched torus $\mathbb{T}_i$. Thus $\mathrm{Per}(f^r)$ is one of the sets of Theorem 5(b).

When f^r is topologically equivalent to the identity we have $\mathrm{Per}(f^r) = \{1\}$. Since the branching point z is the unique fixed point of f the other points in $\mathbb{T}_i$ different from z are periodic of period r for f, and then we get $\mathrm{Per}(f) = \{1, r\}$.

If f^r is topologically equivalent to a rotation R_n, and $(f^r)^n = id$, we have $\mathrm{Per}(f^r) = \{1, n\}$. The fact that the branching point z is the unique fixed point of f implies that the other points in $\mathbb{T}_i$ different from z are periodic of period n for f^r and then, these points, are periodic of period nr for f. Thus we get $\mathrm{Per}(f) = \{1, nr\}$.

When f^r is topologically equivalent to a reflection, and $(f^r)^2 = id$, we have $\mathrm{Per}(f^r) = \{1, 2\}$. If the pinched torus flower surface has only one handle, i.e. the pinched torus, the plane π of the reflection is the $xy-$plane which contains the branching point J and equator E, see Figure 3. If the pinched torus flower surface has more than one handle, the reflection is a simultaneous reflection in each handle as it is defined in the pinched torus. Hence the points $x \in \pi \cap \mathbb{T}_i \setminus \{z\}$ are fixed by the f^r, and then they are periodic of period r for f. Furthermore, the other points $x \in \mathbb{T}_i \setminus \pi \cap \mathbb{T}_i$ are periodic of period 2 for f^r, and then they are periodic of period $2r$ to f. Therefore, we conclude that $\mathrm{Per}(f) = \{1, r, 2r\}$.

Considering f^r topologically equivalent to a rotation R_n followed by a reflection, and $(f^r)^{2n} = id$, we get $\mathrm{Per}(f^r) = \{1, n, 2n\}$. As in the previous case, here the reflection is with respect to a topological plane π that contains the branching point z and the rotation R_n is with respect to a curve γ as in Figure 3. Therefore the branching point z remains as the unique fixed point of f, the points in $\pi \cap \mathbb{T}_i \setminus \{z\}$ are periodic of period n for f^r, and then they are periodic of period nr for f, the other points in $\mathbb{T}_i \setminus \pi \cap \mathbb{T}_i$ are periodic of period $2n$ for f^r when n is odd, and periodic of period n for f^r when n is even, and then they are periodic of period $2nr$ of f when n is odd, and periodic of period nr of f when n is even. Hence $\mathrm{Per}(f) = \{1, nr, 2nr\}$ when n is odd and $\mathrm{Per}(f) = \{1, nr\}$ when n is even. □

With the same arguments and considerations we prove a similar result for a pinched sphere flower.

Lemma 8. *Let $f : S \to S$ be a periodic homeomorphism of a pinched sphere flower surface S with r petals denoted by $\mathbb{S}_1, \mathbb{S}_2, \ldots, \mathbb{S}_r$. Suppose that $f(\mathbb{S}_i) = \mathbb{S}_{i+1}$ for all $i = 1, 2, \ldots, r-1$ and $f(\mathbb{S}_r) = \mathbb{S}_1$. Then $\mathrm{Per}(f)$ is either $\{1, r\}$, or $\{1, nr\}$, or $\{1, r, 2r\}$, or $\{1, nr, 2nr\}$.*

Remark 1. *We observe that in Lemmas 7 and 8 the iterate f^r is a periodic homeomorphism defined on a pinched torus $\mathbb{T}$ and on a pinched sphere $\mathbb{S}$, respectively. Then we can apply Proposition 4(b) for f^r and we conclude that for f^r there exist a periodic homeomorphism $F : \mathbb{S}^2 \to \mathbb{S}^2$ such that $f^r|_{\mathbb{T}\setminus\{J\}} = D \circ F \circ D^{-1}|_{\mathbb{S}^2\setminus\{P_N, P_S\}}$, and a periodic homeomorphism $G : \mathbb{S}^2 \to \mathbb{S}^2$ such that $f^r|_{\mathbb{S}\setminus\{J\}} = D \circ G \circ D^{-1}|_{\mathbb{S}^2\setminus\{P_N, P_S\}}$ respectively.*

A consequence of these lemmas is the next result.

A *mixed flower surface* is a flower surface with a unique branching point z and a finite number of handles of at least two types among sphere, pinched sphere and pinched torus. See in Figure 6 a mixed flower surface with 5 petals where 2 of them are spheres and the other 3 are pinched torus.

Theorem 9. *Let $f : S \to S$ be a periodic homeomorphism of a mixed flower surface S with r petals denoted by $P_1, P_2, \ldots, P_r$. Then the set $\mathrm{Per}(f)$ is $\bigcup_{l=1}^{s} \mathrm{Per}(f|_{\mathcal{C}_l})$, where $\mathcal{C}_l = \{P_{i_1}, P_{i_2}, \ldots, P_{i_{r_l}}\}$ is an invariant set by f formed by r_l petals such that $f(P_{i_k}) = P_{i_{k+1}}$, for $k = 1, 2, \ldots, r_l - 1$ and $f(P_{i_{r_l}}) = P_{i_1}$, for $l = 1, 2, \ldots, s$, with the positive integers r_l satisfying $r_1 + r_2 + \ldots + r_s = r$, and each set $\mathrm{Per}(f|_{\mathcal{C}_l})$ is a set as in the Lemma either 6, or 7 or 8 if the set $\mathcal{C}_l$ is either a sphere flower, or a pinched torus flower, or a pinched sphere flower, respectively.*

Proof. First we note that different types of pinched manifolds are not homeomorphic each other. Then there exist invariant sets $\mathcal{C}_l = \{P_{i_1}, P_{i_2}, \ldots, P_{i_{r_l}}\}$ by f formed by r_l petals of the same type such that $f(P_{i_k}) = P_{i_{k+1}}$, for $k = 1, 2, \ldots, r_l - 1$ and $f(P_{i_{r_l}}) = P_{i_1}$, for $l = 1, 2, \ldots, s$, with the positive integers r_l satisfying $r_1 + r_2 + \ldots + r_s = r$, and each set $\mathrm{Per}(f|_{\mathcal{C}_l})$ is a set as in the Lemma either 6, or 7 or 8 if the set $\mathcal{C}_l$ is either a sphere flower, or a pinched torus flower, or a pinched sphere flower, respectively. □

4. r–Lips Surface

A pinched surface with only two vertices z and w and $r > 1$ handles having every handle the vertices z and w as endpoints is called an *r–lips surface*. Note that in an r–lips surface each closed handle is a topological sphere $\mathbb{S}^2$. See a 4–lips surface in Figure 7.

Lemma 10. *Let $f : S \to S$ be a periodic homeomorphism of an r–lips surface S with vertices z and w, and let $\mathbb{S}^2_1, \mathbb{S}^2_2, \ldots, \mathbb{S}^2_r$ be the handles of S*

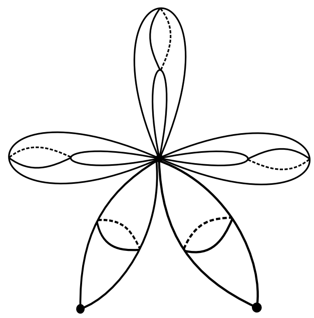

FIGURE 6. A mixed flower surface.

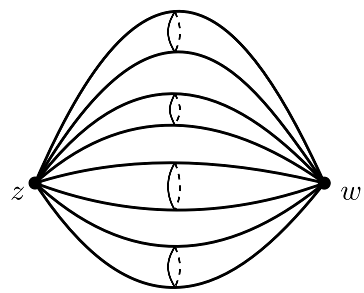

FIGURE 7. A 4–lips surface.

such that $f(\mathbb{S}_i^2) = \mathbb{S}_{i+1}^2$, *for* $i = 1, 2, \ldots, r-1$, *and* $f(\mathbb{S}_r^2) = \mathbb{S}_1^2$. *Then the set* $\mathrm{Per}(f)$ *is*

(a) *either* $\{1, r\}$, *or* $\{1, rn\}$, *or* $\{1, r, 2r\}$ *if* $f(z) = z$;
(b) *or* $\{2, r\}$, *or* $\{2, rn\}$, *or* $\{2, r, 2r\}$ *if* $f(z) = w$ *and* r *is even;*
(c) *or* $\{2, 2r\}$, *or* $\{2, 2rn\}$, *or* $\{2, 2r, 4r\}$ *if* $f(z) = w$ *and* r *is odd.*

Proof. By Proposition 2 we can have $f(z) = z$ or $f(z) = w$. Initially we assume that $f(z) = z$, and then $f(w) = w$, and consequently we get $1 \in \mathrm{Per}(f)$.

The hypotheses imply that $f^r|_{\mathbb{S}_i^2} : \mathbb{S}_i^2 \to \mathbb{S}_i^2$ is a periodic homeomorphism of the sphere $\mathbb{S}_i^2$, with the branching points z and w as fixed points. Thus, $\mathrm{Per}(f^r)$ is one of the sets from Theorem 5(a), except the sets $\{2, n, 2n\}$ **and** $\{2, n\}$ because here we cannot have f^r topologically equivalent to a rotation R_n followed by a reflection since if f^r were we would have $f^r(z) \neq z$, which is a contradiction with $f(z) = z$.

When f^r is topologically equivalent to the identity we get that $\mathrm{Per}(f^r) = \{1\}$. Since the branching points z and w are fixed points of f and the other points in $\mathbb{S}_i^2$ are fixed points only of f^r, not of f, we conclude that the points $x \in \mathbb{S}_i^2$ different from z and w are periodic of period r of f. Then $\mathrm{Per}(f) = \{1, r\}$.

If f^r is topologically equivalent to a rotation R_n we have $\mathrm{Per}(f^r) = \{1, n\}$. Since the branching points z and w are the unique fixed points of f we obtain that the other points in $\mathbb{S}^2_i$ are rotated by the rotation R_n and thus they are periodic of period n of f^r, that is, they are periodic of period rn of f. Then $\mathrm{Per}(f) = \{1, rn\}$.

When f^r is topologically equivalent to a reflection we get $\mathrm{Per}(f^r) = \{1, 2\}$. The reflection is with respect to a topological plane π in $\mathbb{S}^2_i$ that contains the branching points z and w. Consequently we get that the points in $\pi \cap \mathbb{S}^2_i$ are fixed by f^r and the other points in $\mathbb{S}^2_i \setminus \pi \cap \mathbb{S}^2_i$ are periodic of period 2 of f^r. Since the unique fixed points of f are the branching points z and w we conclude that the points in $\pi \cap \mathbb{S}^2_i$ different from z and w are periodic of period r of f, and the points in $\mathbb{S}^2_i \setminus \pi \cap \mathbb{S}^2_i$ are periodic of period $2r$ of f. Thus we get $\mathrm{Per}(f) = \{1, r, 2r\}$, and statement (a) is proved.

Considering the case $f(z) = w$ and r even and doing the same analysis as before we obtain the same sets $\mathrm{Per}(f)$ as before just changing the period 1 by period 2 because here the branching points z and w are periodic of period 2 of f, that is, either $\mathrm{Per}(f) = \{2, r\}$ when f^r is topologically equivalent to the identity, or $\mathrm{Per}(f) = \{2, rn\}$ when f^r is topologically equivalent to a rotation R_n, or $\mathrm{Per}(f) = \{2, r, 2r\}$ when f^r is topologically equivalent to a reflection. Thus statement (b) is proved.

If $f(z) = w$ and r is odd, we have that $f^{2r}|_{\mathbb{S}^2_i} : \mathbb{S}^2_i \to \mathbb{S}^2_i$ is a periodic homeomorphism with fixed points z and w. Using the arguments done to prove statement (a) we obtain that either $\mathrm{Per}(f) = \{2, 2r\}$ when f^{2r} is topologically equivalent to the identity, or $\mathrm{Per}(f) = \{2, 2rn\}$ when f^{2r} is topologically equivalent to a rotation R_n or $\mathrm{Per}(f) = \{2, 2r, 4r\}$ when f^{2r} is topologically equivalent to a reflection. Thus statement (c) is proved. $\square$

Theorem 11 (r–lips Theorem)**.** *Let $f : S \to S$ be a periodic homeomorphism of an r–lips surface S with vertices z and w, and let $\mathbb{S}^2_1, \mathbb{S}^2_2, \ldots, \mathbb{S}^2_r$ be the handles of S. Then the set* $\mathrm{Per}(f)$ *is*

(a) $\bigcup_{l=1}^{r} \mathrm{Per}(f|_{\mathbb{S}^2_l})$ *if $f(z) = z$ and $f(\mathbb{S}^2_i) = \mathbb{S}^2_i$ for all $i = 1, 2, \ldots, r$, where* $\mathrm{Per}(f|_{\mathbb{S}^2_l})$ *is one of the sets from Theorem 5*(a)*;*

(b) $\bigcup_{l=1}^{s} \mathrm{Per}(f|_{\mathcal{C}_l})$ *if $f(z) = z$ and $f(\mathbb{S}^2_i) \neq \mathbb{S}^2_i$ for some $i = 1, 2, \ldots, r$, where $\mathcal{C}_l = \{\mathbb{S}^2_{i_1}, \mathbb{S}^2_{i_2}, \ldots, \mathbb{S}^2_{i_{r_l}}\}$ is an invariant set by f formed by r_l petals such that $f(\mathbb{S}^2_{i_k}) = \mathbb{S}^2_{i_{k+1}}$, for $k = 1, 2, \ldots, r_l - 1$ and $f(\mathbb{S}^2_{i_{r_l}}) = \mathbb{S}^2_{i_1}$, for $l = 1, 2, \ldots, s$, with the positive integers r_l satisfying $r_1 + r_2 + \ldots + r_s = r$, where* $\mathrm{Per}(f|_{\mathcal{C}_l})$ *is one of the sets from Lemma 10*(a)*;*

(c) $\bigcup_{l=1}^{r} \mathrm{Per}(f|_{\mathbb{S}^2_l})$ *if $f(z) \neq z$ and $f(\mathbb{S}^2_i) = \mathbb{S}^2_i$ for all $i = 1, 2, \ldots, r$, where* $\mathrm{Per}(f|_{\mathbb{S}^2_l})$ *is one of the sets from Theorem 5*(a)*, except the sets $\{1\}$ and $\{1, n\}$ with $n \neq 2$;*

(d) $\bigcup_{l=1}^{s} \mathrm{Per}(f|_{\mathcal{C}_l})$ *if $f(z) \neq z$ and $f(\mathbb{S}^2_i) \neq \mathbb{S}^2_i$ for some $i = 1, 2, \ldots, r$, where $\mathcal{C}_l = \{\mathbb{S}^2_{i_1}, \mathbb{S}^2_{i_2}, \ldots, \mathbb{S}^2_{i_{r_l}}\}$ is an invariant set by f formed by r_l petals such that $f(\mathbb{S}^2_{i_k}) = \mathbb{S}^2_{i_{k+1}}$, for $k = 1, 2, \ldots, r_l - 1$ and $f(\mathbb{S}^2_{i_{r_l}}) =$*

$\mathbb{S}^2_{i_1}$, *for* $l = 1, 2, \ldots, s$, *with the positive integers* r_l *satisfying* $r_1+r_2+\ldots+r_s = r$, *where* $\mathrm{Per}(f|_{\mathcal{C}_l})$ *is one of the sets from Lemma 10*(b)-(c).

Proof. By Proposition 2 we have either $f(z) = z$ (and consequently $f(w) = w$), or $f(z) = w$ (and consequently $f(w) = z$).

If $f(z) = z$ and $f(\mathbb{S}^2_i) = \mathbb{S}^2_i$ for all $i = 1, 2, \ldots, r$, then statement (a) follows from Theorem 5(a).

In the case $f(z) = z$ and $f(\mathbb{S}^2_i) \neq \mathbb{S}^2_i$ for some $i = 1, 2, \ldots, r$, since each handle $\mathbb{S}^2_i$ must be applied in other handle $\mathbb{S}^2_j$, there exist invariant sets $\mathcal{C}_l = \{\mathbb{S}^2_{i_1}, \mathbb{S}^2_{i_2}, \ldots, \mathbb{S}^2_{i_{r_l}}\}$ by f formed by r_l petals such that $f(\mathbb{S}^2_{i_k}) = \mathbb{S}^2_{i_{k+1}}$, for $k = 1, 2, \ldots, r_l - 1$ and $f(\mathbb{S}^2_{i_{r_l}}) = \mathbb{S}^2_{i_1}$, with the positive integers r_l satisfying $r_1 + r_2 + \ldots + r_s = r$. Applying Lemma 10(a) in each invariant set $\mathcal{C}_l$ statement (b) follows.

Assuming $f(z) = w$ we get that $f^2(z) = z$, and then $2 \in \mathrm{Per}(f)$. If $f(\mathbb{S}^2_i) = \mathbb{S}^2_i$ for all $i = 1, 2, \ldots, r$ we have that $f|_{\mathbb{S}^2_i} : \mathbb{S}^2_i \to \mathbb{S}^2_i$ is a periodic homeomorphism with the branching point z as a periodic point of period 2. Thus $f|_{\mathbb{S}^2_i}$ cannot be topologically equivalent to the identity, or to a rotation R_n with $n \neq 2$, and then, $\mathrm{Per}(f|_{\mathbb{S}^2_i})$ is one of the sets from Theorem 5(a), except the sets $\{1\}$ and $\{1, n\}$ with $n \neq 2$. Hence statement (c) is proved.

Assume $f(z) = w$ and $f(\mathbb{S}^2_i) \neq \mathbb{S}^2_i$ for some $i = 1, 2, \ldots, r$. Since each handle $\mathbb{S}^2_i$ must be applied in other handle $\mathbb{S}^2_j$, there exist invariant sets $\mathcal{C}_l = \{\mathbb{S}^2_{i_1}, \mathbb{S}^2_{i_2}, \ldots, \mathbb{S}^2_{i_{r_l}}\}$ by f formed by r_l petals such that $f(\mathbb{S}^2_{i_k}) = \mathbb{S}^2_{i_{k+1}}$, for $k = 1, 2, \ldots, r_l - 1$ and $f(\mathbb{S}^2_{i_{r_l}}) = \mathbb{S}^2_{i_1}$, with the positive integers r_l satisfying $r_1 + r_2 + \ldots + r_s = r$. Applying Lemma 10(b), (c) in each invariant set $\mathcal{C}_l$ statement (d) follows. □

5. r–Lips with Flower Surface and Surfaces

In what follows we denote by $\mathcal{F}_z$ a mixed flower or just a flower with branching point z, and we denote by $\mathcal{F}_w \cup S \cup \mathcal{F}_z$ a pinched surface with only 2 branching points (the points z and w) formed by an r–lips surface S, a mixed flower (or just flower) $\mathcal{F}_z$ with branching point z and a mixed flower (or just flower) $\mathcal{F}_w$ with branching point w. In the case that a such surface is formed by only an r–lips surface S and a unique mixed flower (or just flower) $\mathcal{F}_z$ we denote it by $S \cup \mathcal{F}_z$ or $\mathcal{F}_z \cup S$. Note that each handle in the r–lips S is not a petal in $\mathcal{F}_z$ (respectively in $\mathcal{F}_w$) because $\mathcal{F}_z$ has a unique branching point z and each handle in the r–lips has two branching points (z and w). We call the surface $\mathcal{F}_w \cup S \cup \mathcal{F}_z$ by *r–lips with flower surfaces*, and the surface $S \cup \mathcal{F}_z$ or $\mathcal{F}_z \cup S$ by *r–lips with flower surface.* See a 4–lips with flower surfaces in Figure 8.

For a mixed flower or a flower $\mathcal{F}_z$ we define the $\mathrm{card}(\mathcal{F}_z) = (a, b, c)$ where a is the number of closed handles in $\mathcal{F}_z$ which are spheres, b is the number of closed handles in $\mathcal{F}_z$ which are pinched torus, and c is the number of closed handles in $\mathcal{F}_z$ which are pinched spheres.

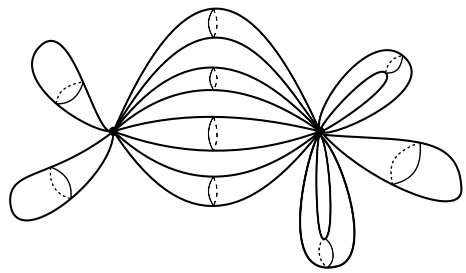

FIGURE 8. A 4–lips with flower surfaces.

Let A be a subset of positive integers, we denote by $2 \cdot A$ the following set $\{2a : a \in A\}$.

Theorem 12 (r-Lips with Flower Surfaces Theorem)**.** *Let $\mathcal{M} = \mathcal{F}_z \cup S \cup \mathcal{F}_w$ be an r–lips with flower surfaces. Let $f : \mathcal{M} \to \mathcal{M}$ be a periodic homeomorphism. Then the set* $\mathrm{Per}(f)$ *is*

(a) $\mathrm{Per}(f|_{\mathcal{F}_z}) \cup \mathrm{Per}(f|_S) \cup \mathrm{Per}(f|_{\mathcal{F}_w})$ *if* $\mathrm{card}(\mathcal{F}_z) \neq \mathrm{card}(\mathcal{F}_w)$ *or if* $\mathrm{card}(\mathcal{F}_z) = \mathrm{card}(\mathcal{F}_w)$ *and $f(z) = z$, where* $\mathrm{Per}(f|_{\mathcal{F}_z})$ (*respectively* $\mathrm{Per}(f|_{\mathcal{F}_w})$) *is one of the sets from Theorem 9 if $\mathcal{F}_z$ is a mixed flower, or one of the sets from Lemmas 6, 7 or 8 if $\mathcal{F}_z$ is a flower, similar definitions for* $\mathrm{Per}(f|_{\mathcal{F}_w})$*, and* $\mathrm{Per}(f|_S)$ *is one of the sets from Theorem 11.*

(b) $\mathrm{Per}(f|_S) \cup 2 \cdot \mathrm{Per}(f^2|_{\mathcal{F}_z}) \cup 2 \cdot \mathrm{Per}(f^2|_{\mathcal{F}_w})$ *if* $\mathrm{card}(\mathcal{F}_z) = \mathrm{card}(\mathcal{F}_w)$ *and $f(z) \neq z$, where* $\mathrm{Per}(f_S)$ *is one of the sets from Theorem 11*(c)-(d), $\mathrm{Per}(f^2|_{\mathcal{F}_z})$ (*respectively* $\mathrm{Per}(f^2|_{\mathcal{F}_w})$) *is one of the sets from Theorem 9.*

Proof. If $\mathrm{card}(\mathcal{F}_z) \neq \mathrm{card}(\mathcal{F}_w)$ we get from Proposition 2 that $f(z) = z$ and $f(w) = w$. In this case we obtain that $\mathcal{F}_z$, $\mathcal{F}_w$ and S are invariant by f. If $\mathrm{card}(\mathcal{F}_z) = \mathrm{card}(\mathcal{F}_w)$ and $f(z) = z$ the sets $\mathcal{F}_z$, $\mathcal{F}_w$ and S remain also invariant by f. Thus, applying Theorem 9 if $\mathcal{F}_z$ (respectively for $\mathcal{F}_w$) is a mixed flower, or Lemmas 6, 7 or 8 if $\mathcal{F}_z$ (respectively for $\mathcal{F}_w$) is a flower, and Theorem 11 for S it follows statement (a).

When $\mathrm{card}(\mathcal{F}_z) = \mathrm{card}(\mathcal{F}_w)$ and $f(z) \neq z$ we obtain from Proposition 2 that $f(z) = w$, $f(w) = z$. Then $f^2(z) = z$ and $f^2(w) = w$. This conclusion together the fact that each petal is applied by f in other petal of the same type imply that the mixed flowers $\mathcal{F}_z$ and $\mathcal{F}_w$ are invariant by f^2 and the r–lips S is invariant by f. Thus $\mathrm{Per}(f|_S)$ is given by one of the period sets of statement (c) or (d) of Theorem 11, $\mathrm{Per}(f^2|_{\mathcal{F}_z})$ and $\mathrm{Per}(f^2|_{\mathcal{F}_w})$ are given by one of the period sets of Theorem 9.

Taking into account that a periodic point $x \in \mathcal{F}_z$ (respectively for $\mathcal{F}_w$) with period n for f^2 is a periodic point of period $2n$ for f we conclude that $\mathrm{Per}(f) = \mathrm{Per}(f|_S) \cup 2 \cdot \mathrm{Per}(f^2|_{\mathcal{F}_z}) \cup 2 \cdot \mathrm{Per}(f^2|_{\mathcal{F}_w})$. This completes the proof of statement (b). □

From Theorem 12 it follows immediately the next result.

Corollary 13. *Let* $\mathcal{M} = \mathcal{F}_z \cup S$ *be an* r*–lips with flower surface. Let* $f : \mathcal{M} \to \mathcal{M}$ *be a periodic homeomorphism. Then the set* $\mathrm{Per}(f) = \mathrm{Per}(f|_{\mathcal{F}_z}) \cup \mathrm{Per}(f|_S)$ *being* $\mathrm{Per}(f|_{\mathcal{F}_z})$ *and* $\mathrm{Per}(f|_S)$ *as in statement* (a) *of Theorem 12.*

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