ON LIMIT CYCLES BIFURCATING FROM THE INFINITY IN DISCONTINUOUS PIECEWISE LINEAR DIFFERENTIAL SYSTEMS

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Abstract. In this paper we consider the linear differential center $(\dot{x}, \dot{y}) = (-y, x)$ perturbed inside the class of all discontinuous piecewise linear differential systems with two zones separated by the straight line $y = 0$. We provide sufficient conditions to ensure the existence of a limit cycle bifurcating from the infinity. The main tools used are the Bendixson transformation and the averaging theory.

1. Introduction and statement of the main result

Establishing the existence of periodic solutions, specially *limit cycles*, i.e. periodic orbits isolated in the set of all periodic orbits, is one of the main problems of the qualitative theory of the differential systems in the plane. A classical way to produce and study limit cycles is perturbing the periodic solutions of a *center*, which is a point having a neighborhood, except itself, filled by periodic solutions. This problem for the smooth differential systems in the plane has been studied intensively, either for limit cycles bifurcating from finite periodic solutions (see, for instance, the hundred of references in the book [6]), and for periodic solutions bifurcating from the infinity (see for instance [19, 9, 20]). In [11] the authors studied the bifurcation of a limit cycle from the infinity for a non–smooth but continuous piecewise differential system.

On the other hand the theory of discontinuous systems has been developing at a fast pace in recent years, with growing importance at the frontier between mathematics, physics, engineering, and the life sciences. Interest stems particularly from discontinuous dynamical models in control theory [3], nonlinear oscillations [2, 15], impact and friction mechanics [5], economics [8, 10], biology [4], and others; recent reviews appear in [18, 14].

There are some works dealing with limit cycles bifurcating from finite periodic solutions in discontinuous differential system, we may cite [13, 16]. As far as we know, up to now, there are no studies about limit cycles bifurcating from the infinity for discontinuous systems.

In this paper we consider the linear differential center $(\dot{x}, \dot{y}) = (-y, x)$ perturbed inside the class of all discontinuous piecewise linear differential systems with two zones separated by the straight line $\Sigma = \{y = 0\}$, i.e. the discontinuous differential system

$$
\begin{aligned}
\dot{x} &= \begin{cases} -y + \varepsilon(a^+ + b^+x + c^+y) + \varepsilon^2(\alpha^+ + \beta^+x + \gamma^+y) & \text{if} \quad y > 0, \\ -y + \varepsilon(a^- + b^-x + c^-y) + \varepsilon^2(\alpha^- + \beta^-x + \gamma^-y) & \text{if} \quad y < 0, \end{cases} \\
\dot{y} &= \begin{cases} x + \varepsilon(k^+ + m^+x + n^+y) + \varepsilon^2(\kappa^+ + \mu^+x + \nu^+y) & \text{if} \quad y > 0, \\ x + \varepsilon(k^- + m^-x + n^-y) + \varepsilon^2(\kappa^- + \mu^-x + \nu^-y) & \text{if} \quad y < 0. \end{cases}
\end{aligned} \tag{1}
$$

Here the dot denotes derivative with respect to the time t.

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Our main objective is to study the limit cycles of system (1) that bifurcate from the infinity when $\varepsilon > 0$ is sufficiently small.

Theorem 1. *We assume that*

$$b^{+} + b^{-} + n^{+} + n^{-} = 0, \quad k^{-} + k^{+} = 0, \quad \textit{and}$$

$$k^{+}\left(2\beta^{-} + 2\left(\beta^{+} + \nu^{-} + \nu^{+}\right) - \left(b^{+} + n^{+}\right)\left(c^{-} - c^{+} - m^{-} + m^{+}\right)\right) < 0.$$

Then, for $\varepsilon > 0$ sufficiently small, there exists a limit cycle $\big(x(t,\varepsilon), y(t,\varepsilon)\big)$ of system (1) such that $\left|\big(x(t,\varepsilon), y(t,\varepsilon)\big)\right| \to \infty$ when $\varepsilon \to 0$ for every $t \in \mathbb{R}$ (see Figure 1).

Theorem 1 is proved in section 3.

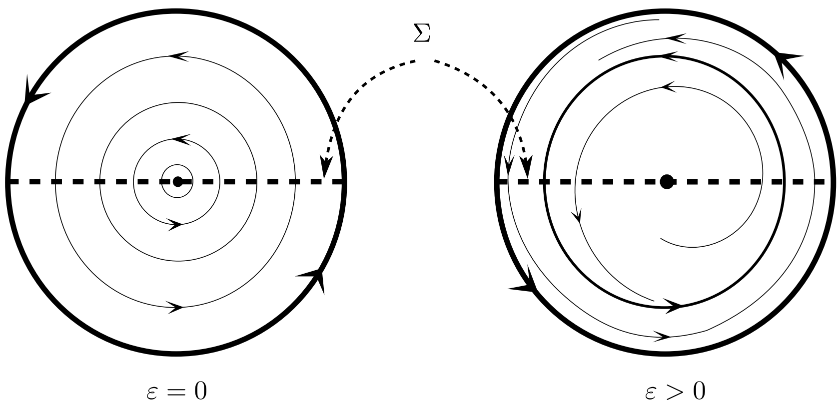

FIGURE 1. Behavior near the infinity of system (1) in the Poincaré disc for $\varepsilon = 0$ and for $\varepsilon > 0$. For a definition of Poincaré disc see for instance Chapter 5 of [7].

Taking $(\dot{x}, \dot{y}) = (-y, x + \varepsilon)$ when $y > 0$, and $(\dot{x}, \dot{y}) = (-y - \varepsilon^2 x, x - \varepsilon)$ when $y < 0$ we have a simple example of a discontinuous piecewise linear differential system for which the hypotheses of Theorem 1 hold (see Figures 2 and 3).

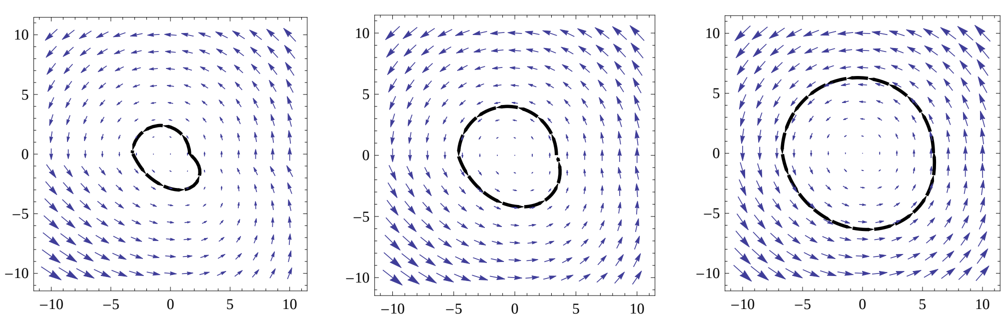

FIGURE 2. Limit cycle of $(\dot{x}, \dot{y}) = (-y, x + \varepsilon)$ for $y > 0$, and $(\dot{x}, \dot{y}) = (-y - \varepsilon^2 x, x - \varepsilon)$ for $y < 0$. The variable ε are assuming the values 0.8, 0.6 and 0.4, respectively. We observe that the amplitude of the periodic orbit becomes bigger when ε goes to 0.

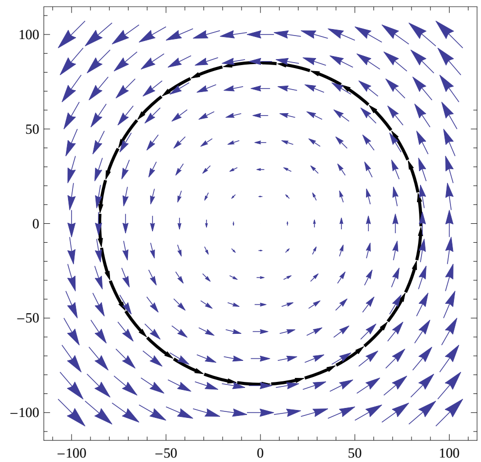

FIGURE 3. The limit cycle for $\varepsilon = 0.03$ of $(\dot{x}, \dot{y}) = (-y, x + \varepsilon)$ for $y > 0$, and $(\dot{x}, \dot{y}) = (-y - \varepsilon^2 x, x - \varepsilon)$ for $y < 0$.

In section 2 we present the basic notations and results on the Bendixson transformation and on the averaging theory for finding limit cycles for discontinuous differential systems that we shall use for proving Theorem 1.

2. Preliminary Results

2.1. **Bendixson transformation.** The study of the Poincaré map in a neighborhood of infinity for planar vector fields, when it is well defined, can be conveniently made by using the Bendixson transformation. This reduces the problem to a similar study in a neighborhood of the origin for the transformed system, see for instance Andronov and others [1]. To fix ideas, we consider the planar system,

$$\begin{aligned} \dot{x} &= f(x, y, \varepsilon), \\ \dot{y} &= g(x, y, \varepsilon), \end{aligned} \tag{2}$$

where f and g are Lipschitz functions in the variables (x, y) and $\varepsilon > 0$ is a small parameter. Through the inversion given by the Bendixson change of variables

$$\begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{x^2 + y^2} \begin{pmatrix} x \\ y \end{pmatrix},$$

we can formulate an equivalent system which behaves in a neighborhood of the origin like system (2) near infinity. Using now the polar coordinates $u = r\cos\theta$, $v = r\sin\theta$, system (2) becomes

$$\begin{aligned} \dot{r} = &\quad -r^2 \left[f\left(\frac{\cos\theta}{r}, \frac{\sin\theta}{r}, \varepsilon\right)\cos\theta + g\left(\frac{\cos\theta}{r}, \frac{\sin\theta}{r}, \varepsilon\right)\sin\theta \right], \\ \dot{\theta} = &\quad -r \left[f\left(\frac{\cos\theta}{r}, \frac{\sin\theta}{r}, \varepsilon\right)\cos\theta - g\left(\frac{\cos\theta}{r}, \frac{\sin\theta}{r}, \varepsilon\right)\cos\theta \right]. \end{aligned} \tag{3}$$

We will be interested in the flow defined in the half-cylinder $\mathbb{R}^+ \times \mathbb{S}^1 = \{(r, \theta) : r \geq 0, \theta \in [-\pi, \pi)\}$.

The Bendixson transformation composed with the polar coordinates gives rise to the following change of variables

$$x = \frac{\cos\theta}{r}, \quad y = \frac{\sin\theta}{r},$$

which we shall call by the *polar Bendixson transformation*.

System (3) has in most cases no sense for $r = 0$, but this difficulty can be normally overcome by a time reparametrization, and typically it suffices to multiply its vector field by an adequate power of r. Also note that, after extending continuously the flow to $r = 0$ if needed, the existence of a periodic orbit at infinity for system (2) is equivalent to have $r = 0$ as a periodic orbit on the cylinder for system (3). To be more precise, we assume in the sequel that system (3) can be extended to the system,

$$\dot{r} = R(r,\theta,\varepsilon), \quad \dot{\theta} = \Theta(r,\theta,\varepsilon), \tag{4}$$

where the functions R and Θ verify the following assumptions:

(A1) Both R and Θ are Lipschitz functions in the variable r and they have period 2π in the variable θ.

(A2) $R(0,\theta,\varepsilon) = 0$ and $\Theta(0,\theta,\varepsilon) \neq 0$ for all $\theta \in \mathbb{S}^1$ and for every $\varepsilon \geq 0$ sufficiently small.

Note that the last assumption implies, for $\varepsilon \geq 0$ sufficiently small, that $r = 0$ is a periodic orbit of system (4) and that it has no equilibrium points in $[0,\rho^*]\times\mathbb{S}^1$ for some $\rho^* > 0$ sufficiently small. This is a sufficient and necessary condition in order that system (2) has a periodic orbit at infinity.

Since $\Theta \neq 0$ in a neighborhood of $r = 0$ for $\varepsilon \geq 0$ sufficiently small, we can transform the differential system (4) into the first order differential equation

$$r' = \frac{dr}{d\theta} = S(r,\theta,\varepsilon) = \frac{R(r,\theta,\varepsilon)}{\Theta(r,\theta,\varepsilon)}, \tag{5}$$

where $r \in [0,\rho^*]$, $\theta \in \mathbb{S}^1$, and $\varepsilon \geq 0$ is a small parameter.

2.2. **Averaging theory.** We consider the following discontinuous differential system

$$r'(\theta) = \varepsilon S_1(\theta,r) + \varepsilon^2 S_2(\theta,r) + \varepsilon^3 T(\theta,r,\varepsilon), \tag{6}$$

with

$$S_i(\theta,r) = \begin{cases} S_i^-(\theta,r) & \text{if} \quad -\pi \leq \theta \leq 0, \\ S_i^+(\theta,r) & \text{if} \quad 0 \leq \theta \leq \pi, \end{cases} \quad \text{and}$$

$$T(\theta,r,\varepsilon) = \begin{cases} T^-(\theta,r,\varepsilon) & \text{if} \quad -\pi \leq \theta \leq 0, \\ T^+(\theta,r,\varepsilon) & \text{if} \quad 0 \leq \theta \leq \pi, \end{cases}$$

where $S_i^- : [-\pi,0]\times[0,\rho^*] \to \mathbb{R}$ and $S_i^+ : [0,\pi]\times[0,\rho^*] \to \mathbb{R}$ for $i = 1,2$ and $T^- : [-\pi,0]\times[0,\rho^*]\times[0,\varepsilon_0] \to \mathbb{R}$ and $T^+ : [0,\pi]\times[0,\rho^*]\times[0,\varepsilon_0] \to \mathbb{R}$ are analytical functions 2π–periodic in the variable θ. The discontinuous set of system (6) consists in the union of the straight lines $\Sigma_0 = \{\theta = 0,\ \rho \geq 0\}$ and $\Sigma_1 = \{\theta = \pi,\ \rho \geq 0\}$.

We observe that the analytical hypotheses of the functions involved in system (6) are not necessary for developing the theory that we shall do in what follows (see for instance [12]).

Let $r(\cdot,\rho,\varepsilon) : [-\pi,\pi] \to \mathbb{R}$ be the solution of the system (6) such that $r(0,\rho) = \rho$. Let $r^+(\cdot,\rho,\varepsilon) : [0,\pi] \to \mathbb{R}$ and $r^-(\cdot,\rho,\varepsilon) : [-\pi,0] \to \mathbb{R}$ be the solutions of the partial systems

$$r'(\theta) = \varepsilon S_1^\pm(\theta,r) + \varepsilon^2 S_2^\pm(\theta,r) + \varepsilon^2 T^\pm(\theta,r,\varepsilon),$$

respectively, such that $r^{\pm}(0,\rho,\varepsilon)=\rho$. Clearly

$$r(\theta,\rho,\varepsilon)=\begin{cases} r^{-}(\theta,\rho,\varepsilon) & \text{if} \quad -\pi\leq\theta\leq 0,\\ r^{+}(\theta,\rho,\varepsilon) & \text{if} \quad 0\leq\theta\leq\pi,\end{cases}$$

for more details see Figure 4.

Now let $\Delta:[0,\rho^*]\times[0,\varepsilon_0]\to\mathbb{R}$ be the *displacement function* defined as $\Delta(\rho,\varepsilon)=r^{+}(\pi,\rho)-r^{-}(-\pi,\rho)$ (see again Figure 4). Given $(\overline{\rho},\overline{\varepsilon})\in(0,\rho^*)\times(0,\varepsilon_0]$ it is easy to prove that the function $r(\cdot,\overline{\rho},\overline{\varepsilon})$ is a periodic solution of system (6) for $\varepsilon=\overline{\varepsilon}$, if and only if $\Delta(\overline{\rho},\overline{\varepsilon})=0$. Moreover $r(\cdot,\overline{\rho},\overline{\varepsilon})$ is a limit cycle of system (6) for $\varepsilon=\overline{\varepsilon}$, if and only if there exist a neighborhood I of $\overline{\rho}$ such that $\Delta(\rho,\overline{\varepsilon})\neq 0$ for every $\rho\in I\backslash\{\overline{\rho}\}$.

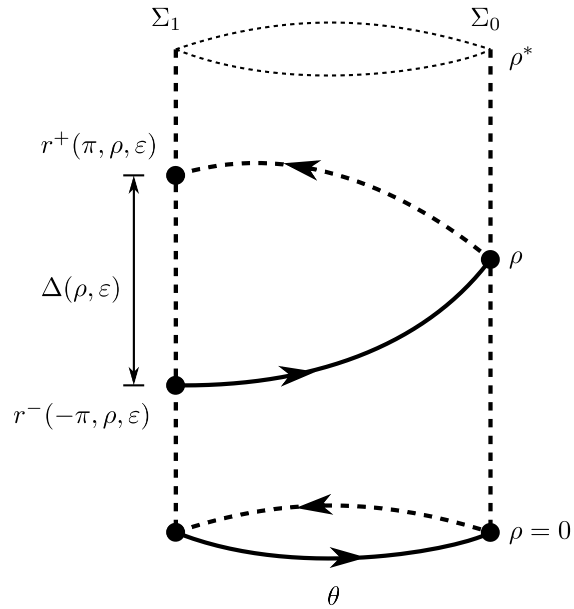

FIGURE 4. For a fixed $\varepsilon>0$ the displacement function Δ related to system (6) evaluated at a point $\rho>0$ is defined as the difference between the position of the first return to Σ_1 in forward time and the position of the first return to Σ_1 in backward time, considering the flow passing through ρ.

We define the functions $\Delta_1,\Delta_2:[0,\rho^*]\to\mathbb{R}$ as

$$\begin{aligned} \Delta_1(\rho) = & \int_0^{\pi}\left[S_1^{+}(\theta,\rho)+S_1^{-}(\theta-\pi,\rho)\right]d\theta, \quad \text{and} \\ \Delta_2(\rho) = & \int_0^{\pi}\left[S_2^{+}(\theta,\rho)+S_2^{-}(\theta-\pi,\rho)\right]d\theta \\ & +\int_0^{\pi}\left[\frac{\partial}{\partial r}S_1^{+}(\theta,\rho)r_1^{+}(\theta,\rho)+\frac{\partial}{\partial r}S_1^{-}(\theta-\pi,\rho)r_1^{-}(\theta-\pi,\rho)\right]d\theta. \end{aligned} \tag{7}$$

Here the functions $r_1^{\pm}:(-\pi,\pi)\times\mathbb{R}^+\to\mathbb{R}$ are defined as

$$r_1^{\pm}(\theta,r)=\int_0^{\theta}S_1^{\pm}(\phi,r)d\phi. \tag{8}$$

The function Δ_1 and Δ_2 are usually called *averaged functions* of order 1 and 2, respectively.

Remark 2. *Let $g : (-\delta_0, \delta_0) \to \mathbb{R}$ be a function defined on the small interval $(-\delta_0, \delta_0)$. We say that $g(\delta) = \mathcal{O}(\delta^\ell)$ for some positive integer ℓ if there exists constants $\delta_1 > 0$ and $\Lambda > 0$ such that $|g(\varepsilon)| \le \Lambda|\delta^\ell|$ for $-\delta_1 \le \delta \le \delta_1$. The symbol $\mathcal{O}$ is one of the Landau's symbols (see for instance* [17]*).*

In order to estimate the zeros of the function $\Delta(\rho,\varepsilon)$ we shall write it in terms of the functions $\Delta_1(\rho)$ and $\Delta_2(\rho)$. This is done in the next proposition.

Proposition 3. *For system* (6) *the equality $\Delta(\rho,\varepsilon) = \varepsilon\Delta_1(\rho) + \varepsilon^2\Delta_2(\rho) + \mathcal{O}(\varepsilon^3)$ holds for $\varepsilon > 0$ sufficiently small.*

Proof. By continuity of the solution $r(\theta,\rho,\varepsilon)$ of system (6) and by compactness of the set $[-\pi,\pi] \times [0,\rho^*] \times [0,\varepsilon_0]$, there exits a compact subset K of $[0,\infty)$ such that $r(\theta,\rho,\varepsilon) \in K$ for all $\theta \in [-\pi,\pi]$, $\rho \in [0,\rho^*]$ and $\varepsilon \in [0,\varepsilon_0]$. Now by the continuity of the function T, $|T(\phi, r(\phi,\rho,\varepsilon),\varepsilon)| \le \max\{|T(\theta,\rho,\varepsilon)| : (\theta,\rho,\varepsilon) \in [-\pi,\pi] \times [0,\rho^*] \times [0,\varepsilon_0]\} = M$. Then

$$\left|\int_{-\pi}^{\theta} T(\phi, r(\phi,\rho,\varepsilon),\varepsilon)d\phi\right| \le \int_{-\pi}^{\pi} |T(\phi, r(\phi,\rho,\varepsilon),\varepsilon)|\, d\phi \le 2\pi M,$$

which implies that $\displaystyle\int_{-\pi}^{\theta} T(\phi, r(\phi,\rho,\varepsilon),\varepsilon)d\phi = \mathcal{O}(1)$. So

$$r^\pm(\theta,\rho,\varepsilon) = \rho + \varepsilon\int_0^\theta S_1^\pm(\phi, r^\pm(\phi,\rho,\varepsilon))d\phi + \varepsilon^2\int_0^\theta S_2^\pm(\phi, r^+(\phi,\rho,\varepsilon),\varepsilon)d\phi + \mathcal{O}(\varepsilon^3).$$

Expanding the functions $\varepsilon \mapsto S_i^\pm(\phi, r^\pm(\phi,\rho,\varepsilon))$ around $\varepsilon = 0$ for $i = 1, 2$ we have

$$r^\pm(\theta,\rho,\varepsilon) = \quad \rho + \varepsilon\int_0^\theta S_1^\pm(\phi,\rho)d\phi + \varepsilon^2\int_0^\theta \left[\frac{\partial}{\partial r}S_1^\pm(\phi,\rho)r_1^\pm(\phi,r) + S_2^\pm(\phi,\rho)\right]d\phi + \mathcal{O}(\varepsilon^3),$$

because

$$\frac{\partial}{\partial\varepsilon}r^\pm(\theta,\rho,\varepsilon)\Big|_{\varepsilon=0} = \int_0^\theta S_1^\pm(\phi,\rho)d\phi = r_1^\pm(\theta,r).$$

So we compute

$$\begin{aligned}
\Delta(\rho,\varepsilon) =\ & r^+(\pi,\rho,\varepsilon) - r^-(-\pi,\rho,\varepsilon)\\
=\ & \varepsilon\left(\int_0^\pi S_1^+(\theta,\rho)d\theta - \int_0^{-\pi} S_1^-(\theta,\rho)d\theta\right) + \varepsilon^2\left(\int_0^\pi \left[\frac{\partial}{\partial r}S_1^\pm(\theta,\rho)r_1^\pm(\theta,r) + S_2^\pm(\theta,\rho)\right]d\theta\right.\\
& \left. - \int_0^{-\pi}\left[\frac{\partial}{\partial r}S_1^\pm(\theta,\rho)r_1^\pm(\theta,r) + S_2^\pm(\theta,\rho,\varepsilon)\right]d\theta\right) + \mathcal{O}(\varepsilon^3)\\
=\ & \varepsilon\left(\int_0^\pi S_1^+(\theta,\rho)d\theta + \int_{-\pi}^{0} S_1^-(\theta,\rho)d\theta\right) + \varepsilon^2\left(\int_0^\pi \left[\frac{\partial}{\partial r}S_1^\pm(\theta,\rho)r_1^\pm(\theta,r) + S_2^\pm(\theta,\rho)\right]d\theta\right.\\
& \left. + \int_{-\pi}^{0}\left[\frac{\partial}{\partial r}S_1^\pm(\theta,\rho)r_1^\pm(\theta,r) + S_2^\pm(\theta,\rho,\varepsilon)\right]d\theta\right) + \mathcal{O}(\varepsilon^3)\\
=\ & \varepsilon\left(\int_0^\pi \left[S_1^+(\theta,\rho) + S_1^-(\theta-\pi,\rho)\right]d\theta\right) + \varepsilon^2\left(\int_0^\pi \left[S_2^\pm(\theta,\rho) + S_2^\pm(\theta-\pi,\rho,\varepsilon)\right.\right.\\
& \left.\left. + \frac{\partial}{\partial r}S_1^\pm(\theta,\rho)r_1^\pm(\theta,r) + \frac{\partial}{\partial r}S_1^\pm(\theta-\pi,\rho)r_1^\pm(\theta-\pi,r)\right]d\theta\right) + \mathcal{O}(\varepsilon^3)\\
=\ & \varepsilon\Delta_1(\rho) + \varepsilon^2\Delta_2(\rho) + \mathcal{O}(\varepsilon^3).
\end{aligned}$$

This completes the proof of the proposition. □

In the proof of Theorem 1 we shall use the following lemma.

Lemma 4. *Let a, b and c be real numbers and let $f_\varepsilon(\rho) = \varepsilon a\rho^2 + \varepsilon^2\left(b\rho^2 + c\rho\right) + \mathcal{O}(\varepsilon^3)$. If $f_\varepsilon(0) = 0$ for $\varepsilon \geq 0$ sufficiently small and $a\,c < 0$, then there exists $\varepsilon_0 > 0$ and a curve $\rho(\varepsilon)$ of zeros of the function f_ε, i.e. $f_\varepsilon(\rho(\varepsilon)) = 0$ for all $\varepsilon \in (0, \varepsilon_0]$ with $\rho(\varepsilon) \neq 0$ for every $\varepsilon \in (0, \varepsilon_0]$ such that $\rho(\varepsilon) \to 0$ when $\varepsilon \to 0$. Moreover for each $\varepsilon \in (0, \varepsilon_0]$, $\rho(\varepsilon)$ is an isolated zero of f_ε.*

Proof. From hypothesis $a\,c < 0$, thus for simplicity we assume that $a < 0$ and $c > 0$. For $a > 0$ and $c < 0$ the proof would follow analogously.

Computing the first derivative of f_ε we obtain $f_\varepsilon'(\rho) = 2\varepsilon a\rho + \varepsilon^2(2b\rho + c) + \mathcal{O}(\varepsilon^3)$. Since $f_\varepsilon'(0) = \varepsilon^2 c + \mathcal{O}(\varepsilon^3)$, there exists a small parameter $\varepsilon_1 > 0$ such that $f_\varepsilon'(0) > 0$ for all $\varepsilon \in (0, \varepsilon_1]$.

From hypotheses there exists another small parameter $\varepsilon_2 > 0$ such that $f_\varepsilon(0) = 0$ for all $\varepsilon \in (0, \varepsilon_2]$. Taking $\varepsilon_3 = \min\{\varepsilon_1, \varepsilon_2\}$ we conclude that $f_\varepsilon(0) = 0$ and $f_\varepsilon'(0) > 0$ for all $\varepsilon \in (0, \varepsilon_3]$. Therefore, for each $\varepsilon \in (0, \varepsilon_3]$ there exists $\rho_1(\varepsilon) > 0$ such that $f_\varepsilon(\rho) > 0$, for all $\rho \in (0, \rho_1(\varepsilon)]$.

For a fixed positive real number ρ_2 we have that $f_\varepsilon(\rho_2) = \varepsilon a\rho_2^2 + \mathcal{O}(\varepsilon^2)$. So there exists a small parameter $\varepsilon_4 > 0$ such that $f_\varepsilon(\rho_2) < 0$ for all $\varepsilon \in (0, \varepsilon_4]$. Set $\varepsilon_5 = \min\{\varepsilon_3,\, \varepsilon_4\}$. Clearly $\rho_2 > \rho_1(\varepsilon)$ for every $\varepsilon \in (0, \varepsilon_5]$. Thus, by the Intermediate Value Theorem we obtain that for all $\varepsilon \in (0, \varepsilon_5]$ there exists $\rho(\varepsilon) \in (\rho_1(\varepsilon), \rho_2)$ such that $f_\varepsilon(\rho(\varepsilon)) = 0$.

Since $f_\varepsilon(\rho(\varepsilon)) = \varepsilon a\rho(\varepsilon)^2 + \varepsilon^2\left(b\rho(\varepsilon)^2 + c\rho(\varepsilon)\right) + \mathcal{O}(\varepsilon^3) = 0$, we have that $(a + \varepsilon b)\rho(\varepsilon)^2 + \varepsilon c\rho(\varepsilon) = \mathcal{O}(\varepsilon^2)$. From Remark 2 there exist $\varepsilon_6 > 0$ and $\Lambda > 0$ such that $|(a+\varepsilon b)\rho(\varepsilon)^2 + \varepsilon c\rho(\varepsilon)| \leq \Lambda|\varepsilon^2|$ for $\varepsilon \in (0, \varepsilon_6]$. Moreover taking ε_7 sufficiently small we have that $a + \varepsilon b < 0$ for every $\varepsilon \in (0, \varepsilon_7]$. In particular, taking $\varepsilon_0 = \min\{\varepsilon_5,\, \varepsilon_6,\, \varepsilon_7\}$, the inequality $-(a+\varepsilon b)\rho(\varepsilon)^2 - \varepsilon c\rho(\varepsilon) < \Lambda\,\varepsilon^2$ holds for $\varepsilon \in (0, \varepsilon_0]$. So

$$\varepsilon\left(\frac{-c + \sqrt{c^2 - 4a\Lambda}}{2a}\right) + \mathcal{O}(\varepsilon^2) < \rho(\varepsilon) < -\varepsilon\left(\frac{c + \sqrt{c^2 + 4a\Lambda}}{2a}\right) + \mathcal{O}(\varepsilon^2).$$

We conclude then $\rho(\varepsilon) \to 0$ when $\varepsilon \to 0$. This completes the proof of the theorem.

□

3. Proof of Theorem 1

In this section using the tools presented in section 2 we shall prove our main result.

Proof of Theorem 1. Applying the polar Bendixson transformation (**??**) to system (1) we compute $(R, \Theta) = (R^+, \Theta^+)$ for $y > 0$, and $(R, \Theta) = (R^-, \Theta^-)$ for $y < 0$, as

$$\begin{aligned} R^\pm(0,\theta,\varepsilon) = \quad & \varepsilon\Big(r^2\left(-a^\pm\cos\theta - k^\pm\sin\theta\right) + \frac{1}{2}r\left(-\left(c^\pm + m^\pm\right)\sin(2\theta)\right. \\ & \left. + \left(n^\pm - b^\pm\right)\cos(2\theta) - b^\pm - n^\pm\right)\Big) + \varepsilon^2\left(r^2\left(-\alpha^\pm\cos\theta\right.\right. \\ & \left. -\kappa^\pm\sin\theta\right) + \frac{1}{2}r\left(-\left(\gamma^\pm + \mu^\pm\right)\sin(2\theta) + \left(\nu^\pm - \beta^\pm\right)\cos(2\theta)\right. \\ & \left.\left. -\beta^\pm - \nu^\pm\right)\Big), \quad \text{and} \\[1em] \Theta^\pm(0,\theta,\varepsilon) = \quad & 1 + \varepsilon\cos\theta\Big(r\left(k^\pm - a^\pm\tan\theta\right) - \sin\theta\left(c^\pm\tan\theta + b^\pm - n^\pm\right) + m^\pm\cos\theta\Big) \\ & + \varepsilon^2\cos\theta\Big(r\left(\kappa^\pm - \alpha^\pm\tan\theta\right) - \sin\theta\left(\gamma^\pm\tan\theta + \beta^\pm - \nu^\pm\right) + \mu^\pm\cos\theta\Big). \end{aligned}$$

It is easy to see that the assumptions $(A1)$ and $(A2)$ hold for the functions $R^{\pm}$, $\Theta^{\pm}$. Thus $r=0$ is a periodic solution for both systems $\dot r = S^+(\theta,r,\varepsilon)$ and $\dot r = S^-(\theta,r,\varepsilon)$, so $r=0$ is also a periodic solution of the discontinuous piecewise system

$$\dot r = \begin{cases} S^-(\theta,r,\varepsilon) & \text{if} \quad -\pi \le \theta \le 0, \\ S^+(\theta,r,\varepsilon) & \text{if} \quad 0 \le \theta \le \pi, \end{cases} \tag{9}$$

which is equivalent to system (1).

Computing now the Taylor series of $S^{\pm}(\theta,r,\varepsilon)$ near $\varepsilon=0$ we have that system (1) is equivalent to system (6) by taking

$$\begin{aligned} S_1^{\pm}(\theta,r) = & -r\left(b^{\pm}\cos^2\theta + (c^{\pm}+m^{\pm})\cos\theta\sin\theta + n^{\pm}\sin^2\theta\right) \\ & -r^2\left(a^{\pm}\cos\theta + k^{\pm}\sin\theta\right), \\ S_2^{\pm}(\theta,r) = & \; r\sec^2\theta\left(\cos^4\theta\left(r\left(k^{\pm}-a^{\pm}\tan\theta\right) - \sin\theta\left(c^{\pm}\tan\theta + b^{\pm} - n^{\pm}\right) + m^{\pm}\cos\theta\right)\right. \\ & \cdot\left(r\left(a^{\pm}+k^{\pm}\tan\theta\right) + \sin\theta\left(c^{\pm}+n^{\pm}\tan\theta + m^{\pm}\right) + b^{\pm}\cos\theta\right) \\ & -\cos^2\theta\left(\cos\theta\left(\alpha^{\pm}r + (\gamma^{\pm}+\mu^{\pm})\sin\theta\right) + \beta^{\pm}\cos^2\theta + \sin\theta\left(\kappa^{\pm}r\right.\right. \\ & \left.\left.\left. +\nu^{\pm}\sin\theta\right)\right)\right). \end{aligned}$$

Let $f_\varepsilon(\rho) = \Delta(\varepsilon,\rho)$. Since $r=0$ is a periodic solution of system (9) we have that $f_\varepsilon(0) = \Delta(0,\varepsilon) = 0$ for every $\varepsilon \in (0,\varepsilon_1)$. From Proposition 3 and using the hypothesis $b^+ + b^- + n^+ + n^- = 0$ we compute $\Delta(\varepsilon,\rho) = \varepsilon\Delta_1(\rho) + \varepsilon^2\Delta_2(\rho) + \mathcal{O}(\varepsilon^2)$, with $\Delta_1(\rho) = a\rho^2$ and $\Delta_2(\rho) = b\rho^3 + c\rho^2 + d\rho$, where

$$\begin{aligned} a = & \; -4k^+, \\ b = & \; (b^+ + n^+)\left(3\pi k^+ - 2\left(a^- + a^+\right)\right) + 2\left(k^+\left(m^- + m^+\right) + \kappa^- - \kappa^+\right), \quad \text{and} \\ c = & \; -\frac{\pi}{4}\left(-\left(b^+ + n^+\right)\left(c^- - c^+ - m^- + m^+\right) + 2\beta^- + 2\left(\beta^+ + \nu^- + \nu^+\right)\right). \end{aligned}$$

From hypotheses we have

$$a\,c = \pi k^+\left(2\beta^- + 2\left(\beta^+ + \nu^- + \nu^+\right) - \left(b^+ + n^+\right)\left(c^- - c^+ - m^- + m^+\right)\right) < 0.$$

Hence applying Lemma 4 we conclude that, for $\varepsilon > 0$ sufficiently small, there exists a periodic solution $\big(x(t,\varepsilon), y(t,\varepsilon)\big)$ of system (1) such that $\left|\big(x(0,\varepsilon), y(0,\varepsilon)\big)\right| = 1/\rho(\varepsilon) \to \infty$ when $\varepsilon \to 0$. Moreover this periodic solution is a limit cycle. So this completes the proof of the theorem. □

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