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The Period Adding and Incrementing Bifurcations: From Rotation Theory to Applications*

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Abstract. This survey article is concerned with the study of bifurcations of discontinuous piecewise-smooth maps, with a special focus on the one-dimensional case. We review the literature on circle maps and quasi-contractions and provide paths through this literature to prove sufficient conditions for the occurrence of two types of bifurcation scenarios involving rich dynamics. The first scenario consists of the appearance of periodic orbits whose symbolic sequences and "rotation" numbers follow a Farey tree structure; the periods of the periodic orbits are given by consecutive addition. This is called the *period adding* bifurcation, and the proof of its existence relies on results for maps on the circle. In the second scenario, symbolic sequences are obtained by consecutive attachment of a given symbolic block, and the periods of periodic orbits are incremented by a constant term. This is called the *period incrementing* bifurcation, and its proof relies on results for maps on the interval. We also discuss the expanding cases, as some of the partial results found in the literature also hold when these maps lose contractiveness. The higher-dimensional case is also discussed by means of *quasi-contractions*. We provide applied examples in control theory, power electronics, and neuroscience, where these results can be used to obtain precise descriptions of their dynamics.

Key words. piecewise-smooth maps, discontinuous circle maps, period adding, devil's staircase, Farey tree, period incrementing

AMS subject classifications. 37E05, 37E10, 37G15, 37N25, 37N35

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