DIVERGENCE AND POINCARÉ–LIAPUNOV CONSTANTS FOR ANALYTIC DIFFERENTIAL SYSTEMS

MAITE GRAU[1] AND JAUME LLIBRE[2]

Abstract. We consider a planar autonomous real analytic differential system with a monodromic singular point p. We deal with the center problem for the singular point p. Our aim is to highlight some relations between the divergence of the system and the Poincaré–Liapunov constants of p when these are defined.

1. Introduction and statement of the main results

Let O be the origin of coordinates of $\mathbb{R}^2$ and let $\mathcal{U}_O$ be a neighborhood of O. We consider two real analytic functions $P(x,y)$ and $Q(x,y)$ in $\mathcal{U}_O$ which vanish at O. In this work we deal with the analytic differential systems of the form

$$\dot{x} = P(x,y), \quad \dot{y} = Q(x,y), \tag{1}$$

where the dot denotes derivative with respect to an independent real variable t.

When all the orbits of system (1) in a punctured neighborhood of the singular point O are periodic, we say that the origin is a *center*. If the orbits of system (1) in a punctured neighborhood of O spiral to O when $t \to +\infty$ or $t \to -\infty$, then we say that the origin is a *focus*. In the first case ($t \to +\infty$), we say that it is *stable* and in the second case ($t \to -\infty$), we say that it is *unstable*. If the origin is either a focus or a center, we say that it is a *monodromic* singular point. The *center problem* consists in distinguishing when a monodromic singular point is either a center or a focus. *In the sequel we assume that the origin of system* (1) *is monodromic.*

As usual we define the *divergence of system* (1), and we denote it by $\mathrm{div}(x,y)$, as the function

$$\mathrm{div}(x,y) = \frac{\partial P}{\partial x}(x,y) + \frac{\partial Q}{\partial y}(x,y).$$

System (1) is said to be *Hamiltonian* if $\mathrm{div}(x,y) \equiv 0$. In such a case there exists a neighborhood of the origin $\mathcal{U}_O$ and an analytic function $H : \mathcal{U}_O \subseteq$

1991 *Mathematics Subject Classification.* 34C25, 37G10, 34A34, 34C05.

Key words and phrases. center problem, Poincaré–Liapunov constants, divergence, Hamiltonian.

$\mathbb{R}^2 \to \mathbb{R}$, called the Hamiltonian, such that

$$P(x,y) \;=\; -\frac{\partial H}{\partial y} \quad \text{and} \quad Q(x,y) \;=\; \frac{\partial H}{\partial y}.$$

We note that the level curves of H are formed by orbits of system (1). A Hamiltonian system (1) with a monodromic singular point at O necessarily has a center at the origin because an analytic function cannot contain a spiral as level curve (unless the analytic function be constant).

Our aim is to highlight some other results relating the divergence of system (1) with the solution of the center problem.

Given a real analytic function $f : \mathcal{U}_O \subseteq \mathbb{R}^2 \to \mathbb{R}$, where $\mathcal{U}_O$ is a neighborhood of the origin $O = (0,0)$, we consider its Taylor expansion at O:

$$f(x,y) \;=\; f_d(x,y) \;+\; \mathcal{O}_{d+1}(x,y),$$

where $d \geq 0$ is an integer and $f_d(x,y)$ is a non–zero homogeneous polynomial of degree d. We say that f is of *sign definite* if $f_d(x,y) \geq 0$ or $f_d(x,y) \leq 0$ for all $(x,y) \in \mathbb{R}^2$. When $f_d(x,y) \geq 0$ (resp. $f_d(x,y) \leq 0$) for all $(x,y) \in \mathbb{R}^2$ we say that f is *positive definite* (resp. *negative definite*). It is clear that a necessary condition for $f(x,y)$ to be of sign definite is that d is even.

Our first result is the following one.

Proposition 1. *Assume that the origin of an analytic differential system* (1) *is a monodromic singular point. If the divergence* $\mathrm{div}(x,y)$ *of system* (1) *is of sign definite, then the origin of system* (1) *is a focus; either unstable if the divergence is positive definite or stable if it is negative definite.*

This result is proved in section 2. We remark that in the case that the origin of system (1) is a strong focus, then the divergence $\mathrm{div}(0,0) \neq 0$ and the stability of the focus is given by the sign of the number $\mathrm{div}(0,0)$. The previous proposition is a generalization of this fact for any monodromic singular point. See for instance Theorem 2.15 of [6], or [8], for the definitions of these classical concepts.

Assume that the origin of system (1) is a monodromic singular point, but not a strong focus. It is well-known that, after a linear change of variables and a constant scaling of the time variable (if necessary), the system can be written in one of the following three forms:

$$\begin{array}{lll} \dot{x} \;=\; -y + F_1(x,y), & \dot{y} \;=\; x + F_2(x,y), \\ \dot{x} \;=\; y + F_1(x,y), & \dot{y} \;=\; F_2(x,y), \\ \dot{x} \;=\; F_1(x,y), & \dot{y} \;=\; F_2(x,y), \end{array} \tag{2}$$

where $F_1(x,y)$ and $F_2(x,y)$ are real analytic functions without constant and linear terms defined in a neighborhood of the origin. In what follows the origin of an analytic differential system (1) is called *linear type*, *nilpotent* or *degenerate* if after an affine change of variables and a scaling of the time it can be written as the first, second and third system of (2), respectively.

There are several tools for determining whether the origin of system (1) is monodromic. In the case that the linear part of system (1) has two complex conjugate eigenvalues with non-zero imaginary part, we have that the origin is monodromic and it can be written in the linear type form (first system of (2)). Any other configuration of non-zero eigenvalues implies that the origin is not monodromic, see for more details [6]. If the linear part of system (1) is not identically zero and has two zero eigenvalues, then we can decide when the origin is a monodromic singular point by the Andreev's theorem (see [4], or Theorem 3.5 of [6]), and the system can be written in the nilpotent form (second system of (2)). The statement of Andreev's result is given in Theorem 7. Finally, when the linear part of system (1) is identically zero, then the system writes in the degenerate form (third system of (2)). Using the blow-up technique one can determine whether the origin is monodromic, see for instance Chapter 3 of [6]. For system (1) let $P(x,y) = P_n(x,y) + \mathcal{O}_{n+1}(x,y)$ and $Q(x,y) = Q_m(x,y) + \mathcal{O}_{m+1}(x,y)$, where $n \geq 1$ and $m \geq 1$ are integers and $P_n(x,y)$ and $Q_m(x,y)$ are non–zero homogeneous polynomials of degrees n and m respectively, formed by the lowest order terms of $P(x,y)$ and $Q(x,y)$, respectively. Define the real polynomial

$$(3) \qquad \Delta(x,y) = \begin{cases} yP_n(x,y) - xQ_m(x,y) & \text{if } n = m, \\ yP_n(x,y) & \text{if } n < m, \\ -xQ_m(x,y) & \text{if } n > m. \end{cases}$$

A sufficient condition for a system in the degenerate form (third system of (2)) to have a monodromic singular point at the origin is that $\Delta(x,y) = 0$ only if $(x,y) = (0,0)$. In this case we say that the origin has *no characteristic directions*. A necessary condition in order that a degenerate system (third system of (2)) has a monodromic singular point at the origin is that $\Delta(x,y)$ is of sign definite.

Let Σ be an analytic transversal section at O, that is, an analytic arc transverse to the flow of the system such that $O \in \partial\Sigma$, the boundary of Σ. We consider a parameter ρ of Σ such that $\rho = 0$ corresponds to the origin of coordinates and Σ is parameterized by the interval $(0, \rho^*)$ with $\rho^* > 0$. Given a point ρ in Σ we consider the orbit of system (1) with ρ as initial condition. Due to the fact that the origin is monodromic, if ρ is close enough to O and we follow the orbit for positive values of the time t, it will cut Σ again at some point. We define the *Poincaré map* $\mathcal{P} : \Sigma \to \Sigma$ being $\mathcal{P}(\rho)$ the point in Σ corresponding to the first cut with Σ of the orbit through ρ in positive time. It is clear that the origin of system (1) is a center if and only if the Poincaré map is the identity. For systems with the linear type form (first system of (2)) and for systems with the nilpotent form (second system of (2)), it is possible to find a parametrization of Σ such that the

Poincaré map is analytic in $\rho = 0$ and writes as

$$\mathcal{P}(\rho) = \rho + \sum_{i=1}^{\infty} \alpha_i \, \rho^i,$$

where α_i are algebraic expressions in the coefficients of F_1 and F_2. There are systems with the degenerate form (third system of (2)) for which such a parametrization is also possible, for instance the ones which do not have characteristic directions, see for instance [8].

The stability of the origin is clearly given by the sign of the first non-zero α_i (that is, it is unstable if $\alpha_i > 0$, and stable if $\alpha_i < 0$), and if all the α_i are zero then the origin is a center. Indeed the even–indexed terms α_{2k} are algebraic expressions of the previous α_i. Therefore the interesting expressions are the ones with odd index, i.e. the α_{2k+1}'s. Moreover we define the $(k+1)$-th *Poincaré–Liapunov constant* as the expression α_{2k+1} modulus the vanishing of all the previous ones. For a system in the degenerate form (third system of (2)) we cannot ensure to have a good parametrization of Σ such that the Poincaré map can be expanded as Taylor series in a neighborhood of $\rho = 0$. In case that we have it, we can define the Poincaré–Liapunov constants analogously.

Our first main result deals with the analytic differential systems (1) with linear type form (first system of (2)).

Theorem 2. *Consider an analytic differential system* (1) *whose origin is of linear type. Denote by* $\mathrm{div}_d(x, y)$ *the lowest order terms of the divergence* $\mathrm{div}(x, y)$ *of the system. Assume that*

$$\alpha_{d+1} = \frac{1}{d+2} \int_0^{2\pi} \mathrm{div}_d \left(\cos t, \sin t\right) \, dt \neq 0. \tag{4}$$

Then the origin is a focus whose first non–zero Poincaré–Liapunov constant is α_{d+1}.

This result is proved in section 2. We remark that in case that $\mathrm{div}_d(x, y)$ is just one monomial of sign definite, the integral (4) does not need to be computed because α_{d+1} is the coefficient of this monomial times a positive constant. We also remark that Theorem 2 can only give a nonzero α_{d+1} when system (1) is of linear type plus nonlinearities starting from terms of odd degree, that is d is even in the notation of Theorem 2.

The following result follows easily from Theorem 2.

Corollary 3. *Consider the system*

$$\dot{x} = -y + P_s(x, y), \quad \dot{y} = x + Q_s(x, y), \tag{5}$$

where $P_s(x, y)$ *and* $Q_s(x, y)$ *are homogeneous polynomials of odd degree* s. *The first Poincaré–Liapunov constants of system* (5) *are* $\alpha_i = 0$ *for* $i = 1, 2, \ldots, s-1$ *and*

$$\alpha_s = \frac{1}{s+1} \int_0^{2\pi} \mathrm{div}(\cos t, \sin t) \, dt.$$

For instance, as it was proved by Sibirsky [15], see also [14] and the references therein, by an affine change of coordinates, any cubic system of the form (5), i.e. $s = 3$, can be written

$$
\begin{aligned}
\dot{x} &= -y-(\omega+\theta-a)x^3-(\eta-3\mu)x^2y-(3\omega-3\theta+2a-\xi)xy^2 \\
&\quad -(\mu-\nu)y^3, \\
\dot{y} &= x+(\mu+\nu)x^3+(3\omega+3\theta+2a)x^2y+(\eta-3\mu)xy^2 \\
&\quad +(\omega-\theta-a)y^3,
\end{aligned} \tag{6}
$$

where ω, θ, a, η, μ, ξ, ν are real parameters. It can be shown that this family has the following set of Poincaré–Liapunov constants (except by the product of positive constants):

$$\alpha_3 = \xi,\ \alpha_5 = \nu a,\ \alpha_7 = \omega\theta a,\ \alpha_9 = \theta a^2\eta,\ \alpha_{11} = \theta\left[4(\mu^2+\theta^2)-a^2\right]a^2.$$

The divergence of system (6) gives $\operatorname{div}(x,y) = 5a(x^2-y^2)+\xi y^2$ and as a consequence of Corollary 3 we also obtain that $\alpha_3 = \pi\xi/4$.

The second main result deals for the analytic differential systems (1) in the nilpotent form (second system of (2)).

Theorem 4. *Consider an analytic differential system* (2) *whose origin is a nilpotent singular point. Denote by* $\operatorname{div}_d(x,y)$ *the lowest order terms of the divergence* $\operatorname{div}(x,y)$ *of the system. Define*

$$V_{d+1}(\varepsilon) = \int_0^{2\pi/\sqrt{\varepsilon}} \operatorname{div}_d\left(\cos(\sqrt{\varepsilon}\,t),\ -\sqrt{\varepsilon}\sin(\sqrt{\varepsilon}\,t)\right)dt, \tag{7}$$

where $\varepsilon > 0$ *and define the constant* v_{d+1} *by the development* $V_{d+1}(\varepsilon) = \frac{v_{d+1}}{\sqrt{\varepsilon}} + \mathcal{O}(\varepsilon)$.

(a) *If the origin is a center, then* $v_{d+1} = 0$.
(b) *If* $v_{d+1} > 0$ (*resp.* $v_{d+1} < 0$), *then the origin is an unstable* (*resp. stable*) *focus.*

This result is proved in section 2.

The following examples are applications of Theorem 4. Next system is a particular case of Example 3 (p. 542) of [11]. We reobtain the result in a shorter way.

Example 5. *The nilpotent singular point of the system*

$$\dot{x} = y, \quad \dot{y} = -x^3+cy^3+bx^3y, \tag{8}$$

where $b, c \in \mathbb{R}$, *is a center if and only if* $c = 0$. *If* $c > 0$ (*resp.* $c < 0$), *the origin is an unstable* (*resp. stable*) *focus.*

The proof of this example is given in section 2.

Next system is Example 4 (p. 543) of [11]. We obtain the first necessary condition in order to have a center.

Example 6. *Consider the nilpotent singular point at the origin of the system*

$$\dot{x} = y + Ax^2y + Bxy^2 + Cy^3, \quad \dot{y} = -x^3 + Px^2y + Kxy^2 + Ly^3, \tag{9}$$

where $A, B, C, P, K, L \in \mathbb{R}$. *If* $P > 0$ (*resp.* $P < 0$), *then the origin of system* (9) *is an unstable* (*resp. stable*) *focus.*

The proof of this example is given in section 2. In [11] the authors show that system (9) has a center at the origin if and only if $P = 0$, $B + 3L = 0$ and $L(A + K) = 0$. We get the first necessary condition as a consequence of Theorem 4.

The stability of the origin of a system (2) in nilpotent form has been studied in [2, 3]. See also the references therein. We provide a relationship between the stability of the origin and the divergence of the system. In order to state our next result we recall the definition of Andreev number as it was used in [8]. This definition is related to the singular point at the origin of an analytic differential system in nilpotent form (second system of (2)). The following theorem is due to Andreev [4] and it solves the monodromy problem for the origin of this system.

Theorem 7. [4] *Let* $y = F(x)$ *be the solution of* $y + F_1(x, y) = 0$ *passing through* $(0, 0)$. *Define the functions* $f(x) = F_2(x, F(x)) = ax^\alpha + \mathcal{O}(x^{\alpha+1})$ *with* $a \neq 0$ *and* $\alpha \geq 2$ *and* $\phi(x) = (\partial F_1/\partial x + \partial F_2/\partial y)(x, F(x))$. *We have that either* $\phi(x) = bx^\beta + \mathcal{O}(x^{\beta+1})$ *with* $b \neq 0$ *and* $\beta \geq 1$ *or* $\phi(x) \equiv 0$. *Then, the origin of the system* (2) *in nilpotent form is monodromic if, and only if,* $a < 0$, $\alpha = 2n - 1$ *is an odd integer and one of the following conditions holds:*

(i) $\beta > n - 1$.
(ii) $\beta = n - 1$ *and* $b^2 + 4an < 0$.
(iii) $\phi(x) \equiv 0$.

Definition 8. *We consider a system* (2) *in the nilpotent form with the origin as a monodromic singular point. We define its* Andreev number $n \geq 2$ *as the corresponding integer value given in Theorem* 7.

Also as it appears in [8], we consider system (2) in nilpotent form and we assume that the origin is a nilpotent monodromic singular point with Andreev number n. Then, the change of variables

$$(x, y) \mapsto (x, y - F(x)),$$

where $F(x)$ is defined in Theorem 7, and the scaling

$$(x, y) \mapsto (\xi\, x, -\xi\, y),$$

with $\xi = (-1/a)^{1/(2-2n)}$, brings system (2) in nilpotent form into the following analytic form for monodromic nilpotent singularities

$$\dot{x} = y\,(-1 + X_1(x, y)), \quad \dot{y} = f(x) + y\,\phi(x) + y^2\, Y_0(x, y), \tag{10}$$

where $X_1(0, 0) = 0$, $f(x) = x^{2n-1} + \mathcal{O}(x^{2n})$ with $n \geq 2$ and either $\phi(x) \equiv 0$ or $\phi(x) = bx^\beta + \mathcal{O}(x^{\beta+1})$ with $\beta \geq n - 1$. We remark that we have relabeled the functions $f(x)$, $\phi(x)$ and the constant b with respect to the ones corresponding to system (2). We recall, cf. Theorem 7, that when $\beta = n-1$ we also have that $b^2 - 4n < 0$.

We say that a real polynomial $p(x,y)$ is $(1,n)$-*quasi homogeneous of weight degree* w if $p(\lambda x, \lambda^n y) = \lambda^w p(x,y)$ for all $(x,y) \in \mathbb{R}^2$ and for all $\lambda \in \mathbb{R}$. Consider system (10) with Andreev number n and let $\operatorname{div}(x,y)$ be its divergence. We define $\operatorname{div}_d^{(1,n)}(x,y)$ as the $(1,n)$-quasi homogeneous terms of div with the lowest weight degree d. The following results deals with a differential system (10) except in the case that the function $\phi(x)$ defined in Theorem 7 is $\phi(x) = bx^{n-1} + \mathcal{O}(x^n)$ with $b \neq 0$. The stability of the origin in the former case is provided in [2, 3].

Theorem 9. *Consider an analytic differential system* (10) *with Andreev number n and assume that the function $\phi(x)$ defined in Theorem* 7 *is either $\phi(x) \equiv 0$ or $\phi(x) = bx^\beta + \mathcal{O}(x^{\beta+1})$ with $b \neq 0$ and $\beta > n-1$. Denote by $u(\theta) = \sqrt[2n]{\cos^{2n}\theta + n\sin^2\theta}$ and by $\operatorname{div}_d^{(1,n)}(x,y)$ the $(1,n)$-quasi homogeneous terms of lowest order d of the divergence $\operatorname{div}(x,y)$ of the system. Assume that*

$$\alpha = \int_0^{2\pi} \operatorname{div}_d^{(1,n)}(\cos\theta, \sin\theta)\, \frac{\cos^2\theta + n\sin^2\theta}{u(\theta)^{d+n+1}}\, d\theta \neq 0. \tag{11}$$

Then the origin is a focus which is stable (*resp. unstable*) *if $\alpha < 0$* (*resp. $\alpha > 0$*).

This result is proved in section 2. We consider system (10) instead of directly system (2) in nilpotent form because we will need that the weighted blow-up $x = r\cos\theta$, $y = r^n \sin\theta$ takes the system to one with $\dot{\theta} = \vartheta(\theta) r^{n-1} + \mathcal{O}(r^n)$ with $\vartheta(\theta)$ a trigonometric rational function with $\vartheta(\theta) > 0$ for all $\theta \in [0, 2\pi)$. An analogous result could be stated for the original system (2) in nilpotent form if this condition is satisfied.

The following example is a particular case of the system studied in Theorem D of [2].

Example 10. *Consider the nilpotent singular point at the origin of the system*

$$\dot{x} = -y + a_{02}y^2 + a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3, \quad \dot{y} = x^3, \tag{12}$$

where $a_{ij} \in \mathbb{R}$. If $a_{30} > 0$ (*resp. $a_{30} < 0$*)*, then the origin of system* (12) *is an unstable* (*resp. stable*) *focus.*

The proof of this example is given in section 2.

Our next main result deals with a system of linear type (first system of (2)) or in degenerate form (third system of (2)) with a monodromic singular point at the origin. We assume that the origin has no characteristic directions. As we have already stated, this means that the polynomial $\Delta(x,y)$ defined in (3) is such that $\Delta(x,y) = 0$ only if $(x,y) = (0,0)$. We remark that in this case the degree of the lowest order terms of $P(x,y)$ and $Q(x,y)$ must coincide, that is, $P(x,y) = P_n(x,y) + \mathcal{O}_{n+1}(x,y)$ and $Q(x,y) = Q_n(x,y) + \mathcal{O}_{n+1}(x,y)$ where $P_n(x,y)$ and $Q_n(x,y)$ are nonzero homogeneous polynomials of degree n formed by all the terms of this degree in $P(x,y)$ and $Q(x,y)$. We define

$$v(\theta) \;=\; \exp\left[\int_0^{\theta} \frac{\cos\sigma P_n(\cos\sigma,\sin\sigma) + \sin\sigma Q_n(\cos\sigma,\sin\sigma)}{\cos\sigma Q_n(\cos\sigma,\sin\sigma) - \sin\sigma P_n(\cos\sigma,\sin\sigma)}\, d\sigma\right]. \tag{13}$$

Theorem 11. *Consider an analytic differential system* (1) *whose origin is monodromic and has no characteristic directions. Denote by* $\mathrm{div}_d(x,y)$ *the lowest order terms of degree* d *of the divergence* $\mathrm{div}(x,y)$ *of the system. Assume that* $v(2\pi) = 1$ *and*

$$\alpha \;=\; \int_0^{2\pi} \frac{\mathrm{div}_d\,(\cos\theta,\sin\theta)\;\, v(\theta)^{d-n+1}}{\cos\theta Q_n(\cos\theta,\sin\theta) - \sin\theta P_n(\cos\theta,\sin\theta)}\, d\theta \;\neq\; 0. \tag{14}$$

Then the origin is a focus which is stable (*resp. unstable*) *if* $\alpha < 0$ (*resp.* $\alpha > 0$).

This result is proved in section 2. We remark that Theorem 2 is a particular case of Theorem 11. In Theorem 2 we compute the exact value of a Poincaré–Liapunov constant for a system of linear type whereas Theorem 11 also deals with systems in degenerate form.

We also remark that, as it is shown in the proof of Theorem 11, if $v(2\pi) > 1$ then the origin is an unstable focus and if $v(2\pi) < 1$ then the origin is a stable focus. The statement of Theorem 11 can be useful to establish the stability of the origin in case that $v(2\pi) = 1$.

The following example is a particular case of Example 3 in [8]. Now, we study the stability of the origin.

Example 12. *Consider the degenerate singular point at the origin of the system*

$$\begin{aligned} \dot{x} \;&=\; -y(x^2+y^2) \;+\; x^3(\lambda_1 x^2 + \lambda_2(x^2+y^2)), \\ \dot{y} \;&=\; x(x^2+y^2) \;+\; x^2 y(\lambda_1 x^2 + \lambda_2(x^2+y^2)), \end{aligned} \tag{15}$$

where $\lambda_1, \lambda_2 \in \mathbb{R}$. *If* $3\lambda_1 + 4\lambda_2 > 0$ (*resp.* $3\lambda_1 + 4\lambda_2 < 0$), *then the origin of system* (15) *is an unstable* (*resp. stable*) *focus.*

The proof of this example is given in section 2.

2. Proofs of the results

Consider a positively oriented, piecewise smooth, simple closed curve C in a plane and denote by D be the region bounded by C. We assume that $C \cup D$ is contained in the region where system (1) is analytic and we recall that Green's Theorem establishes that

$$\oint_C P(x,y)dy - Q(x,y)dx \;=\; \iint_D \mathrm{div}(x,y)\, dxdy. \tag{16}$$

Proof of Proposition 1. Theorem 1 in page 258 of [12], which is a consequence of Green's Theorem, establishes that if the divergence of a system (1) is not identically zero and does not change sign in a simply connected region in $\mathbb{R}^2$, then there is no closed orbit lying entirely in this simply connected region. If the divergence of system (1) is of sign definite, then there is a neighborhood $\mathcal{U}_O$ of the origin in which $\mathrm{div}(x, y) \geq 0$ or $\mathrm{div}(x, y) \leq 0$ for all $(x, y) \in \mathcal{U}_O$. If the origin is a center, then there is a continuum of periodic orbits completely contained in $\mathcal{U}_O$ which contradicts the aforementioned Theorem 1 in page 258 of [12]. Hence, the origin is a focus.

We are going to prove that if $\mathrm{div}(x, y)$ is positive definite, then the origin of (1) is an unstable focus. The corresponding proof when $\mathrm{div}(x, y)$ is negative definite is analogous. Let us consider a transversal section Σ whose boundary contains the origin O and a neighborhood $\mathcal{U}_O$ of the origin such that $\mathrm{div}(x, y) \geq 0$ for all $(x, y) \in \mathcal{U}_O$. We only consider the part of Σ contained in $\mathcal{U}_O$. We fix a point ρ in Σ and we consider the point in Σ corresponding to its image by the Poincaré map $\mathcal{P}(\rho)$. If ρ is close enough to the origin, we can ensure that $\mathcal{P}(\rho)$ is contained in $\mathcal{U}_O$. We define the closed curve C formed by the arc of the orbit from ρ to $\mathcal{P}(\rho)$ together with the arc of Σ between these two points. We denote by ℓ the arc of Σ between the points ρ and $\mathcal{P}(\rho)$. Since Σ is a transversal section, we have that all the orbits of (1) cross ℓ in the same direction, either inside or outside the region D. The origin is stable if the orbits cross ℓ in the inside direction and unstable otherwise. We consider the left–hand side of the formula (16) and we write it as the sum of the two arcs which form C:

$$\oint_C P dy - Q dx \, = \, \int_{C \setminus \ell} P dy - Q dx \, + \, \int_\ell P dy - Q dx.$$

Since the curve $C \setminus \ell$ is an orbit of system (1) we have that

$$\int_{C \setminus \ell} P(x, y) dy - Q(x, y) dx \, = \, 0.$$

On the other hand, since $\mathrm{div}(x, y) \geq 0$ for all $(x, y) \in D$, we have that the right–hand side of the formula (16) is positive, which implies that

$$\int_\ell P(x, y) dy - Q(x, y) dx \, > \, 0.$$

This implies that all the orbits of (1) cross ℓ in the outside direction and, thus, the origin of (1) is unstable. □

A change of coordinates $(x, y) \to (u(x, y), v(x, y))$ is said to be *tangent to the identity* if $u(x, y)$ and $v(x, y)$ are real functions in a neighborhood of the origin of the form $u(x, y) = x + \tilde{u}(x, y)$, $v(x, y) = y + \tilde{v}(x, y)$, where $\tilde{u}(x, y)$ and $\tilde{v}(x, y)$ have neither constant nor linear terms. We consider changes of coordinates whose regularity is analytic or smooth depending on the regularity of $u(x, y)$ and $v(x, y)$

Next paragraph deals with normal form theory for systems (2) whose origin is linear type. See, for instance, [1] or Chapter 4 of [6] (and references therein) for more information about this topic.

In [5], see also [13], it is shown that a system of linear type form (first case of (2)) can be transformed to the so-called *Birkhoff normal form*:

$$\begin{aligned} \dot{u} &= -v + S_1(u^2+v^2)u - S_2(u^2+v^2)v, \\ \dot{v} &= u + S_2(u^2+v^2)u + S_1(u^2+v^2)v, \end{aligned} \tag{17}$$

where $S_1(z)$ and $S_2(z)$ are smooth functions with $S_2(0) = 0$. It is also shown that the origin of a linear type system (2) is a center if and only if the corresponding function $S_1(z)$ is identically zero. Indeed the coefficients of $S_1(z)$ are the Poincaré–Liapunov constants associated to a system in the linear type form (first case of (2)). In particular, the Poincaré map associated to the system in linear type form (2) is

$$\mathcal{P}(\rho) = \rho + \alpha_{2k+1}\rho^{2k+1} + \mathcal{O}(\rho^{2k+2}), \tag{18}$$

with $k \geq 1$ and $\alpha_{2k+1} \neq 0$, if and only if

$$S_1(z) = \frac{\alpha_{2k+1}}{2\pi} z^k + \mathcal{O}(z^{k+1}).$$

In general, even though system (2) is analytic, the functions $S_1(z)$, $S_2(z)$ and the change of coordinates are not convergent. Nevertheless, normal form theory ensures that if the Poincaré map associated to the system (2) in linear type form has the expression (18), then there exists an analytic change of coordinates tangent to the identity such that the system can be transformed into the form

$$\begin{aligned} \begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} &= \begin{pmatrix} -v \\ u \end{pmatrix} + \frac{\alpha_{2k+1}}{2\pi}(u^2+v^2)^k \begin{pmatrix} u \\ v \end{pmatrix} \\ &\quad + \sum_{i=1}^{k} \beta_{2i+1}(u^2+v^2)^i \begin{pmatrix} -v \\ u \end{pmatrix} + \mathcal{O}_{2k+2}(u,v), \end{aligned} \tag{19}$$

where β_{2i+1} are real numbers for $i = \overline{1, r}$.

Proof of Theorem 2. We consider the change of variables which is tangent to the identity and which takes system (2) in linear type form to the system (17). We observe that the divergence of system (17) is a function of u^2+v^2 and takes the form

$$\delta(u^2+v^2) := 2S_1(u^2+v^2) + 2(u^2+v^2)S_1'(u^2+v^2).$$

If we expand $S_1(z)$ in powers of z, we can write

$$S_1(z) = \sum_{j \geq 0} c_j z^j,$$

and as we have already stated, we can identify $c_j = \alpha_{2j+1}/(2\pi)$ where α_{2j+1} is the Poincaré–Liapunov constant associated to system (2) in linear type form (or equivalently of the system (17)). We remark that

$$\begin{aligned}\delta(z) &= 2S_1(z) + 2zS_1'(z) \;=\; 2\sum_{j\geq 0} c_j z^j \;+\; 2z\sum_{j\geq 0} j c_j z^{j-1} \\ &= 2\sum_{j\geq 0}(1+j)c_j z^j \;=\; \frac{2}{2\pi}\sum_{j\geq 0}(1+j)\alpha_{2j+1} z^j.\end{aligned}$$

Therefore, if the first nonzero Poincaré–Liapunov constant of system (17) is α_{2k+1}, we have that the lowest order terms of the divergence of system (17) are $(2k+1)\alpha_{2k+1}(u^2+v^2)^k/\pi$. Thus, the integral which appears in the statement of Theorem 2 for system (17) gives $2(2k+1)\alpha_{2k+1}$. If we denote by $d = 2k$, we get the result for system (17). Lemma 8 (p. 10) of the article [10] gives the relation between the divergences of the systems (2) and (17). We denote $x = F(u,v)$, $y = G(u,v)$ the change of variables which takes the linear type system (2) to the system (17) and by $J(u,v)$ its Jacobian. Since the change is tangent to the identity, we have that $J(u,v) = 1 + \mathcal{O}_1(x,y)$. By Lemma 8 of [10] we have that

$$\operatorname{div}\left(F(u,v),\, G(u,v)\right) \;=\; \delta(u^2+v^2) \;+\; \frac{1}{J(u,v)}\left(\frac{\partial J}{\partial u}\,\dot{u} \;+\; \frac{\partial J}{\partial v}\,\dot{v}\right), \tag{20}$$

where $\operatorname{div}(x,y)$ is the divergence of system (2). Since the change of coordinates is tangent to the identity we have that the lowest order terms of $\operatorname{div}(F(u,v),\, G(u,v))$ are $\operatorname{div}_d(u,v)$.

We consider the solution of system (17) with initial condition the point $(\rho,0)$ with $\rho > 0$ small, and we denote it by $\Phi_t(\rho)$. We denote the components of $\Phi_t(\rho)$ by $(u(t;\rho),\, v(t;\rho))$ and we expand them in powers of ρ. Since $\rho = 0$ corresponds to the singular point at the origin, we have that $(u(t;0),\, v(t;0)) = (0,0)$ for all $t \in \mathbb{R}$. Hence,

$$u(t;\rho) \;=\; u_1(t)\rho \;+\; \mathcal{O}(\rho^2), \quad v(t;\rho) \;=\; v_1(t)\rho \;+\; \mathcal{O}(\rho^2).$$

Since the initial condition of $\Phi_t(\rho)$ is the point $(\rho,0)$, we get that $u_1(0) = 1$ and $v_1(0) = 0$. Equating the lowest order terms in ρ in the equation of the flow

$$\begin{aligned}D_t\Phi_t(\rho) \;=\; &\big(-v + S_1(u^2+v^2)u - S_2(u^2+v^2)v, \\ &u \;+\; S_2(u^2+v^2)u + S_1(u^2+v^2)v\big)_{|(u,v)=\Phi_t(\rho)}\end{aligned}$$

we get $u_1'(t) = -v_1(t)$, $v_1'(t) = u_1(t)$. We deduce that $u_1(t) = \cos t$ and $v_1(t) = \sin(t)$. We have that $\Phi_t(\rho) = \rho(\cos t, \sin t) + \mathcal{O}(\rho^2)$. The Poincaré map associated to system (2) and system (17) is $\mathcal{P}(\rho) = \Phi_{T(\rho)}(\rho)$ for a certain $T(\rho) > 0$ with $T(\rho) = 2\pi + \mathcal{O}(\rho)$. Assume that $\mathcal{P}(\rho) = \rho + \alpha_{2k+1}\rho^{2k+1} + \mathcal{O}(\rho^{2k+2})$. We integrate the second term of the right hand side of (20) evaluated on the orbit $\Phi_t(\rho)$ for t from 0 to $T(\rho)$ and we get

$$\int_0^{T(\rho)} \frac{1}{J(u,v)} \left(\frac{\partial J}{\partial u}\, \dot u \,+\, \frac{\partial J}{\partial v}\, \dot v \right)_{(u,v)\,=\,\Phi_t(\rho)} dt \;=\; \big[\ln J(\Phi_t(\rho)) \big]_0^{T(\rho)}$$

$$= \ln J(\mathcal{P}(\rho), 0) \,-\, \ln J(\rho,0) \;=\; \alpha_{2k+1} \rho^{2k+1} \left(\frac{\partial J}{\partial u}(0,0) \right) \,+\, \mathcal{O}(\rho^{2k+2}).$$

We integrate identity (20) evaluated on the orbit $\Phi_t(\rho)$ for t from 0 to $T(\rho)$ and we note that the lowest order terms in ρ correspond to ρ^{2k}. We deduce that $d = 2k$ and the following identity:

$$\int_0^{2\pi} \mathrm{div}_d(\cos t, \sin t)\, dt \;=\; (d+2)\alpha_{d+1}.$$

□

In the paper [11], see also the references therein, a method to give necessary conditions for a differential system in nilpotent form, second case of (2), to have a center at the origin is described. This method is based on the following result, whose statement is corrected in [7].

Theorem 13 ([7])**.** *Suppose that the origin of the real analytic differential system* (2) *in* nilpotent *form is a center, then there exist analytic functions* M_1 *and* M_2 *starting with terms of degree at least* 2 *in* x *and* y*, such that the system*

$$\dot x \;=\; y + F_1(x,y) + \varepsilon M_1(x,y), \quad \dot y \;=\; -\varepsilon x + F_2(x,y) + \varepsilon M_2(x,y) \tag{21}$$

has a linear type center at the origin for all $\varepsilon > 0$.

In the proof of this result which appears in [7], one can see that the functions $M_1(x,y)$ and $M_2(x,y)$ take the form

$$\begin{aligned} M_1(x,y) &= (x + f(x,y)) \frac{\partial f}{\partial y} \, (1 + U(x,y)), \\ M_2(x,y) &= x \,-\, (x + f(x,y)) \left(1 + \frac{\partial f}{\partial x} \right) (1 + U(x,y)), \end{aligned} \tag{22}$$

where $f(x,y)$ is an analytic function starting with terms of degree at least 2 in x and y and $U(x,y)$ is an analytic function with $U(0,0) = 0$.

Remark that system (21) for $\varepsilon = 0$ is the original differential system (2) in nilpotent form. Remark also that system (21) for $\varepsilon > 0$ has a monodromic singular point at the origin of linear type. As a consequence of Theorem 13, one can compute the Poincaré–Liapunov constants associated to system (21), which will be algebraic functions of ε, and a necessary condition for nilpotent differential system (2) to have a center at the origin is the vanishing of them, for all $\varepsilon > 0$. These Poincaré–Liapunov constants may also depend on the coefficients of the analytic functions $M_1(x,y)$ and $M_2(x,y)$, which can be determined by their vanishing.

Proof of Theorem 4. Taking into account the form of the function M_1 and M_2 given in (22), we note that system (21) writes as

$$\begin{aligned} \dot{x} &= y + F_1(x,y) + \varepsilon(x + f(x,y))\,\frac{\partial f}{\partial y}\,(1 + U(x,y))\,, \\ \dot{y} &= F_2(x,y) - \varepsilon(x + f(x,y))\left(1 + \frac{\partial f}{\partial x}\right)(1 + U(x,y))\,. \end{aligned} \tag{23}$$

Let $\widetilde{\mathrm{div}}(x,y)$ be the divergence of the previous differential system, and let $\mathrm{div}(x,y)$ be the divergence of the original differential system (2) in nilpotent form. Then

$$\widetilde{\mathrm{div}}(x,y) \;=\; \mathrm{div}(x,y) + \mathcal{O}(\varepsilon).$$

and consequently

$$\widetilde{\mathrm{div}}_d(x,y) \;=\; \mathrm{div}_d(x,y) + \mathcal{O}(\varepsilon), \tag{24}$$

where $\mathrm{div}_d(x,y)$ denotes the lowest order terms of the function $\mathrm{div}(x,y)$.

On the other hand, and analogously to the proof of Theorem 2, we consider the solution of system (21) with initial condition the point $(\rho,0)$ with $\rho > 0$ small and we denote it by $\Phi_t(\rho;\varepsilon)$. We denote by $T(\rho;\varepsilon)$ the strictly positive function such that the point $\Phi_{T(\rho;\varepsilon)}(\rho;\varepsilon)$ is the first cut with the transversal section $\Sigma = \{(\rho,0) \,:\, \rho > 0,\, \rho \text{ small}\}$. Here, we remark the dependence on $\varepsilon > 0$. Also analogously to the proof of Theorem 2 it is easy to show that

$$\Phi_t(\rho;\varepsilon) \;=\; \left(\cos\left(\sqrt{\varepsilon}\,t\right),\, -\sqrt{\varepsilon}\,\sin\left(\sqrt{\varepsilon}\,t\right)\right)\rho \;+\; \mathcal{O}(\rho^2),$$

and

$$T(\rho;\varepsilon) \;=\; \frac{2\pi}{\sqrt{\varepsilon}} \;+\; \mathcal{O}(\rho).$$

As a direct consequence of Theorem 2, we have that the expression $V_{d+1}(\varepsilon)$ is one of the Poincaré–Liapunov constants associated to system (23). Finally the expression of $V_{d+1}(\varepsilon)$ from (24) is the expression (7) which appears in the statement of Theorem 4 with an additional $\mathcal{O}(\varepsilon)$. Then, the sign of the coefficient v_{d+1} is the sign of expression (7), and statement (b) follows. A necessary condition for the origin to be a center is that $v_{d+1} = 0$, which proves statement (a). □

Recall that a system (1) is said to be *time–reversible* when it is invariant under the transformation $(x,y,t) \to (-x,y,-t)$. A system with such a symmetry and with a monodromic singular point at O has a center at the origin because a spiral cannot be symmetric.

Proof of Example 5. It can be shown that the origin of system (8) is monodromic by using Andreev's theorem [4], see Theorem 7. The divergence of this system is $\mathrm{div}(x,y) = 3cy^2 + bx^3$. Since $\mathrm{div}(x,y)$ is sign definite when $c \neq 0$, by Proposition 1, a necessary condition for system (8) to have a center at the origin is $c = 0$. Indeed, if $c > 0$ (resp. $c < 0$), the origin is an unstable (resp. stable) focus. We remark that when $c = 0$, the system is time–reversible and thus, it has a center at the origin. □

System (8) is a particular case of Example 3 in [11]:

$$\dot{x} \,=\, y, \quad \dot{y} \,=\, -x^3 + ay^2 + bx^3y + cy^3,$$

where $a, b, c \in \mathbb{R}$. In [11], the authors prove that a necessary condition for this system to have a center at the origin is $ab(ab + 3c) = 0$ and that the condition $a = c = 0$ is also sufficient. Our Theorem 4 (nor Proposition 1) gives no result in case that $a \neq 0$ because the divergence is $\mathrm{div}(x, y) = 2ay + 3cy^2 + bx^3$ whose lowest order terms give $\mathrm{div}_1(x, y) = 2ay$ and, thus,

$$V_2(\varepsilon) \,=\, \int_0^{2\pi/\sqrt{\varepsilon}} 2a\sqrt{\varepsilon} \sin\left(\sqrt{\varepsilon}\, t\right)\, dt \,=\, 0.$$

Proof of Example 6. The origin of system (9) is monodromic as a consequence of Andreev's theorem [4], see Theorem 7. The divergence of this system is a homogeneous polynomial of degree 2: $\mathrm{div}(x, y) = Px^2 + 2(A + K)xy + (B + 3L)y^2$. Thus, the expression of $V_3(\varepsilon)$ given in Theorem 4 gives

$$V_3(\varepsilon) \,=\, \frac{\pi}{\sqrt{\varepsilon}} \left(P + (B + 3L)\varepsilon\right).$$

We get the result as a direct consequence of Theorem 4. □

In the paper [9], which is the seminal work for [11], systems (2) with a degenerate singular point at the origin are also considered. For instance, the following system appears in page 414 of [9]:

$$\begin{aligned} \dot{x} &= 12\lambda x^3 - 20\lambda xy^2 + 9\mu y^3 - 9x^2y - 25y^3, \\ \dot{y} &= 12\lambda x^2y - 20\lambda y^3 + 9x^3 + 25xy^2, \end{aligned}$$

where $\lambda, \mu \in \mathbb{R}$ and $\mu < 25/9$. A classical result about homogeneous centers establishes that the origin of this system is a center if and only if $\lambda = 0$ or $\mu = 0$. In order to see this system as a limit of a linear type center, the authors of [9] consider the system

$$\begin{aligned} \dot{x} &= -\varepsilon y \,+\, 12\lambda x^3 - 20\lambda xy^2 + 9\mu y^3 - 9x^2y - 25y^3, \\ \dot{y} &= \varepsilon x \,+\, 12\lambda x^2y - 20\lambda y^3 + 9x^3 + 25xy^2, \end{aligned}$$

with $\varepsilon > 0$. The analogous statement to Theorem 4 for this system is that the condition

$$\int_0^{2\pi/\varepsilon} \mathrm{div}\left(\cos(\varepsilon t),\, \sin(\varepsilon t)\right) dt \,=\, 0,$$

is necessary in order that the origin of the system is a center, under the assumption that it is a degenerate center which is the limit of linear type centers. This condition writes as

$$\int_0^{2\pi/\varepsilon} 16\left(3\lambda\cos^2(\varepsilon t) - 5\lambda\sin(\varepsilon t)^2 + 2\cos(\varepsilon t)\sin(\varepsilon t)\right) dt \,=\, -\frac{32\lambda}{\varepsilon}\,\pi.$$

Thus, we obtain that a necessary condition for the origin to be a center is that $\lambda = 0$ whereas the centers with $\mu = 0$ are not limit of linear type centers, analogously to the result of [9].

In order to prove Theorem 9 we need to recall the following well-known result, which we include a proof for, for the sake of completeness.

Lemma 14. *Let $\rho > 0$ and $F(r,\theta)$ be a real analytic function 2π-periodic in θ and defined in a neighborhood of $r = 0$. Let $r(\theta;\rho)$ be the solution of the following Cauchy problem*

$$\frac{\partial r}{\partial \theta}(\theta;\rho) = F\left(r(\theta;\rho),\theta\right), \text{ with initial condition } r(0,\rho) = \rho.$$

Then,

$$\frac{\partial r}{\partial \rho}(\theta;\rho) = \exp\left[\int_0^\theta \frac{\partial F}{\partial r}\left(r(\sigma;\rho),\sigma\right) d\sigma\right].$$

Proof. We differentiate the identities

$$\frac{\partial r}{\partial \theta}(\theta;\rho) = F\left(r(\theta;\rho),\theta\right), \quad r(0,\rho) = \rho$$

with respect to ρ and we have that:

$$\frac{\partial}{\partial \theta}\left(\frac{\partial r}{\partial \rho}(\theta;\rho)\right) = \frac{\partial F}{\partial r}\left(r(\theta;\rho),\theta\right)\left(\frac{\partial r}{\partial \rho}(\theta;\rho)\right) \quad \text{and} \quad \frac{\partial r}{\partial \rho}(0;\rho) = 1.$$

Hence,

$$\int_0^\theta \frac{\partial F}{\partial r}\left(r(\sigma;\rho),\sigma\right)\, d\sigma = \int_0^\theta \frac{\frac{\partial}{\partial \sigma}\left(\frac{\partial r}{\partial \rho}(\sigma;\rho)\right)}{\frac{\partial r}{\partial \rho}(\sigma;\rho)}\, d\sigma = \ln\left(\frac{\partial r}{\partial \rho}(\sigma;\rho)\right)\Big|_{\sigma=0}^{\sigma=\theta}.$$

Since $\frac{\partial r}{\partial \rho}(0;\rho) = 1$, we deduce that

$$\int_0^\theta \frac{\partial F}{\partial r}\left(r(\sigma;\rho),\sigma\right)\, d\sigma = \ln\left(\frac{\partial r}{\partial \rho}(\theta;\rho)\right).$$

□

Proof of Theorem 9. We consider system (10) with Andreev number n and the weighted blow-up $x = r\cos\theta$, $y = r^n \sin\theta$. The Jacobian of this weighted blow up is $J(r,\theta) = r^n(\cos^2\theta + n\sin^2\theta)$. We have that

$$\dot{r} = \frac{r^n\psi(\theta) + \mathcal{O}(r^{n+1})}{\cos^2\theta + n\sin^2\theta)}, \quad \dot{\theta} = \frac{r^{n-1}\varphi(\theta) + \mathcal{O}(r^n)}{\cos^2\theta + n\sin^2\theta}$$

where $\psi(\theta)$ and $\varphi(\theta)$ are trigonometric polynomials and

$$\varphi(\theta) = \cos^{2n}\theta + n\sin^2\theta \quad \text{and} \quad \psi(\theta) = \varphi'(\theta)/(-2n).$$

Recall that we are under the assumption that the function $\phi(x)$ defined in Theorem 7 is either $\phi(x) \equiv 0$ or $\phi(x) = bx^\beta + \mathcal{O}(x^{\beta+1})$ with $\beta > n-1$. Note that $\varphi(\theta) > 0$ for all θ. For the rest of the proof, we denote by $R(r,\theta)$

and $\Theta(r,\theta)$ the functions such that $\dot r = R(r,\theta)$ and $\dot\theta = \Theta(r,\theta)$. Given $\rho > 0$, we note that we can consider the ordinary differential equation

$$\frac{dr}{d\theta} = \frac{R(r,\theta)}{\Theta(r,\theta)}, \tag{25}$$

whose solution $r(\theta;\rho)$, with initial condition $r(0;\rho) = \rho$, gives rise to the Poincaré map associated to system (10) in a neighborhood of the origin by $\mathcal{P}(\rho) = r(2\pi;\rho)$. By the expressions of $\psi(\theta)$ and $\varphi(\theta)$ it is easy to see that

$$r(\theta;\rho) = \frac{\rho}{u(\theta)} + \mathcal{O}(\rho^2) \quad \text{with} \quad u(\theta) = \sqrt[2n]{\cos^{2n}\theta + n\sin^2\theta}. \tag{26}$$

We denote by $\delta(r,\theta)$ the divergence of the ordinary differential equation (25), that is,

$$\delta(r,\theta) = \frac{\partial}{\partial r}\left(\frac{R(r,\theta)}{\Theta(r,\theta)}\right).$$

Given any function $G(r,\theta)$, we denote by $G'(r,\theta)$ the following expression:

$$G'(r,\theta) = \frac{\partial G}{\partial r}(r,\theta)\left(\frac{R(r,\theta)}{\Theta(r,\theta)}\right) + \frac{\partial G}{\partial \theta}(r,\theta).$$

We denote by $\text{div}(x,y)$ the divergence of system (10). Some long but straightforward computations show that

$$\text{div}(r\cos\theta, r^n\sin\theta) = \delta(r,\theta)\Theta(r,\theta) + \Theta'(r,\theta) + \frac{J'(r,\theta)}{J(r,\theta)}\Theta(r,\theta).$$

Thus,

$$\frac{\text{div}(r(\theta;\rho)\cos\theta, r(\theta;\rho)^n\sin\theta)}{\Theta(r(\theta;\rho),\theta)} = \delta(r(\theta;\rho),\theta) + \frac{\Theta'(r(\theta;\rho),\theta)}{\Theta(r(\theta;\rho),\theta)} + \frac{J'(r(\theta;\rho),\theta)}{J(r(\theta;\rho),\theta)}.$$

We integrate the previous identity with respect to θ from 0 to 2π and we get

$$\begin{aligned}\int_0^{2\pi} \frac{\text{div}(r(\theta;\rho)\cos\theta, r(\theta;\rho)^n\sin\theta)}{\Theta(r(\theta;\rho),\theta)}\,d\theta &= \int_0^{2\pi}\delta(r(\theta;\rho),\theta)\,d\theta + \\ &+ \Big[\ln\Theta(r,\theta)\Big]_{(r,\theta)=(\rho,0)}^{(r,\theta)=(\mathcal{P}(\rho),2\pi)} + \Big[\ln J(r,\theta)\Big]_{(r,\theta)=(\rho,0)}^{(r,\theta)=(\mathcal{P}(\rho),2\pi)},\end{aligned} \tag{27}$$

where we have used that $r(2\pi;\rho) = \mathcal{P}(\rho)$.

Let us assume that the origin of system (10) is a focus and its Poincaré map is

$$\mathcal{P}(\rho) = \rho + \alpha_k\rho^k + \mathcal{O}(\rho^{k+1}),$$

with $k \geq 1$ and $\alpha_k \neq 0$. Given any function $G(r,\theta)$ defined in a neighborhood of $r = 0$ and 2π-periodic in θ, we have that

$$G(\mathcal{P}(\rho),2\pi) - G(\rho,0) = \alpha_k\rho^k\,\frac{\partial G}{\partial r}(0,0) + \mathcal{O}(\rho^{k+1}).$$

From Lemma 14 we have that

$$\int_0^{2\pi}\delta(r(\theta;\rho),\theta)\,d\theta = \ln\left(\frac{\partial r}{\partial\rho}(2\pi;\rho)\right).$$

Since $r(2\pi;\rho) \,=\, \rho + \alpha_k \rho^k + \mathcal{O}(\rho^{k+1})$, we have that

$$\frac{\partial r}{\partial \rho}(2\pi;\rho) \,=\, 1 + \alpha_k \rho^{k-1} + \mathcal{O}(\rho^k)$$

and, thus,

$$\ln\left(\frac{\partial r}{\partial \rho}(2\pi;\rho)\right) \,=\, \begin{cases} \ln(1+\alpha_1) \,+\, \mathcal{O}(\rho) & \text{if } k=1, \\ \alpha_k \rho^{k-1} \,+\, \mathcal{O}(\rho^k) & \text{if } k>1. \end{cases}$$

We note that the development in powers of ρ of left-hand side of (27) is

$$\int_0^{2\pi} \frac{\operatorname{div}(r(\theta;\rho)\cos\theta, r(\theta;\rho)^n \sin\theta)}{\Theta(r(\theta;\rho),\theta)}\, d\theta \,=\, \alpha \rho^{d-n+1} \,+\, \mathcal{O}(\rho^{d-n+2}),$$

where

$$\alpha \,=\, \int_0^{2\pi} \operatorname{div}_d^{(1,n)}(\cos\theta,\sin\theta)\, \frac{\cos^2\theta + n\sin^2\theta}{u(\theta)^{d+n+1}}\, d\theta$$

is the value appearing in (11). Thus, if $\alpha \neq 0$, we have that $k-1 = d-n+1$ and the Poincaré map is

$$\mathcal{P}(\rho) \,=\, \begin{cases} e^{\alpha}\rho + \mathcal{O}(\rho^2) & \text{if } k=1 \\ \rho + \alpha\rho^k + \mathcal{O}(\rho^{k+1}) & \text{if } k>1. \end{cases}$$

□

Proof of Example 10. It is easy to see that the Andreev number of system (12) is $n=2$. Even though the system is not written in the form (10), the weighted blow-up $x = r\cos\theta$, $y = r^2\sin\theta$ gives

$$\dot{r} \,=\, \frac{\cos^3\theta\sin\theta - \cos\theta\sin\theta}{\cos^2\theta + 2\sin^2\theta}\, r^2 \,+\, \mathcal{O}(r^3), \quad \dot{\theta} \,=\, \frac{\cos^4\theta + 2\sin^2\theta}{\cos^2\theta + 2\sin^2\theta}\, r \,+\, \mathcal{O}(r^2).$$

Therefore, we can use the value α given in Theorem 9 to show the stability of the origin analogously. We note that the divergence of system (10) is

$$\operatorname{div}(x,y) \,=\, 3a_{30}x^2 + 2a_{21}xy + a_{12}y^2,$$

and its $(1,2)$-quasihomogeneous terms of lowest order are $\operatorname{div}_2^{(1,2)}(x,y) \,=\, 3a_{30}x^2$. We note that the integrand in the expression of α of Theorem 9 is a_{30} product a strictly positive function. Therefore, the sign of α coincides with the sign of a_{30}. □

Proof of Theorem 11. The proof of this result is analogous to the proof of Theorem 9 in many points. We are going highlight the main differences. We consider system (1) and the polar blow-up $x = r\cos\theta$, $y = r\sin\theta$. The Jacobian of this blow up is $J(r,\theta) \,=\, r$. We have that

$$\begin{aligned} \dot{r} \;&=\; (\cos\theta P_n(\cos\theta,\sin\theta) + \sin\theta Q_n(\cos\theta,\sin\theta))\, r^n \,+\, \mathcal{O}(r^{n+1}), \\ \dot{\theta} \;&=\; (\cos\theta Q_n(\cos\theta,\sin\theta) - \sin\theta P_n(\cos\theta,\sin\theta))\, r^{n-1} \,+\, \mathcal{O}(r^n). \end{aligned}$$

For the rest of the proof, we denote by $R(r,\theta)$ and $\Theta(r,\theta)$ the functions such that $\dot{r} \,=\, R(r,\theta)$ and $\dot{\theta} \,=\, \Theta(r,\theta)$. Given $\rho > 0$, we note that we can consider the analogous ordinary differential equation to (25) whose solution $r(\theta;\rho)$, with initial condition $r(0;\rho) = \rho$, gives rise to the Poincaré map associated to system (1) in a neighborhood of the origin by $\mathcal{P}(\rho) = r(2\pi;\rho)$. By the expressions of $R(r,\theta)$ and $\Theta(r,\theta)$ it is easy to see that

$$r(\theta;\rho) = v(\theta)\rho + \mathcal{O}(\rho^2) \tag{28}$$

with $v(\theta)$ the one given in (13). As before, we denote by $\delta(r,\theta)$ the divergence of the ordinary differential equation (25). Again, straightforward computations show that

$$\operatorname{div}(r\cos\theta, r\sin\theta) = \delta(r,\theta)\Theta(r,\theta) + \Theta'(r,\theta) + \frac{J'(r,\theta)}{J(r,\theta)}\Theta(r,\theta),$$

where recall that now $J(r,\theta) = r$. We substitute r by $r(\theta;\rho)$ in the previous identity and we integrate it with respect to θ from 0 to 2π in order to get

$$\begin{aligned}\int_0^{2\pi} \frac{\operatorname{div}(r(\theta;\rho)\cos\theta, r(\theta;\rho)\sin\theta)}{\Theta(r(\theta;\rho),\theta)}\, d\theta &= \int_0^{2\pi} \delta(r(\theta;\rho),\theta)\, d\theta + \\ + \Big[\ln\Theta(r,\theta)\Big]_{(r,\theta)=(\rho,0)}^{(r,\theta)=(\mathcal{P}(\rho),2\pi)} &+ \Big[\ln J(r,\theta)\Big]_{(r,\theta)=(\rho,0)}^{(r,\theta)=(\mathcal{P}(\rho),2\pi)},\end{aligned} \tag{29}$$

where we have used that $r(2\pi;\rho) = \mathcal{P}(\rho)$.

Let us assume that the origin of system (1) is a focus and its Poincaré map is

$$\mathcal{P}(\rho) = \rho + \alpha_k\rho^k + \mathcal{O}(\rho^{k+1}),$$

with $k \geq 1$ and $\alpha_k \neq 0$. We have that

$$\frac{\partial r}{\partial\rho}(2\pi;\rho) = 1 + \alpha_k\rho^{k-1} + \mathcal{O}(\rho^k)$$

and, thus,

$$\ln\left(\frac{\partial r}{\partial\rho}(2\pi;\rho)\right) = \begin{cases} \ln(1+\alpha_1) + \mathcal{O}(\rho) & \text{if } k=1, \\ \alpha_k\rho^{k-1} + \mathcal{O}(\rho^k) & \text{if } k>1. \end{cases}$$

We note that the development in powers of ρ of left-hand side of (29) is

$$\int_0^{2\pi} \frac{\operatorname{div}(r(\theta;\rho)\cos\theta, r(\theta;\rho)\sin\theta)}{\Theta(r(\theta;\rho),\theta)}\, d\theta = \alpha\rho^{d-n+1} + \mathcal{O}(\rho^{d-n+2}),$$

where α is the value appearing (14). Thus, if $\alpha \neq 0$, we have that $k-1 = d-n+1$ and the Poincaré map is

$$\mathcal{P}(\rho) = \begin{cases} e^{\alpha}\rho + \mathcal{O}(\rho^2) & \text{if } k=1 \\ \rho + \alpha\rho^k + \mathcal{O}(\rho^{k+1}) & \text{if } k>1. \end{cases}$$

□

Proof of Example 12. Since $P_n(x,y) = -y(x^2+y^2)$, $Q_n(x,y) = x(x^2+y^2)$ and $n=3$, we have that $\cos\theta Q_3(\cos\theta,\sin\theta) - \sin\theta P_3(\cos\theta,\sin\theta) = 1$ and $v(\theta) = 1$. Note that the divergence of the system is $\operatorname{div}(x,y) = 6x^2(\lambda_1 x^2 +$

$\lambda_2(x^2+y^2))$. Thus, a straightforward computation gives that the value of α which appears in Theorem 11 is

$$\alpha \;=\; \int_0^{2\pi} \mathrm{div}(\cos\theta, \sin\theta)\, d\theta \;=\; \frac{3\pi}{2}\,(3\lambda_1+4\lambda_2)$$

and the statement follows. □

Acknowledgements

The first author is partially supported by a MINECO/FEDER grant number MTM2011-22877 and by an AGAUR (Generalitat de Catalunya) grant number 2014SGR 1204. The second author is partially supported by a MINECO/FEDER grant MTM2008-03437 and MTM2013-40998-P, an AGAUR grant number 2013SGR-568, an ICREA Academia, the grants FP7-PEOPLE-2012-IRSES 318999 and 316338, FEDER-UNAB-10-4E-378.

References

[1] A. Algaba and M. Reyes, *Characterizing isochronous points and computing isochronous sections*, J. Math. Anal. Appl. **355** (2009), 564–576.

[2] M.J. Álvarez and A. Gasull, *Monodromy and Stability for nilpoten critical points*, Int. J. Bifurcation and Chaos **15** (2005), 1253–1265.

[3] M.J. Álvarez and A. Gasull, *Generating limit cycles from a nilpotent critical point via normal forms*, J. Math. Anal. Appl. 318 (2006) 271–287.

[4] A. Andreev, *Investigation on the behaviour of the integral curves of a system of two differential equations in the neighborhood of a singular point*, Translation Amer. Math. Soc. **8** (1958), 187–207.

[5] G. Belitskiĭ, *Smooth equivalence of germs of vector fields with one zero or a pair of purely imaginary eigenvalues*, Funct. Anal. Appl. **20** (1986), 253–259.

[6] F. Dumortier, J. Llibre and J.C. Artés, Qualitative theory of planar differential systems. Universitext. Springer Verlag, Berlin, 2006.

[7] I.A. García, H. Giacomini, J. Giné and J. Llibre, *Analytic nilpotent centers as limits of nondegenerated centers revisited.* Preprint.

[8] I.A. García, H. Giacomini and M. Grau, *Generalized Hopf bifurcation for planar vector fields via the inverse integrating factor*, J. Dynam. Differential Equations **23** (2011), 251–281.

[9] H. Giacomini, J. Giné and J. Llibre, *The problem of distinguishing between a center and a focus for nilpotent and degenerate analytic systems*, J. Differential Equations **227** (2006), 406–426.

[10] H. Giacomini and M. Grau, *On the stability of limit cycles for planar differential systems.* J. Differential Equations **213** (2005), 368–388.

[11] J. Giné and J. Llibre, *A method for characterizing nilpotent centers.* J. Math. Anal. Appl. **413** (2014), 537–545.

[12] L. Perko, *Differential equations and dynamical systems.* Third edition. Texts in Applied Mathematics, **7**. Springer-Verlag, New York, 2001.

[13] H. Poincaré, *Sur l'intégration des équations différentielles du premier ordre et du premier degré I and II,* Rend. Circ. Mat. Palermo **5** (1891) 161–191; Rend. Circ. Mat. Palermo **11** (1897) 193–239.

[14] D. Schlomiuk, *Algebraic and geometric aspects of the theory of polynomial vector fields.* Bifurcations and periodic orbits of vector fields (Montreal, PQ, 1992), 429–467, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 408, Kluwer Acad. Publ., Dordrecht, 1993.

[15] K.S. Sibirskiĭ, *On the number of limit cycles in the neighborhood of a singular point.* (Russian) Differencial'nye Uravnenija **1** (1965) 53–66. English translation: Differential Equations **1** (1965), 36–47.

[1] Departament de Matemàtica, Universitat de Lleida, Avda. Jaume II, 69; 25001 Lleida, Catalonia, Spain

E-mail address: mtgrau@matematica.udl.cat

[2] Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain

E-mail address: jllibre@mat.uab.cat