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A CHEBYSHEV CRITERION FOR ABELIAN INTEGRALS

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ABSTRACT. We present a criterion that provides an easy sufficient condition in order for a collection of Abelian integrals to have the Chebyshev property. This condition involves the functions in the integrand of the Abelian integrals and can be checked, in many cases, in a purely algebraic way. By using this criterion, several known results are obtained in a shorter way and some new results, which could not be tackled by the known standard methods, can also be deduced.

1. INTRODUCTION AND STATEMENT OF THE RESULT

The second part of *Hilbert's 16th problem* [13] asks about the maximum number and location of limit cycles of a planar polynomial vector field of degree d. Solving this problem, even in the case $d = 2$, seems to be out of reach at the present state of knowledge (see the works of Ilyashenko [17] and Li Jibin [20] for a survey of the recent results on the subject). Our paper is concerned with a weaker version of this problem, the so-called *infinitesimal Hilbert's 16th problem*, proposed by Arnold [1]. Let ω be a real 1-form with polynomial coefficients of degree at most d. Consider a real polynomial H of degree $d+1$ in the plane. A closed connected component of a level curve $H = h$ is denoted by γ_h and is called an *oval* of H. These ovals form continuous families (see Figure 2), and the infinitesimal Hilbert's 16th problem is to find an upper bound $V(d)$ of the number of real zeros of the *Abelian integral*

$$I(h) = \int_{\gamma_h} \omega.$$

The bound should be uniform with respect to the choice of the polynomial H, the family of ovals $\{\gamma_h\}$ and the form ω. It should depend on the degree d only. (In the literature an Abelian integral is usually the integral of a rational 1-form over a continuous family of algebraic ovals. Throughout the paper, by an abuse of language, we use the name Abelian integral also in case the functions are analytic.)

Zeros of Abelian integrals are related to limit cycles in the following way. Consider a small deformation of a Hamiltonian vector field $X_\varepsilon = X_H + \varepsilon Y$, where

$$X_H = -H_y\partial_x + H_x\partial_x \text{ and } Y = P\partial_x + Q\partial_y.$$

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