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Bifurcation of critical periods from Pleshkan's isochrones

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Abstract

Pleshkan proved in 1969 that, up to a linear transformation and a constant rescaling of time, there are four isochrones in the family of cubic centers with homogeneous nonlinearities $\mathscr{C}_3$. In this paper we prove that if we perturb any of these isochrones inside $\mathscr{C}_3$, then at most two critical periods bifurcate from its period annulus. Moreover, we show that, for each $k = 0, 1, 2$, there are perturbations giving rise to exactly k critical periods. As a byproduct, we obtain a partial result for the analogous problem in the family of quadratic centers $\mathscr{C}_2$. Loud proved in 1964 that, up to a linear transformation and a constant rescaling of time, there are four isochrones in $\mathscr{C}_2$. We prove that if we perturb three of them inside $\mathscr{C}_2$, then at most one critical period bifurcates from its period annulus. In addition, for each $k = 0, 1$, we show that there are perturbations giving rise to exactly k critical periods. The quadratic isochronous center that we do not consider displays some peculiarities that are discussed at the end of the paper.

1. *Introduction and statement of the results*

In this paper we study the period function of the centers of planar vector fields with homogeneous nonlinearities, more concretely, differential systems of the following form:

$$\begin{aligned} \dot{x} &= -y + P_n(x, y), \\ \dot{y} &= x + Q_n(x, y), \end{aligned} \tag{1}$$

where $P_n(x, y)$ and $Q_n(x, y)$ are homogeneous real polynomials of degree n. We denote by $\mathscr{H}_n$ the family of vector fields of the above form and by $\mathscr{C}_n$ the subset of those systems in $\mathscr{H}_n$ with a center at the origin. Finally, $\mathcal{I}_n$ stands for the set of nonlinear isochrones in $\mathscr{H}_n$. Accordingly $\mathcal{I}_n \subset \mathscr{C}_n \subset \mathscr{H}_n$. We restrict ourselves to the cases $n = 2$ and $n = 3$, which are the degrees for which the centers and the isochrones have been classified.

Recall that the *period annulus* of a center is the biggest punctured neighborhood foliated by periodic orbits and in what follows we shall denote it by $\mathcal{P}$. Compactifying $\mathbb{R}^2$ to the Poincaré disk, the boundary of $\mathcal{P}$ has two connected components: the center itself and a polycycle. We call them, respectively, the *inner* and *outer boundary* of the period annulus. The *period function* of the center assigns to each periodic orbit γ in $\mathcal{P}$ its period. If all the periodic orbits in $\mathcal{P}$ have the same period, then the center is called *isochronous.* Since the period function is defined on the set of periodic orbits in $\mathcal{P}$, usually the first step is to parametrize this set, for instance $\{\gamma_s\}_{s\in(0,1)}$, and then one can study the qualitative properties of the period function by means of the map $s \mapsto$ period of γ_s, which is analytic on $(0, 1)$. The *critical periods* are the critical points of this function, and their number, character (maximum or minimum) and distribution do not depend on the particular parametrization of the set of periodic orbits used. We are interested in the *bifurcation* of critical periods, roughly speaking, the disappearance

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