PERIODS OF CONTINUOUS MAPS ON CLOSED SURFACES

JUAN LUIS GARCÍA GUIRAO[1] AND JAUME LLIBRE[2]

Abstract. The objective of the present work is to present what information on the set of periodic points of a continuous self–map on a closed surface can be obtained using the action of this map on the homological groups of the closed surface.

1. Introduction

Along this work by a *closed surface* we denote a connected compact surface with or without boundary, orientable or not. More precisely, *an orientable connected compact surface without boundary of genus* $g \geq 0$, $\mathbb{M}_g$, is homeomorphic to the sphere if $g = 0$, to the torus if $g = 1$, or to the connected sum of g copies of the torus if $g \geq 2$. *An orientable connected compact surface with boundary of genus* $g \geq 0$, $\mathbb{M}_{g,b}$, is homeomorphic to $\mathbb{M}_g$ minus a finite number $b > 0$ of open discs having pairwise disjoint closure. In what follows $\mathbb{M}_{g,0} = \mathbb{M}_g$.

A non–orientable connected compact surface without boundary of genus $g \geq 1$, $\mathbb{N}_g$, is homeomorphic to the real projective plane if $g = 1$, or to the connected sum of g copies of the real projective plane if $g > 1$. *A non–orientable connected compact surface with boundary of genus* $g \geq 1$, $\mathbb{N}_{g,b}$, is homeomorphic to $\mathbb{N}_g$ minus a finite number $b > 0$ of open discs having pairwise disjoint closure. In what follows $\mathbb{N}_{g,0} = \mathbb{N}_g$.

Let $f : \mathbb{X} \to \mathbb{X}$ be a continuous map on a closed surface $\mathbb{X}$. A point $x \in \mathbb{X}$ is periodic of period n if $f^n(x) = x$ and $f^k(x) \neq x$ for $k = 1, \ldots, n-1$. We denote by $\mathrm{Per}(f)$ the set of periods of all periodic points of f. The aim of the present paper is to provide some information on $\mathrm{Per}(f)$.

Let A be an $n \times n$ complex matrix. A $k \times k$ *principal submatrix* of A is a submatrix lying in the same set of k rows and columns, and a $k \times k$ *principal minor* is the determinant of such a principal submatrix. There are $\binom{n}{k}$ different $k \times k$ principal minors of A, and the sum of these is denoted by $E_k(A)$. In particular, $E_1(A)$ is the trace of A, and $E_n(A)$ is the determinant of A, denoted by $\det(A)$.

It is well known that the characteristic polynomial of A is given by

$$\det(tI - A) = t^n - E_1(A)t^{n-1} + E_2(A)t^{n-2} - \ldots + (-1)^n E_n(A).$$

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Our main result is state in the following theorem.

Theorem 1. *Let $\mathbb{X}$ be a closed surface and let $f : \mathbb{X} \to \mathbb{X}$ be a continuous map and let A and (d) be the integral matrices of the endomorphisms $f_{*i} : H_i(\mathbb{X}, \mathbb{Q}) \to H_i(\mathbb{X}, \mathbb{Q})$ induced by f on the i–th homology group of $\mathbb{X}$, $i = 1, 2$.*

If $\mathbb{X}$ is either $\mathbb{M}_{g,b}$ with $b > 0$, or $\mathbb{N}_{g,b}$ with $b \geq 0$, then the following statements hold.

(a) *If $E_1(A) \neq 1$, then $1 \in \mathrm{Per}(f)$.*
(b) *If $E_1(A) = 1$ and $E_2(A) \neq 0$, then $\mathrm{Per}(f) \cap \{1, 2\} \neq \emptyset$.*

If $\mathbb{X} = \mathbb{M}_{g,b}$ with $b = 0$, then the following statement hold.

(c) *If $E_1(A) \neq 1 + d$, then $1 \in \mathrm{Per}(f)$.*
(d) *If $E_1(A) = 1 + d$ and $E_2(A) \neq d^2 + 3d + 1$, then $\mathrm{Per}(f) \cap \{1, 2\} \neq \emptyset$.*

If $\mathbb{X} = \mathbb{M}_{g,b}$ with $b > 0$, then the following statement hold.

(e) *If $2g + b - 1 \geq 3$, $E_1(A) = 1$, $E_2(A) = 0$ and k is the smallest integer of the set $\{3, 4, ..., 2g + b - 1\}$ such that $E_k(A) \neq 0$, then $\mathrm{Per}(f)$ has a periodic point of period a divisor of k.*

If $\mathbb{X} = \mathbb{N}_{g,b}$ with $b \geq 0$, then the following statement hold.

(f) *If $g + b - 1 \geq 3$, $E_1(A) = 1$, $E_2(A) = 0$ and k is the smallest integer of the set $\{3, 4, ..., g + b - 1\}$ such that $E_k(A) \neq 0$, then $\mathrm{Per}(f)$ has a periodic point of period a divisor of k.*

Theorem 1 is proven in section 2.

Similar results tote ones obtained in Theorem 1 but for homeomorphisms on closed surfaces where obtained by Franks and Llibre in [3], and by the authors in [4].

2. Proof of Theorem 1

Let $f : \mathbb{X} \to \mathbb{X}$ be a continuous map and let $\mathbb{X}$ be either $\mathbb{M}_{g,b}$ or $\mathbb{N}_{g,b}$. Then the *Lefschetz number* of f is defined by

$$L(f) = \mathrm{trace}(f_{*0}) - \mathrm{trace}(f_{*1}) + \mathrm{trace}(f_{*2}).$$

For continuous self–maps f defined on $\mathbb{X}$ the Lefschetz fixed point theorem states (see for instance [1]).

Theorem 2. *If $L(f) \neq 0$ then f has a fixed point.*

With the objective of studying the periodic points of f we shall use the Lefschetz numbers of the iterates of f, i.e. $L(f^n)$. Note that if $L(f^n) \neq 0$ then f^n has a fixed point, and consequently f has a periodic point of period a divisor of n. In order to study the whole sequence $\{L(f^n)\}_{n\geq 1}$ it is defined the formal *Lefschetz zeta function* of f as

$$Z_f(t) = \exp\left(\sum_{n=1}^{\infty} \frac{L(f^n)}{n} t^n\right). \tag{1}$$

The Lefschetz zeta function is in fact a generating function for the sequence of the Lefschetz numbers $L(f^n)$.

Let f be a continuous self–map defined on $\mathbb{M}_{g,b}$ or $\mathbb{N}_{g,b}$, respectively. For a closed surface the homological groups with coefficients in $\mathbb{Q}$ are linear vector spaces over $\mathbb{Q}$. We recall the homological spaces of $\mathbb{M}_{g,b}$ with coefficients in $\mathbb{Q}$, i.e.

$$H_k(\mathbb{M}_{g,b}, \mathbb{Q}) = \mathbb{Q} \oplus \overset{n_k}{\cdots} \oplus \mathbb{Q},$$

where $n_0 = 1$, $n_1 = 2g$ if $b = 0$, $n_1 = 2g + b - 1$ if $b > 0$, $n_2 = 1$ if $b = 0$, and $n_2 = 0$ if $b > 0$; and the induced linear maps $f_{*k} : H_k(\mathbb{M}_{g,b}, \mathbb{Q}) \to H_k(\mathbb{M}_{g,b}, \mathbb{Q})$ by f on the homological group $H_k(\mathbb{M}_{g,b}, \mathbb{Q})$ are $f_{*0} = (1)$, $f_{*2} = (d)$ where d is the *degree* of the map f if $b = 0$, $f_{*2} = (0)$ if $b > 0$, and $f_{*1} = A$ where A is an $n_1 \times n_1$ integral matrix (see for additional details [6, 7]).

We recall that the homological groups of $\mathbb{N}_{g,b}$ with coefficients in $\mathbb{Q}$, i.e.

$$H_k(\mathbb{N}_{g,b}, \mathbb{Q}) = \mathbb{Q} \oplus \overset{n_k}{\cdots} \oplus \mathbb{Q},$$

where $n_0 = 1$, $n_1 = g + b - 1$ and $n_2 = 0$; and the induced linear maps are $f_{*0} = (1)$ and $f_{*1} = A$ where A is an $n_1 \times n_1$ integral matrix (see again for additional details [6, 7]).

From the work of Franks in [2] we have for a continuous self–map of a closed surface that its Lefschetz zeta function is the rational function

$$Z_f(t) = \frac{\det(I - tf_{*1})}{\det(I - tf_{*0})\det(I - tf_{*2})},$$

where in $I - tf_{*k}$ the I denotes the $n_k \times n_k$ identity matrix, and $\det(I - tf_{*2}) = 1$ if $f_{*2} = (0)$. Then for a continuous map $f : \mathbb{M}_{g,b} \to \mathbb{M}_{g,b}$ we have

$$Z_f(t) = \begin{cases} \dfrac{\det(I - tA)}{(1-t)(1-dt)} & \text{if } b = 0, \\ \dfrac{\det(I - tA)}{1-t} & \text{if } b > 0, \end{cases} \tag{2}$$

and for a continuous map $f : \mathbb{N}_{g,b} \to \mathbb{N}_{g,b}$ we have

$$Z_f(t) = \frac{\det(I - tA)}{1-t}. \tag{3}$$

Proof of Theorem 1. Combining the expressions (1) and (2) if $\mathbb{X} = \mathbb{M}_{g,b}$ and $b > 0$, and the expressions (1) and (3) if $\mathbb{X} = \mathbb{N}_{g,b}$ with $b \geq 0$, we obtain the following equalities

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{L(f^n)}{n}t^n &= \log(Z_f(t))\\
&= \log\left(\frac{\det(I-tA)}{1-t}\right)\\
&= \log\left(\frac{1-E_1(A)t+E_2(A)t^2-...+(-1)^mE_m(A)t^m}{1-t}\right)\\
&= \log(1-E_1(A)t+E_2(A)t^2-...)-\log(1-t)\\
&= \left(-E_1(A)t+\left(E_2(A)-\frac{E_1(A)^2}{2}\right)t^2-...\right)-\left(-t-\frac{t^2}{2}-...\right)\\
&= (1-E_1(A))t+\left(\frac{1}{2}-\frac{E_1(A)^2}{2}+E_2(A)\right)t^2+O(t^3).
\end{aligned}$$

Here $n_1 = 2g+b-1$ if $\mathbb{X} = \mathbb{M}_{g,b}$ with $b > 0$, or $n_1 = g+b-1$ if $\mathbb{X} = \mathbb{N}_{g,b}$ with $b \geq 0$. Therefore we have

$$L(f) = 1-E_1(A) \quad \text{and} \quad L(f^2) = 1-E_1(A)^2+2E_2(A).$$

Hence, if $E_1(A) \neq 1$ then $L(f) \neq 0$, and by Theorem 2 statement (a) follows.

If $E_1(A) = 1$ and $E_2(A) \neq 0$, then $L(f^2) = 2E_2(A) \neq 0$, and again by Theorem 2 we get that $\mathrm{Per}(f) \cap \{1,2\} \neq \emptyset$. So statement (b) is proved.

Let $\mathbb{X} = \mathbb{M}_{g,b}$ with $b = 0$. By (1) and (2) with $b = 0$ we obtain the following equalities

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{L(f^n)}{n}t^n &= \log(Z_f(t))\\
&= \log\left(\frac{\det(I-tA)}{(1-t)(1-dt)}\right)\\
&= \log\left(\frac{1-E_1(A)t+E_2(A)t^2-...+(-1)^mE_m(A)t^m}{(1-t)(1-dt)}\right)\\
&= \log(1-E_1(A)t+E_2(A)t^2-...)-\log((1-t)(1-dt))\\
&= \left(-E_1(A)t+\left(E_2(A)-\frac{E_1(A)^2}{2}\right)t^2-...\right)\\
&\quad -\left(-(1+d)t-\left(\frac{d^2+1}{2}\right)t^2-...\right)\\
&= (1+d-E_1(A))t+\left(E_2(A)-\frac{E_1(A)^2}{2}-\frac{d^2+1}{2}\right)t^2+O(t^3).
\end{aligned}$$

Here $n_1 = 2g$. Therefore we have

$$L(f) = 1+d-E_1(A), \quad \text{and} \quad L(f^2) = 2E_2(A)-E_1(A)^2-(d^2+1).$$

Hence, if $E_1(A) \neq 1+d$ then $L(f) \neq 0$, and by Theorem 2 statement (c) follows.

If $E_1(A) = 1+d$ and $E_2(A) \neq d^2+d+1$, then $L(f^2) = 2E_2(A)-2(d^2+d+1) \neq 0$, and again by Theorem 2 we get that $\mathrm{Per}(f) \cap \{1,2\} \neq \emptyset$. So statement (d) is proved.

Assume now that $\mathbb{X} = \mathbb{M}_{g,b}$ with $b > 0$, $2g+b-1 \geq 3$, $E_1(A) = 1$, $E_2(A) = 0$ and k is the smallest integer of the set $\{3, 4, ..., 2g+b-1\}$ such that $E_k(A) \neq 0$. Therefore

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{L(f^n)}{n} t^n &= \log\left(\frac{1-t+(-1)^k E_k(A)t^k + ... + (-1)^{b-1}E_{2g+b-1}(A)t^{2g+b-1}}{1-t}\right)\\ &= \log\left(1+\frac{(-1)^k E_k(A)t^k + ... + (-1)^{b-1}E_{2g+b-1}(A)t^{2g+b-1}}{1-t}\right)\\ &= (-1)^k E_k(A)t^k + O(t^{k+1}).\end{aligned}$$

Hence, $L(f) = ... = L(f^{k-1}) = 0$ and $L(f^k) = (-1)^k k E_k(A) \neq 0$. So, from Theorem 2, it follows the statement (e).

Suppose that $\mathbb{X} = \mathbb{N}_{g,b}$ with $b \geq 0$, $g+b-1 \geq 3$, $E_1(A) = 1$, $E_2(A) = 0$ and k is the smallest integer of the set $\{3, 4, ..., g+b-1\}$ such that $E_k(A) \neq 0$. Therefore

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{L(f^n)}{n} t^n &= \log\left(\frac{1-t+(-1)^k E_k(A)t^k + ... + (-1)^{g+b-1}E_{g+b-1}(A)t^{g+b-1}}{1-t}\right)\\ &= \log\left(1+\frac{(-1)^k E_k(A)t^k + ... + (-1)^{g+b-1}E_{g+b-1}(A)t^{g+b-1}}{1-t}\right)\\ &= (-1)^k E_k(A)t^k + O(t^{k+1}).\end{aligned}$$

Again $L(f) = ... = L(f^{k-1}) = 0$ and $L(f^k) = (-1)^k k E_k(A) \neq 0$. Therefore, from Theorem 2, it follows the statement (f). ☐

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[1] Departamento de Matemática Aplicada y Estadística. Universidad Politécnica de Cartagena, Hospital de Marina, 30203-Cartagena, Región de Murcia, Spain.
E-mail address: juan.garcia@upct.es

[2] Departament de Matemàtiques. Universitat Autònoma de Barcelona, Bellaterra, 08193-Barcelona, Catalonia, Spain
E-mail address: jllibre@mat.uab.cat