PERIODS OF CONTINUOUS MAPS ON SOME COMPACT SPACES

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Abstract. The objective of this paper is to provide information on the set of periodic points of a continuous self–map defined in the following compact spaces: $\mathbb{S}^n$ (the n–dimensional sphere), $\mathbb{S}^n \times \mathbb{S}^m$ (the product space of the n–dimensional with the m–dimensional spheres), $\mathbb{C}P^n$ (the n–dimensional complex projective space) and $\mathbb{H}P^n$ (the n–dimensional quaternion projective space). We use as main tool the action of the map on the homology groups of these compact spaces.

1. Introduction

Let $f : \mathbb{X} \to \mathbb{X}$ be a continuous map on a compact space $\mathbb{X}$. A point $x \in \mathbb{X}$ is *periodic of period* n if $f^n(x) = x$ and $f^k(x) \neq x$ for $k = 1, \ldots, n-1$. We denote by $\mathrm{Per}(f)$ the *set of periods* of all periodic points of f. The aim of the present paper is to provide some information on $\mathrm{Per}(f)$ for some compact spaces. More precisely, we shall present results for the spaces $\mathbb{X} \in \Delta$, where Δ is the set formed by the spaces: $\mathbb{S}^n$ (the n–dimensional sphere), $\mathbb{S}^n \times \mathbb{S}^m$ (the product space of the n–dimensional with the m–dimensional spheres), $\mathbb{C}P^n$ (the n–dimensional complex projective space) and $\mathbb{H}P^n$ (the n–dimensional quaternion projective space).

The statement of our main results are the following ones.

Theorem 1. *Let f be a continuous self–map on $\mathbb{S}^n$ of degree D. Then the following statements hold.*

(a) *If n is even and $D = -1$, then $\mathrm{Per}(f) \cap \{1, 2\} \neq \emptyset$.*
(b) *If n is odd and $D \neq 1$, then $\mathrm{Per}(f) \cap \{1\} \neq \emptyset$.*

Theorem 2. *Let f be a continuous self–map on $\mathbb{S}^n \times \mathbb{S}^n$ of degree D, and let $f_{*n} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in \mathbb{Z}$, the action of f on the n–th homology group $H_n(\mathbb{S}^n \times \mathbb{S}^n, \mathbb{Q}) \approx \mathbb{Q} \oplus \mathbb{Q}$. Then the following statements hold.*

(a) *Assume n is even.*
  (a.1) *If $1 + a - d + D \neq 0$, then $\mathrm{Per}(f) \cap \{1\} \neq \emptyset$.*
  (a.2) *If $1 + a - d + D = 0$ and $1 + a^2 + 2bc + d^2 + D^2 \neq 0$, then $\mathrm{Per}(f) \cap \{1, 2\} \neq \emptyset$.*

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(a.3) *If* $1 + a - d + D = 1 + a^2 + 2bc + d^2 + D^2 = 0$ *and* $1 + a^3 + 3abc + 3bcd + d^3 + D^3 \neq 0$ *then* $\mathrm{Per}(f) \cap \{1, 3\} \neq \emptyset$.
(a.4) *If* $D \neq -1$, *then* $\mathrm{Per}(f) \cap \{1, 2, 3\} \neq \emptyset$.
(a.5) *If* $D = -1$, *then* $\mathrm{Per}(f) \cap \{1, 2, 3, 4\} \neq \emptyset$.

(b) *Assume* n *is odd.*
(b.1) *If* $1 - a - d + D \neq 0$, *then* $\mathrm{Per}(f) \cap \{1\} \neq \emptyset$.
(b.2) *If* $1 - a - d + D = 0$ *and* $1 - a^2 - 2bc - d^2 + D^2 \neq 0$, *then* $\mathrm{Per}(f) \cap \{1, 2\} \neq \emptyset$.

Theorem 3. *Let* $f : \mathbb{S}^n \times \mathbb{S}^m \to \mathbb{S}^n \times \mathbb{S}^m$ *with* $n \neq m$ *be a continuous map of degree* D, $f_{*n} = (a)$, $f_{*m} = (b)$ *with* $a, b \in \mathbb{Z}$. *Here for* $k = n, m$, f_{*k} *denotes the action on the* k*–th homology group* $H_k(\mathbb{S}^n \times \mathbb{S}^m, \mathbb{Q}) \approx \mathbb{Q}$. *Then the following statements hold.*

(a) *Assume that* n *and* m *are even.*
(a.1) *If* $(a, b) \neq (-1, -1)$, *then* $\mathrm{Per}(f) \cap \{1\} \neq \emptyset$.
(a.2) *If* $(a, b) = (-1, -1)$, *then* $\mathrm{Per}(f) \cap \{1, 2\} \neq \emptyset$.

(b) *Assume that* n *and* m *are odd. Then* $(a, b) = (1, 1)$, *then* $\mathrm{Per}(f) \cap \{1\} \neq \emptyset$.

(c) *Assume that* n *is odd and* m *is even. Then* $(a, b) = (1, -1)$, *then* $\mathrm{Per}(f) \cap \{1\} \neq \emptyset$.

Theorem 4. *Let* $f : \mathbb{X} \to \mathbb{X}$ *be a continuous map and let* $\mathbb{X}$ *be either* $\mathbb{C}P^n$ *or* $\mathbb{H}P^n$. *If* f_{*k} *denotes the action on the* k*–th homology group* $H_k(\mathbb{X}, \mathbb{Q}) \approx \mathbb{Q}$, *with* $k = 2n$ *if* $\mathbb{X} = \mathbb{C}P^n$, *and* $k = 4n$ *if* $\mathbb{X} = \mathbb{H}P^n$ *the actions* $f_{*2n} = f_{*4n} = (a^n)$.

(a) *If* n *is odd and* $a = -1$, *then* $\mathrm{Per}(f) \cap \{1, 2\} \neq \emptyset$.
(b) *If the assumptions of statement (a) do not hold, then* $\mathrm{Per}(f) \cap \{1\} \neq \emptyset$.

These four theorems are proved in the next section.

Similar results on the spaces of Δ for C^1 self-maps where stated in [5], for transversal self-maps in [10] and [6] except for the case of the n–dimensional sphere which is monographically studied in [8]. Also results of the same kind for continuous–self maps on compact surfaces where obtained in [7].

2. Proofs of Theorems 1, 2, 3 and 4

Assume that $\mathbb{X} \in \Delta$ with dimension n and let $f : \mathbb{X} \to \mathbb{X}$ be a continuous map, there exist $n + 1$ induced linear maps $f_{*k} : H_k(\mathbb{X}, \mathbb{Q}) \to H_k(\mathbb{X}, \mathbb{Q})$ for $k = 0, 1, \ldots, n$ by f. Every linear map f_{*k} is given by an $n_k \times n_k$ matrix with integer entries, where n_k is the dimension of $H_k(\mathbb{X}, \mathbb{Q})$.

In this setting is defined the *Lefschetz number* $L(f)$ as

$$L(f) = \sum_{k=0}^{n} (-1)^k \mathrm{trace}(f_{*k}).$$

The importance of this notion is given by the existence of a result connecting the algebraic topology with the fixed point theory called the *Lefschetz Fixed*

Point Theorem which establishes the existence of a fixed point if $L(f) \neq 0$, see for instance [1].

Since our aim is to obtain information on the set of periods of f for continuous self–maps of Δ, it is useful to have information on the whole sequence $\{L(f^m)\}_{m=0}^{\infty}$ of the Lefschetz numbers of all iterates of f. Thus we define the *Lefschetz zeta function* of f as

$$\mathcal{Z}_f(t) = \exp\left(\sum_{k=1}^{\infty} \frac{L(f^k)}{k} t^k\right). \tag{1}$$

This function generates the whole sequence of Lefschetz numbers, and it may be independently computed through

$$\mathcal{Z}_f(t) = \prod_{k=0}^{n} \det(I_{n_k} - t f_{*k})^{(-1)^{k+1}}, \tag{2}$$

where I_{n_k} is the $n_k \times n_k$ identity matrix, and we take $\det(I_{n_k} - t f_{*k}) = 1$ if $n_k = 0$. Note that the expression (2) is a rational function in t. So the information on the infinite sequence of integers $\{L(f^k)\}_{k=0}^{\infty}$ is contained in two polynomials with integer coefficients, for more details see [3].

In short the Lefschetz zeta function is a good tool for studying the existence of periodic points and we shall see here, and also for studying the non existence of such points as was shown in [4, 9].

Proof of Theorem 1. For $n \geq 1$ let $f : \mathbb{S}^n \to \mathbb{S}^n$ be a continuous map. The homological groups of $\mathbb{S}^n$ over $\mathbb{Q}$ and the induced linear maps are of the form

$$H_q(\mathbb{S}^n, \mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } q \in \{0, n\}, \\ 0 & \text{otherwise,} \end{cases}$$

where $f_{*0} = (1)$, $f_{*i} = (0)$ for $i = 1, ..., n-1$ and $f_{*n} = (D)$ where D is the degree of the map f, see for more details [2].

From (2) we have that

$$\mathcal{Z}_f(t) = \frac{(1 - Dt)^{(-1)^{n+1}}}{1 - t}.$$

If n is even, then

$$\sum_{k=1}^{\infty} \frac{L(f^k)}{k} t^k = \sum_{k=1}^{\infty} \frac{1 + D^k}{k} t^k,$$

so from (1) we get $L(f^k) = 1 + D^k$. Hence if $D = -1$, $L(f^k) = 0$ if k is odd and $L(f^k) = 2$ if k is even. Consequently, by the Lefschetz fixed point theorem we have $\mathrm{Per}(f) \cap \{1, 2\} \neq \emptyset$. This proves statement (a).

If n is odd, then

$$\sum_{k=1}^{\infty} \frac{L(f^k)}{k} t^k = \sum_{k=1}^{\infty} \frac{1 - D^k}{k} t^k,$$

so from (1) we get $L(f^k) = 1 - D^k$. Hence, if $D = 1$ then $L(f^k) = 0$ for $k \geq 1$, if $D = -1$ then $L(f^k) = 0$ if k is even and $L(f^k) = 2$ if k is odd, if $D \neq \pm 1$ then $L(f^k) \neq 0$ for $k \geq 1$. Consequently, by the Lefschetz fixed point theorem we have $\mathrm{Per}(f) \cap \{1\} \neq \emptyset$. This proves statement (b). $\square$

Proof of Theorem 2. Let f be a continuous self–map of $\mathbb{S}^n \times \mathbb{S}^n$. We know that the induced linear maps are $f_{*0} = (1)$, $f_{*n} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in \mathbb{Z}$, $f_{*2n} = (D)$, where D is the degree of the map f and $f_{*i} = (0)$ for $i \in \{0, ..., 2n\}$, $i \neq 0, n, 2n$ (see for more details [2]). From (2) the Lefschetz zeta function of f is

$$\mathcal{Z}_f(t) = \frac{p(t)^{(-1)^{n+1}}}{(1-t)(1-Dt)}. \tag{3}$$

where $p(t) = 1 - (a + d)t + (ad - bc)t^2$.

If n even from the definition of the Lefschetz zeta function in (1) in (3) we have that

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{L(f^k)}{k} t^k = & \log\left(\frac{1}{(1-(a+d)t+(ad-bc)t^2)(1-t)(1-Dt)}\right) \\
= & (1 + a + d + D)t + \frac{1}{2}(1 + a^2 + 2bc + d^2 + D^2)t^2 \\
& + \frac{1}{3}(1 + a^3 + 3abc + 3bcd + d^3 + D^3)t^3 \\
& + \frac{1}{4}(1 + a^4 + 4a^2bc + 2b^2c^2 + 4abcd + 4bcd^2 + d^4 + D^4)t^4 + \dots
\end{aligned}$$

If $L(f) = 1 + a + d + D \neq 0$, then $\mathrm{Per}(f) \cap \{1\} \neq \emptyset$, and statement (a.1) is proved. If $L(f) = 0$ and $L(f^2) = 1 + a^2 + 2bc + d^2 + D^2 \neq 0$, then $\mathrm{Per}(f) \cap \{1, 2\} \neq \emptyset$, and it follows statement (a.2). If $L(f) = L(f^2) = 0$ and $L(f^3) = 1 + a^3 + 3abc + 3bcd + d^3 + D^3 \neq 0$, then $\mathrm{Per}(f) \cap \{1, 3\} \neq \emptyset$, proving statement (a.3). If $L(f) = L(f^2) = L(f^3) = 0$, then $D = -1$ consequently statement (a.4) is proved. If $D = -1$, the system $L(f) = L(f^2) = L(f^3) = L(f^4) = 0$ has no solutions in the variables a, b, c, d and statement (a.5) is proved.

If n is odd, from (1) and (3) we have that

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{L(f^k)}{k} t^k = & \log\left(\frac{1-(a+d)t+(ad-bc)t^2}{(1-t)(1-Dt)}\right) \\
= & (1 - a - d + D)t + \frac{1}{2}(1 - a^2 - 2bc - d^2 + D^2)t^2 \\
& + \frac{1}{3}(1 - a^3 - 3abc - 3bcd - d^3 + D^3)t^3 \dots
\end{aligned}$$

We note that if $L(f) = 1-a-d+D = 0$ and $L(f^2) = 1-a^2-2bc-d^2+D^2 = 0$, then $\mathcal{Z}_f(t) = 0$ and in this case we do not have information on the periods of the map f.

If $L(f) \neq 0$, then statement (b.1) follows. If $L(f) = 0$ and $L(f^2) \neq 0$, then $\mathrm{Per}(f) \cap \{1, 2\} \neq \emptyset$ and statement (b.2) is proved. This completes the proof of the theorem. $\square$

Proof of Theorem 3. Let f be a continuous self–map of $\mathbb{S}^n \times \mathbb{S}^m$ with $n \neq m$. It is known that the induced linear maps are $f_{*0} = (1)$, $f_{*n} = (a)$, $f_{*m} = (b)$ with $a, b \in \mathbb{Z}$, $f_{*n+m} = (D)$, where $D \in \mathbb{Z}$ is the degree of the map f and $f_{*i} = (0)$ for $i \in \{0, ..., n+m\}$, $i \neq 0, n, m, n+m$ (see for more details [2]).

By Poincaré duality, or again by a direct consideration with the cup-product, we have $\deg(f) = D = ab$, see [11].

From (2) the Lefschetz zeta function of f is of the form

$$\mathcal{Z}_f(t) = \frac{(1-at)^{(-1)^{n+1}}(1-bt)^{(-1)^{m+1}}(1-abt)^{(-1)^{n+m+1}}}{1-t}. \tag{4}$$

Let f be an orientation preserving homeomorphism, n and m even. Therefore the degree D of f is 1. By (1) and (4) we have that

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{L(f^k)}{k} t^k = &\quad \log\left(\frac{1}{(1-t)(1-at)(1-bt)(1-abt)}\right) \\ = &\quad \sum_{k=1}^{\infty} \frac{1 + a^k + b^k + a^k b^k}{k} t^k\end{aligned}$$

Therefore, $L(f^k) = 1 + a^k + b^k + a^k b^k$. If $(a, b) \neq (-1, -1)$, then $L(f) \neq 0$ and this proves statement (a.1). If $(a, b) = (-1, -1)$, then $L(f^2) = 2$ and $\mathrm{Per}(f) \cap \{1, 3\} \neq \emptyset$ proving statement (a.2).

Assume that n and m even. Therefore, from the definition of the Lefschetz zeta function (1) and (4) we have that

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{L(f^k)}{k} t^k = &\quad \log\left(\frac{(1-at)(1-bt)}{(1-t)(1-abt)}\right) \\ = &\quad \sum_{k=1}^{\infty} \frac{1 - a^k - b^k + a^k b^k}{k} t^k\end{aligned}$$

Therefore, $L(f^k) = 1 - a^k - b^k + a^k b^k$. If $(a, b) \neq (1, 1)$, then $L(f) \neq 0$ and this shows statement (b). If $(a, b) = (1, 1)$, then $L(f^k) = 0$ for $k \geq 1$.

Assume that n odd and m even. Therefore, from the definition of the Lefschetz zeta function (1) and (4) we have that

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{L(f^k)}{k} t^k = &\quad \log\left(\frac{(1-at)(1-abt)}{(1-t)(1-bt)}\right) \\ = &\quad \sum_{k=1}^{\infty} \frac{1 - a^k + b^k - a^k b^k}{k} t^k\end{aligned}$$

Therefore, $L(f^k) = 1 - a^k + b^k - a^k b^k$. If $(a, b) \neq (1, -1)$, then $L(f) \neq 0$ and this shows statement (c). If $(a, b) = (1, -1)$, then $L(f^k) = 0$ for $k \geq 1$, ending the proof of the theorem. $\square$

Proof of Theorem 4. Let f be a continuous self–map of $\mathbb{C}P^n$ with $n \geq 1$. We know that the induced linear maps are $f_{*q} = (a^{\frac{q}{2}})$ for $q \in \{0, 2, 4, ..., 2n\}$ with $a \in \mathbb{Z}$, and $f_{*q} = (0)$ otherwise (see for more details [12, Corollary 5.28]).

From (2) the Lefschetz zeta function of f has the form

$$\mathcal{Z}_f(t) = \left(\prod_q (1 - a^{q/2} t) \right)^{-1}, \tag{5}$$

where q runs over $\{0, 2, 4, ..., 2n\}$.

Let f be a continuous self–map of $\mathbb{H}P^n$ with $n \geq 1$. We know that the induced linear maps are $f_{*q} = (a^{\frac{q}{4}})$ for $q \in \{0, 4, 8, ..., 4n\}$ with $a \in \mathbb{Z}$, and $f_{*q} = (0)$ otherwise (see for more details [12, Corollary 5.33]).

From (2) the Lefschetz zeta function of f has the form

$$\mathcal{Z}_f(t) = \left(\prod_q (1 - a^{q/4} t) \right)^{-1}, \tag{6}$$

where q runs over $\{0, 4, 8, ..., 4n\}$.

By (1), (5) and (6) we have that

$$\sum_{k=1}^{\infty} \frac{L(f^k)}{k} t^k = \sum_{k=1}^{\infty} \frac{a^{k(n+1)-1}}{a^k - 1} t^k$$

Therefore $L(f^k) = \dfrac{a^{k(n+1)-1}}{a^k - 1}$. Hence it is easy to check first that $L(f) = 0$ if and only if n is odd and $a = -1$, and second that $L(f^2) = 1 + a^2 + \ldots + a^{2n} \neq 0$. From these two facts the statements (a) and (b) follow. □

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