ON THE PERIODS OF A CONTINUOUS SELF–MAP ON A GRAPH

JUAN LUIS GARCÍA GUIRAO[1] AND JAUME LLIBRE[2]

ABSTRACT. Let G be a graph and f be a continuous self–map on G. We present new and known results (from another point of view) on the periods of the periodic orbits of f using mainly the action of f on its homology, or the shape of the graph G.

1. Introduction and statement of the main results

A *discrete dynamical system* (G, f) is formed by a continuous map $f : G \to G$ where G is a topological space.

A point $x \in G$ is *periodic* of *period* k if $f^k(x) = x$ and $f^i(x) \neq x$ if $0 < i < k$. If $k = 1$, then x is called a *fixed point*. $\text{Per}(f)$ denotes the *set of periods* of all the periodic points of f.

The *orbit* of the point $x \in G$ is the set $\{x, f(x), f^2(x), \ldots, f^n(x), \ldots\}$ where by f^n we denote the composition of f with itself n times. To knowledge the behavior of all different kind of orbits of f is to study *the dynamics of the map* f.

Many times the periodic points play an important role for understanding the dynamics of a discrete dynamical system. One of the best known results in this direction is the paper *Period three implies chaos* for continuous interval maps, see [8].

Here a *graph* G is a compact connected space containing a finite set V such that $G \setminus V$ has finitely many open connected components, each one homeomorphic to the interval $(0, 1)$, called *edges* of G, and the points of V are called the *vertexes* of G. The edges are disjoint from the vertexes, and the vertexes are at the boundary of the edges.

In this paper we shall work with a graph G. Our goal is to study the periods of the periodic points of the continuous maps $f : G \to G$.

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The *degree* of a vertex V of a graph G is the number of edges having V in its boundary, if an edge has both boundaries in V then we count this edge twice. An *endpoint* of a graph G is a vertex of degree one. A *branching point* of a graph G is a vertex of degree at least three.

The homological spaces of G with coefficients in $\mathbb{Q}$ are denoted by $H_k(G, \mathbb{Q})$. Since G is a graph $k = 0, 1$. A continuous map $f : G \to G$ induces linear maps $f_{*k} : H_k(G, \mathbb{Q}) \to H_k(G, \mathbb{Q})$. We only work with graphs, so $H_0(G, \mathbb{Q}) \approx \mathbb{Q}$ and f_{*0} is the identity map. A subset of G homeomorphic to a circle is a *circuit*. It is known that $H_1(G, \mathbb{Q}) \approx \mathbb{Q}^m$ being m the number of the independent circuits of G in the sense of the homology. Here f_{*1} is a $m \times m$ matrix A with integer entries. For more details on this homology see for instance [12].

If A be a $m \times m$ matrix, then a submatrix lying in the same set of k rows and columns is a $k \times k$ *principal submatrix* of A. The determinant of a principal submatrix is a $k \times k$ *principal minor*. The sum of the $\binom{n}{k}$ different $k \times k$ principal minors of A is denoted by $E_k(A)$. Note that $E_m(A)$ is the determinant of A and $E_1(A)$ is the trace of A. Of course the characteristic polynomial of A is given by

$$\det(tI - A) = t^m - E_1(A)t^{m-1} + E_2(A)t^{m-2} - \ldots + (-1)^m E_m(A). \tag{1}$$

The biggest modulus of the eigenvalues of the matrix A is called the *spectral radius* of A and it is denoted by $\mathrm{sp}(A)$.

Our main results are the following ones.

Theorem 1. *Let G be a graph, $f : G \to G$ be a continuous map, and A be the integral matrix of the endomorphism $f_{*1} : H_1(G, \mathbb{Q}) \to H_1(G, \mathbb{Q})$ induced by f on the first homology group of G. The following statements hold.*

(a) *If $E_1(A) \neq 1$, then $1 \in \mathrm{Per}(f)$.*
(b) *If $E_1(A) = 1$ and $E_2(A) \neq 0$, then $\mathrm{Per}(f) \cap \{1, 2\} \neq \emptyset$.*
(c) *If $E_1(A) = 1$, $E_2(A) = ... = E_{k-1}(A) = 0$ and $E_k(A) \neq 0$ for $k = 3, \ldots, m$ then $\mathrm{Per}(f)$ intersection the set of all the divisors of k is not empty.*

Theorem 1 is proved in section 2 using the Lefschetz fixed point theory. The next result is an immediate consequence of Theorem 1.

Corollary 2. *Under the assumptions of Theorem 1, if the characteristic polynomial of the matrix A is different from $1 - t$, then $\mathrm{Per}(f) \cap \{1, 2, \ldots, m\} \neq \emptyset$.*

Let k be a positive integer we denote by $\mathrm{god}(k)$ the *greatest odd divisor* of k. Let S be a subset of positive integer, the *pantheon* of S is the set $\{god(k) : k \in S\}$.

Theorem 3. *Let G be a graph, $f : G \to G$ be a continuous map, and A be the integral matrix of the endomorphism $f_{*1} : H_1(G, \mathbb{Q}) \to H_1(G, \mathbb{Q})$ induced by f on the first homology group of G. If $\mathrm{sp}(f_{*1}) > 1$, then f has infinitely many periods. More precisely, there is an $n \in \mathbb{N}$ such that $\{kn : k \in \mathbb{N}\} \subset \mathrm{Per}(f)$ and the pantheon of $\mathrm{Per}(f)$ is infinite.*

Theorem 4. *Let G be a graph with v vertexes, e endpoints, s edges and at least one branching point. Let $f : G \to G$ be a continuous map having all the branching point of G fixed. If for some period n of f, $\mathrm{god}(n) > e + 2s - 2v + 2$ then f has infinitely many periods. More precisely, there is an $n \in \mathbb{N}$ such that $\{kn : k \in \mathbb{N}\} \subset \mathrm{Per}(f)$ and the pantheon of $\mathrm{Per}(f)$ is infinite.*

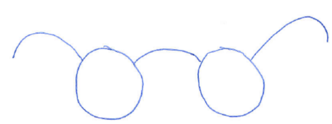

FIGURE 1. The glasses graph.

From Theorem 4 we can deduce many results similar to the one given in the seminal paper *Period three implies chaos* for self–continuous maps on the interval in the sense of having infinitely many periods.

Corollary 5. *The following map f have infinitely many periods if:*

(a) *f is a continuous self-map on the graph having the shape of the letter Y with the branching point fixed and having a period n such that $\mathrm{god}(n) > 3$;*
(b) *f is a continuous self-map on the graph having the shape of the number 8 or on the graph having the shape of the letter θ with the branching points fixed and having a period n such that $\mathrm{god}(n) > 4$;*
(c) *f is a continuous self-map on the glasses graph having the shape of the graph described in Figure 1 with the branching points fixed and having a period n such that $\mathrm{god}(n) > 2$.*

Theorems 3, 4 and Corollary 5 are proved in section 3.

We note that this paper is a kind of survey with new results. That is, Theorem 1 is completely new, but Theorems 3 and 4 essentially follow combining known results on the continuous self-maps on graphs as we will see in their proofs.

2. Proof of Theorem 1

Let $f : G \to G$ be a continuous map on the graph G. The *Lefschetz number* of f is defined by

$$L(f) = \mathrm{trace}(f_{*0}) - \mathrm{trace}(f_{*1}).$$

The Lefschetz Fixed Point Theorem states: *If* $L(f) \neq 0$ *then* f *has a fixed point* (see for instance [5]).

In order to control the whole sequence of the Lefschetz numbers of the iterates of f, i.e. $\{L(f^n)\}_{n\geq 1}$, we use the formal *Lefschetz zeta function* of f defined by

$$Z_f(t) = \exp\left(\sum_{n=1}^{\infty} \frac{L(f^n)}{n} t^n\right). \tag{2}$$

It is known that for a continuous self–map of a graph G the Lefschetz zeta function is the rational function

$$Z_f(t) = \frac{\det(I - t f_{*1})}{\det(I - t f_{*0})} = \frac{\det(I - tA)}{1-t}, \tag{3}$$

where A is the integer matrix defined by f_{*1}, for a proof see Franks [6].

Since $\det(I - tA) = t^n \det\left(\frac{1}{t} I - A\right)$, from (1) we get

$$\det(I - tA) = 1 - E_1(A)t + E_2(A)t^2 - \ldots + (-1)^m E_m(A) t^m.$$

From (2) and (3) we obtain

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{L(f^n)}{n} t^n &= \log(Z_f(t)) \\
&= \log\left(\frac{\det(I - tA)}{1-t}\right) \\
&= \log\left(\frac{1 - E_1(A)t + E_2(A)t^2 - \ldots + (-1)^m E_m(A)t^m}{1-t}\right) \\
&= (1 - E_1(A))t + \frac{1}{2}\big(1 - E_1(A)^2 + 2E_2(A)\big)t^2 \\
&= \frac{1}{3}\big(1 - E_1(A)^3 + 3E_1(A)E_2(A) - 3E_3(A)\big)t^3 + O(t^4).
\end{aligned} \tag{4}$$

So we have that

$$L(f) = 1 - E_1(A), \quad \text{and} \quad L(f^2) = 1 - E_1(A)^2 + 2E_2(A).$$

Therefore if $E_1(A) \neq 1$ than $L(f) \neq 0$ and by the Lefschetz Fixed Point Theorem statement (a) is proved. If $E_1(A) = 1$ and $E_2(A) \neq 0$, then $L(f^2) = 2E_2(A) \neq 0$, again by the Lefschetz Fixed Point Theorem statement (b) follows.

Working with the expression (4) when $E_1(A) = 1$, $E_2(A) = ... = E_{k-1}(A) = 0$ and $E_k(A) \neq 0$ for $k = 3, \ldots, m$ we obtain that $L(f^k) = (-1)^k k E_k(A) \neq 0$, hence by the Lefschetz Fixed Point Theorem statement (c) is proved. This completes the proof of Theorem 1.

3. Proof of Theorems 3, 4 and Corollary 5

Let G be a graph, and let $f : G \to G$ bea continuous map. One way of measuring the complexity of the dynamics of the map f is through the notion of topological entropy. Here we introduce the topological entropy using the definition of Bowen [4].

Since a graph G is a subset of $\mathbb{R}^3$, we consider the distance between two points of G as the distance of these two points in $\mathbb{R}^3$. Now, we define the distance d_n on G by

$$d_n(x, y) = \max_{0 \le i \le n} d(f^i(x), f^i(y)), \quad \forall x, y \in G.$$

A finite set S is called (n, ε)*–separated with respect to* f if for different points $x, y \in S$ we have $d_n(x, y) > \varepsilon$. We denote by S_n the maximal cardinality of an (n, ε)–separated set. Define

$$h(f, \varepsilon) = \limsup_{n \to \infty} \frac{1}{n} \log S_n.$$

Then

$$h(f) = \lim_{\varepsilon \to 0} h(f, \varepsilon).$$

is the *topological entropy* of f.

We have given the definition of Bowen because probably is the shorter one, the classical definition was due to Adler, Konheim and Mc Andrew [1]. See for instance the book of Hasselblatt and Katok [7] and [2] for other equivalent definitions and properties of the topological entropy.

The next result is due to Manning [11].

Theorem 6. *Let* $f : G \to G$ *a continuous map on the graph* G*, then* $\log \max\{1, \mathrm{sp}(f_{*,1})\} \le h(f)$.

There are two different proofs for the next result, see [9] and [3]:

Theorem 7. *Let $f : G \to G$ a continuous map on the graph G. Then the following statements are equivalent:*

(a) *There is an $m \in \mathbb{N}$ such that $\{mn : n \in \mathbb{N}\} \subset \mathrm{Per}(f)$.*
(b) *$h(f) > 0$.*
(c) *The pantheon of $\mathrm{Per}(f)$ is infinite.*

Proof of Theorem 3. Since $\mathrm{sp}(f_{*,1}) > 1$ by Theorem 6 we have that $h(f) > 0$. Then by Theorem 7 Theorem 3 follows. □

The following result can be found in [10].

Theorem 8. *Let $f : G \to G$ a continuous map on the graph G having e endpoints, s edges, v vertexes and at least one branching point. Assume that f has all branching points fixed. Then $\mathrm{god}(n) > e + 2s - 2v + 2$ for some period n of f if and only if $h(f) > 0$.*

Proof of Theorem 4. Under the assumptions of Theorem 4 we have that $\mathrm{god}(n) > e+2s-2v+2$ for some period n of f, so $h(f) > 0$ by Theorem 8. Again by Theorem 7 Theorem 4 is proved. □

Proof of Corollary 5. The proof is a direct consequence of the application of Theorem 4 taking account that $e + 2s - 2v + 2$ is respectively equal to 3 ($e = 3$, $v = 4$ and $s = 3$) for the graph Y; equal to 4 for the graph 8 ($e = 0$, $v = 1$ and $s = 2$) and for the graph θ ($e = 0$, $v = 2$ and $s = 3$), and 2 for the glasses graph ($e = 2$, $v = 6$ and $s = 7$). □

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References

[1] R.L. Adler, A.G. Konheim and M.H. McAndrew, *Topological entropy*, Trans. Amer. Math. Soc. **114** (1965), 309–319.
[2] F. Balibrea, *On problems of Topological Dynamics in non-autonomous discrete systems*, Applied Mathematics and Nonlinear Sciences **1(2)** (2016), 391–404.
[3] A.M. Blokh, *The Spectral Decomposition, Periods of Cycles and Misiurewicz Conjecture for Graph Maps*, 24–31, Lecture Notes in Math., Springer **1514**, 1991.

[4] R. Bowen, *Entropy for group endomorphisms and homogeneous spaces* Trans. Amer. Math. Soc **153** (1971), 401–414; erratum: Trans. Amer. Math. Soc. **181** (1973), 509–510.
[5] R.F. Brown, *The Lefschetz fixed point theorem*, Scott, Foresman and Company, Glenview, IL, 1971.
[6] J. Franks, *Homology and dynamical systems*, CBSM Regional Conf. Ser. in Math. **49**, Amer. Math. Soc., Providence, R.I. 1982.
[7] B. Hasselblatt and A. Katok, *Handbook of dynamical systems*, Vol. 1A. North–Holland, Amsterdam, 2002.
[8] T.Y. Li and J. Yorke, *Period three implies chaos*, Amer. Math. Monthly **82** (1975), 985–992.
[9] J. Llibre and M. Misiurewicz, *Horseshoes, entropy and periods for graph maps*, Topology **52** (1993), l49–164.
[10] J. Llibre and R. Saghin, *Topological entropy and periods of graph maps*, J. Difference Equ. Appl. **18** (2012), 589–598.
[11] A. Manning, *Topological entropy and the first homology group*, in Dynamical systems - Warwick 1974, Lecture Notes in Math **468**, Springer-Verlag, Berlin, 1975, 185–190.
[12] E.H. Spanier, *Algebraic Topology*, Springer–Berlag, New York (1981).

[1] Departamento de Matemática Aplicada y Estadística. Universidad Politécnica de Cartagena, Hospital de Marina, 30203-Cartagena, Región de Murcia, Spain–Corresponding Author–

E-mail address: `juan.garcia@upct.es`

[2]Departament de Matemàtiques. Universitat Autònoma de Barcelona, Bellaterra, 08193-Barcelona, Catalonia, Spain

E-mail address: `jllibre@mat.uab.cat`