Generalized van der Waals Hamiltonian: Periodic orbits and C^1 nonintegrability

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The aim of this paper is to study the periodic orbits of the generalized van der Waals Hamiltonian system. The tool for studying such periodic orbits is the averaging theory. Moreover, for this Hamiltonian system we provide information on its $\mathcal{C}^1$ nonintegrability, i.e., on the existence of a second first integral of class $\mathcal{C}^1$.

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I. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

We study the generalized van der Waals problem given by the Hamiltonian

$$\mathcal{H} = \frac{1}{2}\left(P_1^2 + P_2^2 + P_3^2\right) - \frac{1}{\sqrt{Q_1^2 + Q_2^2 + Q_3^2}} + \left(Q_1^2 + Q_2^2 + \beta^2 Q_3^2\right), \quad (1)$$

depending on the parameter $\beta \in \mathbb{R}$. This Hamiltonian is a generalization of the Hamiltonian which studies the classical dynamics of a hydrogen atom in the presence of a uniform magnetic and quadrupolar electric field. With some restrictions the motion of the system is described by a Hamiltonian system with two degrees of freedom. For more details see Refs. [1–4] and the references therein.

Particular cases connected with problems of physical interest are $\beta = 0$ (the Zeeman effect) and $\beta = \sqrt{2}$, which corresponds to the van der Waals effect (see Elipe *et al.* [5] and references therein). For the values $\beta^2 = 1/4$, 1, and 4, the Hamiltonian system is integrable (see Farrelly *et al.* [6] and Ferrer *et al.* [7,8]).

Introducing the canonical change of coordinates given by the cylindrical coordinates $Q_1 = R\cos\theta$, $Q_2 = R\sin\theta$, and $Q_3 = Z$, the Hamiltonian (1) becomes

$$\mathcal{H} = \frac{1}{2}\left(P_R^2 + \frac{P_\theta^2}{R^2} + P_Z^2\right) - \frac{1}{\sqrt{R^2 + Z^2}} + (R^2 + \beta^2 Z^2). \quad (2)$$

Since the momentum P_θ is a first integral of the Hamiltonian system associated to the Hamiltonian (2), this Hamiltonian system can be reduced to a system with two degrees of freedom. The dynamics of the so-called polar problem (see Elipe [5]) is considered when $P_\theta = 0$. In this case the Hamiltonian (2) reduces to

$$\mathcal{H} = \frac{1}{2}\left(P_R^2 + P_Z^2\right) - \frac{1}{\sqrt{R^2 + Z^2}} + (R^2 + \beta^2 Z^2). \quad (3)$$

Our main objective is to prove analytically the existence of periodic solutions of the Hamiltonian system associated to the Hamiltonian (3), and as a corollary to provide information about the $\mathcal{C}^1$ nonintegrability of such a Hamiltonian system.

In this work we use as a main tool the averaging method of first order to find analytically periodic orbits of the Hamiltonian system associated to the Hamiltonian (2) with $P_\theta = 0$. (See the Appendix for more details on the averaging theory; see also some recent applications of this method to other Hamiltonian systems like the ones studied in Refs. [9,10].) One of the main difficulties in practice for applying the averaging method is to express the differential system in the normal form for applying the averaging theory (see the Appendix). The use of adequate variables in each situation can allow the application of the averaging theory for finding periodic orbits.

For the Hamiltonian system associated to the Hamiltonian (2) with $P_\theta = 0$ we have the following results.

Theorem 1. For every $h < 0$ the Hamiltonian system associated to the generalized van der Waals Hamiltonian $\mathcal{H}$ with $P_\theta = 0$ given by Eq. (3) has a periodic solution in the energy level $\mathcal{H} = h + \sqrt{-2/h}$ if $\beta \notin \{\pm 2, \pm 1/2\}$. Moreover, this periodic solution is linear stable if $\beta \in (-\infty, -2) \cup (-1/2, 1/2) \cup (2, \infty)$ and unstable if $\beta \in (-2, -1/2) \cup (2, 1/2)$.

Using the periodic orbits found in Theorem 1 we study the $\mathcal{C}^1$ nonintegrability in the Liouville-Arnold sense of the polar generalized van der Waals Hamiltonians.

Theorem 2. For the generalized van der Waals Hamiltonian $\mathcal{H}$ with $P_\theta = 0$ given by Eq. (3) and $\beta \notin \{\pm 2, \pm 1/2\}$ its associated Hamiltonian system cannot have a $\mathcal{C}^1$ second first integral G such that the gradients of $\mathcal{H}$ and G are linearly independent at each point of the periodic orbits found in Theorem 1.

Many times the study of the periodic orbits of a Hamiltonian system is made numerically. In general to prove analytically the existence of periodic solutions of a Hamiltonian system is a very difficult task, often impossible. Here, with the averaging

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