HAMILTONIAN POLYNOMIAL DIFFERENTIAL SYSTEMS WITH GLOBAL CENTERS IN $\mathbb{R}^2$

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Abstract. We characterize the polynomial Hamiltonian systems having a global center in $\mathbb{R}^2$, and show that the polynomial Hamiltonian systems of degree $n \geq 3$ having a global center can exhibit one of all kinds of center: linear type, nilpotent or degenerate.

In particular we characterize all the cubic polynomial Hamiltonian systems having a degenerate center, and provide an approach using dynamical systems for characterizing when real algebraic curves $H(x, y) = h$ in $\mathbb{R}^2$ are a continuum of ovals varying $h \in \mathbb{R}$.

1. Introduction and statement of the main results

Let $H(x, y)$ be a polynomial of degree $n+1$ in the variables x and y with coefficients in $\mathbb{R}$, then the polynomial differential system

$$(1) \qquad \dot{x} = -\frac{\partial H(x, y)}{\partial y} = P(x, y), \qquad \dot{y} = \frac{\partial H(x, y)}{\partial x} = Q(x, y),$$

is called a *polynomial Hamiltonian system of degree* n *with Hamiltonian* $H(x, y)$, where n is a positive integer, denoted by $n \in \mathbb{N}$.

The notion of center goes back to Poincaré [17] and Dulac [9]. A *center* is an equilibrium point p of system (1) in the plane $\mathbb{R}^2$, which has a neighborhood U such that p is the unique equilibrium in U and $U \setminus \{p\}$ is filled by periodic orbits (closed orbits or ovals) enclosing p. The center p is *global* if $\mathbb{R}^2 \setminus \{p\}$ is filled by periodic orbits.

To characterize the real algebraic curves $H(x, y) = h$ in $\mathbb{R}^2$ having ovals for a continuum of the values of $h \in \mathbb{R}$, is equivalent to characterize the centers of the Hamiltonian system (1) in $\mathbb{R}^2$ with the polynomial Hamiltonian function $H(x, y)$. For a center of system (1) with Hamiltonian $H(x, y)$ we can define its *period function* $T(h)$ as the period of the periodic orbit contained in the curve $H(x, y) = h$.

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In the qualitative theory of the planar differential equations there are three kind of centers: linear type, nilpotent and degenerate, see for instance [13]. More precisely, after moving the center to the origin of coordinates and making a linear change of variables, and a scaling of the time variable (if necessary), the polynomial differential system in $\mathbb{R}^2$ having a center at the origin can be written in one of the following three forms:

$$\dot{x} = -y + X_2(x,y), \qquad \dot{y} = x + Y_2(x,y),$$

which is called a *linear type center* or an *elementary center*;

$$\dot{x} = y + X_2(x,y), \qquad \dot{y} = Y_2(x,y),$$

which is called a *nilpotent center*;

$$\dot{x} = X_2(x,y), \qquad \dot{y} = Y_2(x,y),$$

which is called a *degenerate center*. Where $X_2(x,y)$ and $Y_2(x,y)$ are polynomials starting at least with terms of second order.

A classical and difficult problem in the qualitative theory of planar polynomial differential systems is the characterization of their centers and of the phase portraits of the systems having some center. Over one century mathematicians have completely solved the problem only for the linear and the quadratic polynomial differential systems. Specially, the linear polynomial differential systems have only a linear center which is global and its period function $T(h)$ is constant. The centers of the quadratic polynomial differential systems and their phase portraits have been classified by Bautin [1], Kapteyn [10, 11], Schlomiuk [18], Vulpe [19], Żołądek [22]. It has been shown that the quadratic polynomial differential systems only exhibit linear type centers. Up to now the classification of the centers for cubic polynomial differential systems and of their phase portraits are still unsolved problems. There are many works on the cubic centers for some different subclasses of cubic polynomial differential systems. For example, the linear type centers of the cubic polynomial systems without quadratic terms have been determined by Malkin [16], Vulpe and Sibirskii [20], Żołądek [23] and references therein. The linear type centers and the nilpotent centers of the cubic polynomial Hamiltonian systems without quadratic terms have been determined by Colak, Llibre and Valls [4–7], together with their phase portraits. The classification of reversible cubic polynomial differential systems with a center has been done by Żołądek [24, 25], and Buzzi et al [2]. How many centers, their distribution and of which type for the cubic polynomial Hamiltonian systems having two invariant straight lines which intersect has been done by Llibre and Xiao [15], they proved that such systems only can have linear type centers and nilpotent centers.

Note that all subclasses of cubic polynomial Hamiltonian differential systems with centers mentioned in the previous paragraph do not have degenerate centers. Cima and Llibre in [3] characterized the phase portraits of the centers of the cubic homogeneous polynomial differential system, all these centers are degenerate. Thus, the cubic polynomial differential systems can exhibit the three kind of centers, linear type, nilpotent and degenerate, while linear systems and quadratic systems have only linear type centers. Then several natural questions are: *Can the cubic polynomial Hamiltonian systems have degenerate centers*? And if the answer is positive: *What are the phase portraits of such Hamiltonian systems*? Moreover, *how to determine if all real algebraic curves* $H(x, y) = h$ *in* $\mathbb{R}^2$ *are ovals?* which is equivalent to ask *what conditions can guarantee that polynomial Hamiltonian systems have a global center? Can the polynomial Hamiltonian systems having a global center exhibit centers of the three kinds respectively and what about their period function, can be monotone?*.

Since a translation in the plane transforms a Hamiltonian system in another Hamiltonian system, we can consider the center at the origin of coordinates by doing a translation. Hence, without loss of generality *we assume in the rest of this paper that the origin* $(0, 0)$ *is an equilibrium point of a Hamiltonian system* (1) *with Hamiltonian* $H(x, y)$ *such that* $H(0, 0) = 0$.

In this paper first we study the conditions in order that the Hamiltonian system (1) has a global center at the origin of $\mathbb{R}^2$, and after we answer the above questions. Before stating our main results, we recall that the polynomial differential systems can be extended analytically to the infinity of $\mathbb{R}^2$, this extension is called the *Poincaré compactification*. Roughly speaking, the Poincaré compactification identify $\mathbb{R}^2$ with the interior of the unit disc centered at the origin and its boundary, the circle $\mathbb{S}^1$, with the infinity of $\mathbb{R}^2$, in the plane $\mathbb{R}^2$ we can go to infinity in as many directions as points have $\mathbb{S}^1$. Then the polynomial differential system can be extended to an analytic differential system in the closed unit disc $\mathbb{D}$, i.e. in particular to the infinity $\mathbb{S}^1$. This closed disc $\mathbb{D}$ is called *the Poincaré disc*. The equilibrium points of the extended differential system in $\mathbb{S}^1$ are called *the infinite equilibria* of the initial polynomial differential system, and the infinite equilibria appear on pairs diametrally opposite on $\mathbb{S}^1$. For more details on this Poincaré compactification see for instance Chapter 5 of [8].

Theorem 1. *Polynomial Hamiltonian systems* (1) *of degree* $2n$ *cannot have global centers for any positive integer* n.

Theorem 2. *Assume that polynomial Hamiltonian system* (1) *has the unique finite equilibrium at* $E_0(0,0)$ *in* $\mathbb{R}^2$. *Then* $E_0(0,0)$ *is a global center if and only if either the infinity of system* (1) *is a periodic orbit, or all the infinite equilibrium points have its local phase portrait formed by one hyperbolic sector having its two separatrices at infinity.*

Moreover, the infinity of system (1) *is a periodic orbit if and only if the highest homogeneous part of the Hamiltonian* $H(x,y)$ *has no real linear factors.*

Theorem 2 shows the necessary and sufficient conditions on that the real algebraic curves $H(x,y)=h$ in $\mathbb{R}^2$ are a continuum of ovals varying $h\in\mathbb{R}$.

Proposition 3. *There exist polynomial Hamiltonian systems* (1) *of degree* $2n+1$ *having global centers either of liner type, or nilpotent, or degenerate for every positive integer* n.

Theorems 1 and 2 and Proposition 3 are proved in section 2.

For the polynomial

$$a+bz+cz^2+dz^3+ez^4, \tag{2}$$

we define

$$\begin{aligned} D_2 =&\ 3d^2-8ce,\\ D_3 =&\ c^2d^2-3bd^3-4c^3e+14bcde-6d^2e-18b^2e^2+16ce^2,\\ D_4 =&\ b^2c^2d^2-4c^3d^2-4b^3d^3+18bcd^3-27d^4-4b^2c^3e+\\ &16c^4e+18b^3cde-80bc^2de-6b^2d^2e+144cd^2e-\\ &27b^4e^2+144b^2ce^2-128c^2e^2-192bde^2+256e^3. \end{aligned} \tag{3}$$

The answers to the above first two questions for the Hamiltonian cubic polynomial systems having degenerate centers are in the next theorem.

Theorem 4. *Let* $H(x,y)$ *be a Hamiltonian polynomial of degree* 4. *Then its associated cubic polynomial Hamiltonian system has a degenerate center at the origin of coordinates if and only if the following conditions (a) and (b) hold.*

(a) *After a linear change of variables and a scaling of the time if necessary the Hamiltonian can be written as*

$$H(x,y) = x^4 + bx^3y + cx^2y^2 + dxy^3 + ey^4 \tag{4}$$

(i.e $H(x,y)$ is a homogeneous polynomial) with $e \neq 0$.

(b) *The coefficients of the polynomial $H(x,y)$ given in* (4) *satisfy one of the following conditions:*
 (i) $D_4 > 0$ *and* $D_3 \leq 0$,
 (ii) $D_4 > 0$ *and* $D_2 \leq 0$,
 (iii) $D_4 = D_3 = 0$ *and* $D_2 < 0$,
 where D_i, $i = 2, 3, 4$, *is defined in* (3).

Moreover, (c) *if the Hamiltonian cubic polynomial systems have a degenerate center, then the degenerate center is global and its period function is strictly monotone.*

We identify the set of all cubic polynomial differential systems, or simply *cubic systems*, with the set of points of the parameter space $\mathbb{R}^{20}$, where the components of each point are the 20 coefficients of a cubic polynomial differential system in a given order. Of course, in order to have a cubic system the coefficients of some of its monomials of degree three must be non-zero. Theorem 4 provides the algebraic subsets of $\mathbb{R}^{20}$ of Hamiltonian cubic polynomial systems having degenerate centers, more precisely there are two subsets of dimension four and one subset of dimension two.

Theorem 4 is proved in section 3, and in the last section we give a brief discussion on the period function of a global center, and study how to determine if the real algebraic curves $H(x,y) = h$ are a continuum of ovals varying $h \in \mathbb{R}$. In general this is an interesting and challenging problem. Our results show that the tools of differential equations can provide an approach to characterize this problem.

2. Global centers for the Hamiltonian systems (1)

In this section first we prove Theorems 1 and 2, and finally we prove Proposition 3.

Proof of Theorem 1. Since the degree of the polynomial Hamiltonian systems (1) is $2n$, the degree of the polynomial Hamiltonian function $H(x,y)$ is $2n+1$, which is odd. Then, we claim that near the infinity the polynomial $H(x,y)$ changes sign at the extrema of every straight line through the origin with director vector (u,v) if $H_{2n+1}(u,v)\neq 0$, where $H_{2n+1}(x,y)$ is the homogeneous part of $H(x,y)$ of degree $2n+1$. From the claim it follows that the curves $H(x,y)=h$ (constant) near the infinity cannot be closed, and consequently the Hamiltonian system associated to $H(x,y)$ cannot have a global center.

Now we prove the claim. Define $h(t)=H(tu,tv)$. Note that $h(t)$ is a polynomial of degree $2n+1$ in the variable t, because $h(t)=H_0(u,v)+H_1(u,v)t+\ldots+H_{2n+1}(u,v)t^{2n+1}$, where $H_j(x,y)$ is the homogeneous part of degree j of $H(x,y)$. Then the sign of $h(t)$ when $t\to+\infty$ is opposite to the sign of $h(t)$ when $t\to-\infty$. Therefore the claim is proved. □

From Theorem 1 it follows that the quadratic and in general the even degree polynomial Hamiltonian systems cannot have global centers in $\mathbb{R}^2$.

To prove Theorem 2, we first give the following lemma, which can be proved by the qualitative analysis directly.

Lemma 5. *Assume that Hamiltonian polynomial (or Hamiltonian analytic) system* (1) *in* $\mathbb{R}^2$ *has an isolated equilibrium at* $E_0(0,0)$. *Then the equilibrium* $E_0(0,0)$ *is a center if and only if the index of this equilibrium is one.*

We are now in the position to prove Theorem 2.

Proof of Theorem 2. Let $H(x,y)=\sum_{j=1}^{2n}H_j(x,y)$, where $H_j(x,y)$ is the homogenous part of degree j of $H(x,y)$. Using the Poincaré transformation

$$u(\tau)=\frac{y(t)}{x(t)},\quad v(\tau)=\frac{1}{x(t)},\quad t=\int v^{2n-2}(\tau)d\tau.$$

Then system (1) becomes

$$\begin{aligned}\frac{du(\tau)}{d\tau}&=v^{2n-1}\left(\frac{\partial H}{\partial x}\left(\frac{1}{v},\frac{u}{v}\right)+u\frac{\partial H}{\partial y}\left(\frac{1}{v},\frac{u}{v}\right)\right)\\&=\sum_{j=2}^{2n}jv^{2n-j}H_j(1,u),\\\frac{dv(\tau)}{d\tau}&=v^{2n}\frac{\partial H}{\partial y}\left(\frac{1}{v},\frac{u}{v}\right)=\sum_{j=2}^{2n}v^{2n+1-j}\frac{\partial H_j}{\partial y}(1,u).\end{aligned}\tag{5}$$

Thus, all infinite equilibria of system (1) are determined by the real roots of $H_{2n}(1,u)=0$ except perhaps for one pair of the infinite equilibria, the ones which can be at the endpoints of the y-axis. And $H_{2n}(1,u)=0$ has either no real roots or at most $2n$ isolated real roots because $H_{2n}(x,y)\neq 0$.

On the other hand, now we study if the endpoints of the y-axis are equilibria, let

$$u(\tau)=\frac{x(t)}{y(t)},\ v(\tau)=\frac{1}{y(t)},\ t=\int v^{2n-2}(\tau)d\tau.$$

Then system (1) becomes

$$\begin{aligned}\frac{du(\tau)}{d\tau}&=-v^{2n-1}\left(u\frac{\partial H}{\partial x}\left(\frac{u}{v},\frac{1}{v}\right)+\frac{\partial H}{\partial y}\left(\frac{u}{v},\frac{1}{v}\right)\right)\\&=-\sum_{j=2}^{2n}jv^{2n-j}H_j(u,1),\\\frac{dv(\tau)}{d\tau}&=-v^{2n}\frac{\partial H}{\partial x}\left(\frac{u}{v},\frac{1}{v}\right)=-\sum_{j=2}^{2n}v^{2n+1-j}\frac{\partial H_j}{\partial x}(u,1).\end{aligned}\tag{6}$$

Hence, all infinite equilibria of system (1) are determined by the real roots of $H_{2n}(1,u)=0$, and the zero root of $H_{2n}(u,1)=0$. This implies that system (1) has no infinite equilibria if and only if the homogeneous polynomial $H_{2n}(x,y)$ has no real linear factors. And system (1) has infinite equilibria if and only if the homogeneous polynomial $H_{2n}(x,y)$ has real linear factors.

The "only if"part of the theorem is directly from the definition of global center and above analysis. We next prove the "if"part of the theorem.

If Hamiltonian polynomial system (1) has no infinity equilibria on the circle $\mathbb{S}^1$ by the Poincaré compactification, then the circle $\mathbb{S}^1$ is a closed orbit of system (1) in the closed unit disc $\mathbb{D}$. Note that system (1) has the unique finite equilibrium at $E_0(0,0)$. Thus, the index of $E_0(0,0)$ is one. By Lemma 5, we know that the equilibrium $E_0(0,0)$ is a center.

On the other hand, if system (1) has infinite equilibria on the circle $\mathbb{S}^1$, then the circle $\mathbb{S}^1$ is a singular closed orbit of system (1) in the closed unit disc $\mathbb{D}$ since all the infinite equilibrium points have its local phase portrait formed by one hyperbolic sector having its two separatrices at infinity. Therefore, the index of $E_0(0,0)$ is one too. Hence, $E_0(0,0)$ is a center by Lemma 5.

We next prove that $E_0(0,0)$ is global center.

If the last either closed orbit or singular closed orbit surrounding the center localized at the equilibrium $E_0(0,0)$ of system (1) is the circle $\mathbb{S}^1$ the theorem is proved. Otherwise there is a last closed orbit Γ surrounding $E_0(0,0)$ that it is not the circle $\mathbb{S}^1$ of the infinity, that is, there exist non-closed orbits of system (1) between $\mathbb{S}^1$ and Γ. We will see that this will provide a contradiction and consequently the theorem will be proved.

Indeed, since Γ is the last periodic orbit surrounding $E_0(0,0)$, we consider the Poincaré map F associated to a transversal section to Γ. The map F is analytic because the Hamiltonian system is polynomial. Note that $E_0(0,0)$ is the unique finite equilibrium point of system (1) and all orbits of system (1) cannot tend to equilibrium points at the circle $\mathbb{S}^1$ by assumptions. In the region between Γ and $\mathbb{S}^1$, we consider an orbit near Γ, which will spiral by the theorem of continuous dependence on the initial conditions. Since in the bounded region limited by Γ all the orbits are periodic, the map F which is defined in the part of the transversal section contained in that bounded region limited by Γ is the identity, but for analyticity the map F is also the identity in the part of the transversal section contained in the region between Γ and $\mathbb{S}^1$, and this is a contradiction because in that region the orbits spiral. □

Proof of Proposition 3. For a given positive integer n, we consider the polynomial Hamiltonian function of degree $2n+2$

$$H_l(x,y) = \frac{1}{2}x^2 + \frac{1}{2}y^2 + \frac{1}{2n+2}x^{2n+2} + \frac{1}{2n+2}y^{2n+2},$$

which associated Hamiltonian system is

$$\begin{array}{rcl} \dot{x} &=& -y(1+y^{2n}) = P(y), \\ \dot{y} &=& x(1+x^{2n}) = Q(x). \end{array} \tag{7}$$

Clearly the origin is the unique finite equilibrium points of system (7). And the homogeneous parts $H_{2n+2}(x,y)$ of maximal degree of $H_l(x,y)$ is

$$\frac{1}{2n+2}x^{2n+2} + \frac{1}{2n+2}y^{2n+2}, \tag{8}$$

which has no real linear factors. Therefore, by Theorem 2, system (7) has a global linear type center at $(0,0)$, that is the level set $H_l(x,y) = h,\ h > 0$ is oval.

Now consider the polynomial Hamiltonian function of degree $2n+2$

$$H_{nc}(x,y) = \frac{1}{2}y^2 + \frac{1}{2n+2}x^{2n+2} + \frac{1}{2n+2}y^{2n+2},$$

whose associated Hamiltonian system is

$$\begin{array}{rcl} \dot{x} & = & -y(1+y^{2n}) = P(y), \\ \dot{y} & = & x^{2n+1} = Q(x). \end{array} \tag{9}$$

$(0,0)$ is a unique equilibrium of system (9). Moreover, the homogeneous parts $H_{2n+2}(x,y)$ of maximal degree of $H_{nc}(x,y)$ is (8), which has no real linear factors. Hence, by Theorem 2, system (9) has a global nilpotent center at $(0,0)$.

Finally let

$$H_c(x,y) = \frac{1}{2n+2}x^{2n+2} + \frac{1}{2n+2}y^{2n+2}.$$

Then the Hamiltonian system with Hamiltonian $H_c(x,y)$ is

$$\begin{array}{rcl} \dot{x} & = & -y^{2n+1} = P(y), \\ \dot{y} & = & x^{2n+1} = Q(x), \end{array} \tag{10}$$

Using the previous arguments we get that system (10) has a global degenerate center at $(0,0)$. This completes the proof of the proposition. □

3. Degenerate center of Hamiltonian cubic polynomial systems

From Proposition 3 we know that there are cubic polynomial Hamiltonian systems which have degenerate centers. In this section we characterize all the Hamiltonian cubic polynomial systems having a degenerate center.

In order to prove Theorem 4 we need the next proposition. This proposition can be proved shortly by using Proposition 4.2 of [3] and the conclusion (iii) of page 136 of [12]. For reader's convenience, we give the detail of proof.

Proposition 6. *Assume that the polynomials $P(x,y)$ and $Q(x,y)$ of system* (1) *are homogeneous of degree $2n-1$. Then the origin of system* (1) *is a center if and only if the homogeneous Hamiltonian $H(x,y)$ of degree $2n$ has no real linear factors. Moreover, the center is global, and* *if $n \geq 2$, then the global center is degenerate and the period function $T(h)$ is strictly monotone.*

Proof. Since $P(x,y) = -\partial H/\partial y$ and $Q(x,y) = \partial H/\partial x$ we have that

$$F(x,y) = xQ(x,y) - yP(x,y), \text{ and } G(x,y) = xP(x,y) + yQ(x,y),$$

are homogeneous functions of degree $2n$, and $F(x,y) = 2nH(x,y)$.

By using the polar coordinates (r,θ) given by $x = r\cos\theta$ and $y = \sin\theta$, system (1) (except the equilibrium point $(0,0)$) becomes

$$\begin{aligned} \dot{r} &= r^{2n-1}G(\cos\theta,\sin\theta), \\ \dot{\theta} &= r^{2n-2}F(\cos\theta,\sin\theta) = 2nr^{2n-2}H(\cos\theta,\sin\theta). \end{aligned} \tag{11}$$

Note that the polynomial $H(x,y)$ has no real linear factors and is homogeneous of degree $2n$. Hence $H(x,y) \neq 0$ for all $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$, that is, the function $H(\theta) = H(\cos\theta,\sin\theta)$ is either positive, or negative, for all $\theta \in \mathbb{R}$. Without loss of generality we can assume that $H(x,y) > 0$ for all $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$, that is $H(\theta) > 0$. Thus system (11) can be written as

$$\frac{dr}{d\theta} = \frac{r}{2n}\frac{G(\cos\theta,\sin\theta)}{H(\cos\theta,\sin\theta)}, \tag{12}$$

which has the solutions

$$r^{2n}(\theta;0,r_0) = r_0^{2n}e^{\displaystyle\int_0^\theta \frac{G(\cos\alpha,\sin\alpha)}{H(\cos\alpha,\sin\alpha)}d\alpha},$$

for any the initial points $(0,r_0)$. Clearly the origin of system (11) is a center if and only if

$$I = \int_0^{2\pi} \frac{G(\cos\theta,\sin\theta)}{H(\cos\theta,\sin\theta)}d\theta = 0. \tag{13}$$

Since

$$\frac{d(\ln H(\cos\theta,\sin\theta))}{d\theta} = -\frac{G(\cos\theta,\sin\theta)}{H(\cos\theta,\sin\theta)},$$

we get that

$$I = \int_0^{2\pi} \frac{G(\cos\theta,\sin\theta)}{H(\cos\theta,\sin\theta)}d\theta = -\ln H(\cos\theta,\sin\theta))|_{\theta=0}^{2\pi} \equiv 0.$$

This complete the proof on the sufficient condition, and the necessary condition comes from Theorem 2 directly.

For any $h > 0$ for the closed orbit $\Gamma_h = \{r^{2n} H(\cos\theta, \sin\theta) = h\}$ we have

$$r(\theta; h) = \left(\frac{h}{H(\theta)}\right)^{\frac{1}{2n}}. \tag{14}$$

We now compute the period function $T(h)$ for the closed orbits Γ_h, i.e.

$$T(h) = \oint_{\Gamma_h} \frac{1}{r^{2n-2} F(\cos\theta, \sin\theta)} d\theta = \frac{1}{2nh^{\frac{n-1}{n}}} \int_0^{2\pi} H^{-\frac{1}{n}}(\theta) d\theta,$$

by the second equation of system (11) and (14). Thus

$$T'(h) = \frac{1-n}{2n^2} h^{\frac{1-2n}{n}} \int_0^{2\pi} H^{-\frac{1}{n}}(\theta) d\theta < 0.$$

So the period function $T(h)$ is strictly monotone if $n \geq 2$.

Moreover, if $n \geq 2$ then system (1) is a homogeneous polynomial system of degree $2n - 1 \geq 3$, and consequently the global center $(0,0)$ is degenerate. This completes the proof. □

The following result is proved in [21].

Proposition 7. *The polynomial* (2) *has no real roots if and only if one of the following three conditions hold.*

(i) $D_4 > 0$ *and* $D_3 \leq 0$*,*
(ii) $D_4 > 0$ *and* $D_2 \leq 0$*,*
(iii) $D_4 = D_3 = 0$ *and* $D_2 < 0$*.*

The expressions of D_k *for* $k = 2, 3, 4$ *are given in* (3)*.*

Proof of Theorem 4. Assume that the Hamiltonian cubic polynomial system with Hamiltonian polynomial $H(x, y)$ of degree 4 has a center at the origin of coordinates. Then in order that the origin be a degenerate center the polynomial $H(x, y)$ must have only monomials of degree 3 and 4.

Since the Hamiltonian $H(x, y)$ is a first integral of its associated Hamiltonian system, the origin of such Hamiltonian system is a center if and only if the polynomial function $H(x, y)$ has at the origin a local maximum or minimum.

Clearly if the polynomial $H(x, y)$ has some monomial of degree 3, then the function $H(x, y)$ cannot have a maximum or minimum at the origin because then in a neighborhood of the orign the function $H(x, y)$

would change its sign and consequently the origin would not be a maximum or mininum of the function $H(x, y)$. Therefore $H(x, y)$ must be a homogeneous polynomial of degree 4. Hence the corresponding Hamiltonian system is a homogeneous cubic polynomial differential system, and Propositons 6 can be applied.

Now we consider an arbitrary homogeneous Hamiltonian polynomial of degree 4

$$H(x, y) = ax^4 + bx^3y + cx^2y^2 + dxy^3 + ey^4.$$

If $ae = 0$, then Hamiltonian system associated to $H(x, y)$ has the invariant straight line $x = 0$, or $y = 0$, and consequently the origin will not be a center. Without loss of generality and after a scaling of the time, if necessary, we can assume that $a = 1$, otherwise we interchange the variables x and y. Hence statement (a) is proved.

We now consider the polynomial (2) with $a = 1$. If this polynomial has a real root r, then the straight line $y - rx = 0$ is invariant by the Hamiltonian system associated to $H(x, y)$, and consequently the origin cannot be a center. So the polynomial (2) has no real roots, and by Proposition 7, statement (b) of the theorem follows.

From Proposition 6 it follows that the degenerate center of this Hamiltonian system is global and its period function is strictly monotone, so statement (c) is proved. □

4. Discussion

In this paper we give the necessary and sufficient conditions for Hamiltonian polynomial systems having a global center. A natural problem arises what property has the period function of periodic orbits surrounding the global center of Hamiltonian polynomial system? From Theorem 1, the Hamiltonian polynomial system having global center should have degree odd, that is, $2n - 1$ with $n \in \mathbb{N}$. When $n = 1$, that is, the linear Hamiltonian system. It is clear that the period function of periodic orbits surrounding the global center of linear Hamiltonian system is constant. From Proposition 6 (the conclusion (iii) of page 136 of [12]), the period function of periodic orbits surrounding the global center of homogenous (and quasi-homogenous) Hamiltonian system with degree $2n-1$ with $n \geq 2$ is strictly monotone. But if the Hamiltonian polynomial is not homogeneous the period function can have critical points as the following example shows.

Example 8. *Assume that Hamiltonian function* $H(x,y) = \frac{1}{8}(x^2+y^2)^4 - \frac{\sqrt{7}}{6}(x^2+y^2)^3 + \frac{1}{2}(x^2+y^2)^2$. *Then the associated Hamiltonian system*

$$\begin{aligned} \dot{x} &= -y(x^2+y^2)\left((x^2+y^2)^2 - \sqrt{7}(x^2+y^2)+2\right), \\ \dot{y} &= x(x^2+y^2)\left((x^2+y^2)^2 - \sqrt{7}(x^2+y^2)+2\right), \end{aligned} \tag{15}$$

has a global center at $(0,0)$, *and its period function* $T(h)$ *has two critical points.*

Proof. System (15) after a rescaling of the time becomes the linear center $\dot{x} = -y$, $\dot{y} = x$, and since $(x^2+y^2)\left((x^2+y^2)^2 - \sqrt{7}(x^2+y^2)+2\right)$ only vanishes at the origin, the origin is the unique equilibrium point of system (15) and it is a global center.

We now consider the periodic function $T(h)$ of system (15). In polar coordinates system (15) writes

$$\dot{r} = 0, \quad \dot{\theta} = r^2\left((r^2 - \frac{\sqrt{7}}{2})^2 + \frac{1}{4}\right). \tag{16}$$

Since $\bar{H} = x^2+y^2$ is also a first integral of system (15), the closed orbits of this system are $\bar{H} = h = r^2$. Therefore the period function of the closed orbit $\bar{H} = h = r^2$ from the second equation is

$$T(h) = T(r^2) = \frac{1}{r^2\left((r^2-\frac{\sqrt{7}}{2})^2+\frac{1}{4}\right)}\oint_{\Gamma_h} d\theta = \frac{2\pi}{r^2\left((r^2-\frac{\sqrt{7}}{2})^2+\frac{1}{4}\right)}.$$

And

$$T'(h) = -\frac{3h^2 - 2\sqrt{7}h + 2}{h\left((h-\frac{\sqrt{7}}{2})^2+\frac{1}{4}\right)}$$

Thus, the function $T(h)$ has the two critical points

$$h_1 = \frac{1}{3}\left(\sqrt{7}+1\right), \; h_2 = \frac{1}{3}\left(\sqrt{7}-1\right).$$

This completes the proof of the example. □

From the example and other computations, an interesting problem arise to us *if the period function of a global center of a Hamiltonian polynomial system has an even number of critical points.* We left it as an open problem.

Consider the family of real algebraic curves $H(x,y) = h$ with h varying in an open interval of $\mathbb{R}$. An interesting and difficult problem is to know when all these algebraic curves are ovals, i.e. closed curves.

Theorem 2 gives a characterization. Our results show that the tools of the differential equations allow to provide such characterization. Using Hamiltonian polynomial system (1), we find that the real algebraic curves $H(x, y) = h$ are closely related to the finite and the infinite equilibria of the Hamiltonian system (1). And this method can be extended for studying the real algebraic hypersurfaces $H(X, Y) = h$ with a continuum of the real h in $\mathbb{R}^{2n}$, where $X = (x_1, \cdots, x_n), Y = (y_1, \cdots, y_n)$, see [14].

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