SIAM J. APPLIED DYNAMICAL SYSTEMS
Vol. 19, No. 4, pp. 2343–2370

On the Number of Limit Cycles in Generalized Abel Equations*

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Abstract. Given $p, q \in \mathbb{Z}_{\geq 2}$ with $p \neq q$, we study generalized Abel differential equations $\frac{dx}{d\theta} = A(\theta)x^p + B(\theta)x^q$, where A and B are trigonometric polynomials of degrees $n, m \geq 1$, respectively, and we are interested in the number of limit cycles (i.e., isolated periodic orbits) that they can have. More concretely, in this context, an open problem is to prove the existence of an integer, depending only on p, q, m, and n and that we denote by $\mathcal{H}_{p,q}(n,m)$, such that the above differential equation has at most $\mathcal{H}_{p,q}(n,m)$ limit cycles. In the present paper, by means of a second order analysis using Melnikov functions, we provide lower bounds of $\mathcal{H}_{p,q}(n,m)$ that, to the best of our knowledge, are larger than the previous ones appearing in the literature. In particular, for classical Abel differential equations (i.e., $p = 3$ and $q = 2$), we prove that $\mathcal{H}_{3,2}(n,m) \geq 2(n+m) - 1$.

Key words. generalized Abel equations, Melnikov theory, second order perturbation, limit cycles

AMS subject classifications. 34C07, 34C05, 37C10

DOI. 10.1137/20M1340083

1. Introduction and statements of main results.

The study of the existence of periodic orbits in ordinary differential equations has been an interesting problem for years in many areas of mathematics, particularly in qualitative theory of differential equations. In this area of interest, when we focus on planar polynomial vector fields, one of the most renowned classical problems arises: to know the number and location of isolated periodic orbits, the so-called limit cycles, in terms of its degree n. The study of this problem began at the end of the 19th century with the seminal works by Poincaré, but takes its name after Hilbert because of his famous list of unsolved problems published in 1900. From the original list of 23 problems, the 16th is still open, in particular, its second part. More precisely (see [26, 36] for details), the "existential" Hilbert's 16th problem is to prove that for any $n \geq 2$ there exists a finite number $\mathcal{H}(n)$ such that any polynomial vector field of degree $\leq n$ has less than $\mathcal{H}(n)$ limit cycles.

*Received by the editors May 22, 2020; accepted for publication (in revised form) by J. Rademacher August 3, 2020; published electronically October 22, 2020.
https://doi.org/10.1137/20M1340083
Funding: This work was supported by the Catalan AGAUR 2017SGR1617 grant, the Spanish Ministerio de Ciencia, Innovacion y Universidades - Agencia Estatal de Investigacion MTM2016-77278-P (FEDER), MTM2017-86795-C3-2-P grants, the European Community H2020-MSCARISE-2017-777911 grant, NSF of China 11771101 grant and China Scholarship Council 201606785007 grant.
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