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Liouvillian integrability of gravitating static isothermal fluid spheres

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We examine the integrability properties of the Einstein field equations for static, spherically symmetric fluid spheres, complemented with an isothermal equation of state, $\rho = np$. In this case, Einstein's equations can be reduced to a nonlinear, autonomous second order ordinary differential equation (ODE) for m/R (m is the mass inside the radius R) that has been solved analytically only for $n = -1$ and $n = -3$, yielding the cosmological solutions by De Sitter and Einstein, respectively, and for $n = -5$, case for which the solution can be derived from the De Sitter's one using a symmetry of Einstein's equations. The solutions for these three cases are of Liouvillian type, since they can be expressed in terms of elementary functions. Here, we address the question of whether Liouvillian solutions can be obtained for other values of n. To do so, we transform the second order equation into an equivalent autonomous Lotka–Volterra quadratic polynomial differential system in $\mathbb{R}^2$, and characterize the Liouvillian integrability of this system using Darboux theory. We find that the Lotka–Volterra system possesses Liouvillian first integrals for $n = -1, -3, -5$, which descend from the existence of invariant algebraic curves of degree one, and for $n = -6$, a new solvable case, associated to an invariant algebraic curve of higher degree (second). For any other value of n, eventual first integrals of the Lotka–Volterra system, and consequently of the second order ODE for the mass function must be non-Liouvillian. This makes the existence of other solutions of the isothermal fluid sphere problem with a Liouvillian metric quite unlikely. © *2014 AIP Publishing LLC*. [http://dx.doi.org/10.1063/1.4897213]

I. INTRODUCTION AND STATEMENT OF THE MAIN RESULT

Exact solutions of Einstein's field equations describing static, spherically symmetric fluid spheres have been sought for decades, both in cosmological contexts and as possible models of the internal structure of neutron stars. The inherent mathematical underdetermination of the problem (three equations for the four variables: density, pressure, and two metric coefficients) has allowed for the derivation of a considerable number of such solutions, even though very few of them satisfy all the requirements for physical acceptability (see, e.g., the classical work by Delgaty and Lake,[1] where the 127 non-cosmological solutions available at the time were classified and tested for physical relevance).

The underdetermination can be removed by specifying an additional physical constraint, for example, an equation of state, but this usually makes it very difficult to derive explicit analytic solutions. The case of a simple isothermal equation of state,

$$\rho = np, \tag{1}$$

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