LIMIT CYCLES IN UNIFORM ISOCHRONOUS CENTERS OF DISCONTINUOUS DIFFERENTIAL SYSTEMS WITH FOUR ZONES

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Abstract. We apply the averaging theory of first order for discontinuous differential systems to study the bifurcation of limit cycles from the periodic orbits of the uniform isochronous center of the differential systems $\dot{x} = -y + x^2$, $\dot{y} = x + xy$, and $\dot{x} = -y + x^2y$, $\dot{y} = x + xy^2$ when they are perturbed inside the class of all discontinuous quadratic and cubic polynomials differential systems, respectively. We consider the case where the plane is split in four zones by the straight lines $x = 0$ and $y = 0$.

In our work we have twice the number of limit cycles obtained in a previous work for discontinuous quadratic systems and 5 more limit cycles than those achieved in a prior result for discontinuous cubic systems with a uniform isochronous center at the origin. Comparing our results with those presented for the continuous quadratic and cubic cases we obtained 8 and 9 more limit cycles respectively.

1. Introduction and Statement of the Main Results

Suppose that $q \in \mathbb{R}^2$ is a center of a polynomial differential system in $\mathbb{R}^2$. Without loss of generality we can assume that q is at the origin of coordinates. Then q is an *isochronous center* if there exists a neighborhood U_q of q such that all periodic orbits in U_q have the same period. An isochronous center is *uniform* if in polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ it can be written as $\dot{r} = G(\theta, r)$, $\dot{\theta} = k$, $k \in \mathbb{R} \setminus \{0\}$, for further details see [9]. A singular point q is a *weak focus* if it is a center for the linearized system at q and this singular point is not a center.

Consider the planar polynomial differential system of degree n

$$\dot{x} = -y + x\, f(x, y), \quad \dot{y} = x + y\, f(x, y), \tag{1}$$

where $f(x, y)$ is a polynomial in x and y of degree $n - 1$ and $f(0, 0) = 0$. This differential system has only one singular point at the origin, which is a center for the linear part of the system. Moreover, the solutions of (1) move around the origin with constant angular speed and therefore, the origin is either a uniform isochronous center or a weak focus.

Isochronicity is important in a myriad of fields such as Physics, Chemistry, Biology and Engineering. It also has relation to the existence and uniqueness of solutions for some perturbation problems. Moreover, isochronicity is relevant in stability theory because the periodic solutions of the central region is Lyapunov stable if and only if the adjoining periodic solutions are isochronous, further details on these topics can be found in [7]. In the last decades, the bifurcation of limit cycles from uniform isochronous centers has attracted attention of several authors, see for instance [1, 10, 11, 15, 17].

In this paper, we investigate the birth of limit cycles from a uniform isochronous center of discontinuous piecewise quadratic and cubic differential systems with four zones formed

2010 *Mathematics Subject Classification.* Primary 34A36, 34C07, 34C25, 34C29, 37G15.

Key words and phrases. limit cycle, averaging theory, uniform isochronous center, discontinuous polynomial system.

when the plane is divided by two perpendicular straight lines. To the best of our knowledge this is the first work that analyzes the bifurcation of limit cycles under these conditions.

More precisely, we split the plane in four quadrants by the straight lines $x = 0$ and $y = 0$. Let $Q_i, i = 1, \ldots, 4$ denote the quadrant i, that is, $Q_1 = \{(x, y) \in \mathbb{R}^2 / x, y > 0\}, Q_2 = \{(x, y) \in \mathbb{R}^2 / x < 0, y > 0\}, Q_3 = \{(x, y) \in \mathbb{R}^2 / x, y < 0\}$ and $Q_4 = \{(x, y) \in \mathbb{R}^2 / x > 0, y < 0\}$.

Applying the averaging theory of first order we investigate the number of limit cycles which can bifurcate from the periodic orbits of the uniform isochronous center of the following quadratic and cubic differential systems.

$$\dot{x} = -y + x^2, \quad \dot{y} = x + xy, \tag{2}$$

$$\dot{x} = -y + x^2 y, \quad \dot{y} = x + xy^2, \tag{3}$$

when they are perturbed inside the classes of the following discontinuous quadratic and cubic polynomial differential systems

$$\mathcal{X}_j(x, y) = Y_j^i(x, y) \text{ if } (x, y) \in Q_i, \tag{4}$$

with $i = 1, \ldots, 4$ denoting the quadrant Q_i, and $j = 2$ (quadratic case) or $j = 3$ (cubic case) that is,

$$Y_2^i(x, y) = \begin{pmatrix} -y + x^2 + \varepsilon p^i(x, y) \\ x + xy + \varepsilon q^i(x, y) \end{pmatrix},$$

$$Y_3^i(x, y) = \begin{pmatrix} -y + x^2 y + \varepsilon r^i(x, y) \\ x + xy^2 + \varepsilon s^i(x, y) \end{pmatrix},$$

where ε is a real small parameter, and

$$p^i = \sum_{j+k=1}^{2} a_{jk}^i x^j y^k, \quad q^i = \sum_{j+k=1}^{2} b_{jk}^i x^j y^k, \quad r^i = \sum_{j+k=1}^{3} c_{jk}^i x^j y^k, \quad s^i = \sum_{j+k=1}^{3} d_{jk}^i x^j y^k.$$

In what follows we state our main results.

Theorem 1. *For $|\varepsilon| \neq 0$ sufficiently small there exist discontinuous piecewise quadratic polynomial differential systems* (4)*, with $j = 2$, which have at least 10 limit cycles bifurcating from the periodic orbits of the uniform isochronous center of system* (2)*.*

Theorem 2. *For $|\varepsilon| \neq 0$ sufficiently small there exist discontinuous piecewise cubic polynomial differential systems* (4)*, with $j = 3$, which have at least 12 limit cycles bifurcating from the periodic orbits of the uniform isochronous center of system* (3)*.*

We remark that the lower bounds for the number of limit cycles provided in Theorems 1 and 2 were obtained using the averaging theory of first order. These results could possibly be improved using higher orders of the averaging method, for further details see [15].

The limit cycles that bifurcate from the origin of (1) when it is perturbed inside some classes of continuous polynomial differential systems have been intensively investigated, see [6] and the several references therein. In [5], using results from Bautin [2], it is proved that at most 2 limit cycles bifurcate from the periodic orbits of the uniform isochronous center of the quadratic differential system (2). Applying the averaging theory, in [4] it is showed that at least 2 limit cycles bifurcate from the periodic orbits of that center, when it is perturbed inside the class of all polynomial differential systems of degree 2. For the cubic polynomial differential systems (3), in [12] it is proved that in both cases of limit cycles bifurcating either from the periodic orbits or from the uniform isochronous center itself, the maximum number of limit cycles is 3, applying averaging theory of order 1 and 6, respectively.

In the real world, a large number of phenomena can only be modeled by discontinuous differential equations, for instance see [3] and the references therein. Hence, the study of limit cycles of discontinuous piecewise differential systems has been significantly increasing in recent years. In [17] it has been proved that at least 5 limit cycles bifurcate from the periodic orbits of the uniform isochronous center of (2) when it is perturbed inside the class of all discontinuous quadratic differential systems with the straight line of discontinuity $y = 0$. In [12] it is showed that using the averaging theory of order 6, the maximum number of limit cycles that can appear in a Hopf bifurcation at the uniform isochronous center of system (1), for $n = 3$, is 5, and this number can be reached. In the same work, the authors proved that for system (3), using the averaging method of first order, the maximum number of limit cycles that can bifurcate from the periodic solutions surrounding the center is 7, and this number can be reached. In both cases studied in [12], the considered discontinuous systems were formed by two cubic polynomial differential systems separated by the straight line $y = 0$.

In other words, in some sense we extend the works presented in the last two paragraphs for continuous and discontinuous quadratic and cubic polynomial differential systems. We recall that the largest number of limit cycles achieved in these previous works were 5 for quadratic differential systems and 7 for cubic differential systems, both in the case of discontinuous polynomial perturbations. Therefore, our work provides results which double the number of limit cycles obtained in previous results for quadratic systems and increase in 5 the number of limit cycles achieved in prior results for cubic systems with a uniform isochronous center.

In short, the results on the number of limit cycles that can bifurcate from the periodic orbits of the uniform isochronous center of quadratic differential system (2) and of the cubic differential system (3) when these systems are perturbed respectively inside the class of all continuous and discontinuous quadratic and cubic polynomial differential systems are summarized in Table 1.

Case	Number of limit cycles for	
	system (2)	system (3)
Continuous	2	3
Discontinuous with 2 zones	5	7
Discontinuous with 4 zones	10	12

TABLE 1. *Number of limit cycles for continuous and discontinuous quadratic and cubic differential systems*

This work is part of a general program for investigating the dynamics of polynomial piecewise discontinuous vector fields in which computer algebra is combined with singularity theory of mappings, algebraic geometry and numerical techniques. This general program aims to find some models where the following questions are addressed. (*i*) When is a typical singularity topologically equivalent to a regular center? (*ii*) How about the isochronicity of such a center? (*iii*) When does a polynomial perturbation in a polynomial differential system produce limit cycles? For further information, see for instance [20].

We remark that there exist some works that address the problem of estimating the number of limit cycles from the periodic orbits of a uniform isochronous center of planar quartic polynomial differential systems, see for instance [14]. Nevertheless, to the extent of our knowledge these works only investigate some particular families of quartic systems.

2. Preliminary results

In this section we present the main results we shall use to investigate the discontinuous piecewise quadratic and cubic differential systems (4). The next result is well-known. Further details about it can be found in [13].

Proposition 3. [13] *Suppose that a differential polynomial system of degree n in $\mathbb{R}^2$ has a center that can be placed without loss of generality at the origin of coordinates. Then this center is uniform isochronous if and only if by applying a rescaling of time and a linear change of variables this system can be written as*

$$\dot{x} = -y + x\, f(x,y), \quad \dot{y} = x + y\, f(x,y),$$

where $f(x,y)$ is a polynomial in x and y of degree $n-1$, and $f(0,0)=0$.

We recall that from Proposition 3 and from the fact that

$$(-y + xf(x,y))^2 + (x + yf(x,y))^2 = (x^2+y^2)(1+f^2(x,y)) > 0 \ \text{ if } \ (x,y) \neq (0,0)$$

it follows that the uniform isochronous center at the origin of coordinates is the only finite singular point of systems (2) and (3) (in fact this result is valid for any system (1)) and hence we only need to analyze the bifurcation of limit cycles from the periodic orbits of such center in those differential systems.

A quadratic polynomial differential system with a uniform isochronous center can always be written under the form (2), after a rescaling of time and a linear change of coordinates, see [19].

For the case of cubic differential systems with a uniform isochronous center, there is the following result due to Collins [8].

Theorem 4. [8] *A cubic polynomial differential system in $\mathbb{R}^2$ with a uniform isochronous center that can be placed without loss of generality at the origin can be reduced to one of the two following expressions.*

$$\dot{x} = -y(1-x^2), \quad \dot{y} = x(1+y^2), \tag{5}$$

$$\dot{x} = -y + x^2 + Ax^2y, \quad \dot{y} = x + xy + Axy^2. \tag{6}$$

where $A \in \mathbb{R}$ is a parameter.

We remark that using these prior results we were able to study the bifurcation of limit cycles from the periodic orbits of any quadratic differential system with a uniform isochronous center, and of every cubic differential systems with a uniform isochronous center which can be reduced to the form (5) after a change of coordinates and a rescaling of time.

The following result is the first-order averaging theory for discontinuous piecewise differential systems developed in [16].

Let $D \subset \mathbb{R}^d$ be an open subset and $\mathbb{S}^1 = \mathbb{R}/T$ for a period $T>0$. Furthermore let (S_n) be a finite sequence of open disjoint subsets of $\mathbb{S}^1 \times D$, with $n = 1, \ldots, M$. We assume that the boundaries of each S_n are piecewise $\mathcal{C}^k-$embedded hypersurfaces, for $k \geq 1$. In addition we suppose that all S_n together cover the set $\mathbb{S}^1 \times D$, and we denote by Σ

the union of all boundaries of S_n. Finally, consider $A \subset \mathbb{S}^1 \times D$ and let $\chi_A(t,x)$ be the characteristic function defined as

$$\chi_A(t,x) = \begin{cases} 1, & \text{if } (t,x) \in A, \\ 0, & \text{if } (t,x) \notin A. \end{cases}$$

Theorem 5. [16] *Consider the following discontinuous piecewise differential system*

$$x'(t) = \varepsilon F_1(t,x) + \varepsilon^2 R(t,x,\varepsilon), \tag{7}$$

with

$$F_1(t,x) = \sum_{j=1}^{M} \chi_{\bar{S}_j}(t,x) F_1^j(t,x),$$

$$R(t,x,\varepsilon) = \sum_{j=1}^{M} \chi_{\bar{S}_j}(t,x) R^j(t,x),$$

where $F_1^j : \mathbb{S}^1 \times D \to \mathbb{R}^d$, $R^j : \mathbb{S}^1 \times D \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^d$ *for* $j = 1, \ldots, M$ *are continuous functions,* $T-$*periodic in the variable* t *and* D *is an open subset of* $\mathbb{R}^d$.
We define the averaging function $f_1 : D \to \mathbb{R}^d$ *as*

$$f_1(z) = \int_0^T F_1(t,z)dt. \tag{8}$$

Moreover, assume the following hypotheses.

- *(HC) There exists* $C \subset D$ *an open bounded subset such that for each* $z \in \bar{C}$ *the curve* $\{(t,z) : t \in \mathbb{S}^1\}$ *reaches transversely the set* Σ *and only at generic points of discontinuity.*
- *(Ha1) For* $j = 1, \ldots, M$ *the continuous functions* F_1^j *and* R^j *are* $T-$*periodic with respect to* t *and locally Lipschitz with respect to* x. *In addition the boundaries of* S_j, *for* $j = 1, \ldots, M$, *are piecewise* $\mathcal{C}^k-$*embedded hypersurfaces,* $k \geq 1$.
- *(Ha2) For* $a^* \in C$ *with* $f_1(a^*) = 0$, *there exists a neighborhood* $U \subset C$ *of* a^* *such that* $f_1(z) \neq 0$ *for all* $z \in \bar{U} \setminus \{a^*\}$ *and* $d_B(f_1, U, 0) \neq 0$.

Then for $|\varepsilon| \neq 0$ *sufficiently small there exists a* $T-$*periodic solution* $x(t,\varepsilon)$ *of system* (7) *such that* $x(0,\varepsilon) \to a^*$ *as* $\varepsilon \to 0$.

We note that in this paper the set Σ referred in Theorem 5 is given by the inverse image of zero by the function $w(x,y) = xy$, i.e., $\Sigma = w^{-1}(0)$.

Consider a planar differential system

$$\dot{x} = P(x,y), \quad \dot{y} = Q(x,y), \tag{9}$$

with $P, Q : \mathbb{R}^2 \to \mathbb{R}$ continuous functions and suppose that this system has a continuous family of period solutions $\{\Gamma_h\} \subset \{(x,y) : \mathcal{H}(x,y) = h, h_1 < h < h_2\}$, where $\mathcal{H}$ is a first integral of (9).

Now suppose that we perturb (9) as follows

$$\dot{x} = P(x,y) + \varepsilon p(x,y), \quad \dot{y} = Q(x,y) + \varepsilon q(x,y), \tag{10}$$

with $p, q : \mathbb{R}^2 \to \mathbb{R}$ continuous functions.

Then for $|\varepsilon| \neq 0$ sufficiently small, in order to study the bifurcation of limit cycles in (10) applying the averaging theory, it is necessary to put this system into the standard configuration (7). The next result provides a method to do that.

Theorem 6. [4] *Consider the unperturbed system* (9) *and its first integral* $\mathcal{H}$. *Assume that for all* (x, y) *in the period annulus formed by the ovals* $\{\Gamma_h\}$, *we have* $xQ(x, y) - yP(x, y) \neq 0$. *Moreover, for all* $R \in (\sqrt{h_1}, \sqrt{h_2})$ *and all* $\theta \in [0, 2\pi)$, *let* $\rho : (\sqrt{h_1}, \sqrt{h_2}) \times [0, 2\pi) \to [0, \infty)$ *be a continuous function such that*

$$H(\rho(R, \theta) \cos\theta, \rho(R, \theta) \sin\theta) = R^2.$$

Then the differential equation which describes the dependence between the square root of the energy $R = \sqrt{h}$ *and the angle* θ *for the perturbed system* (10) *is*

$$\frac{dR}{d\theta} = \varepsilon \frac{\mu(x^2 + y^2)(Qp - Pq)}{2R(Qx - Py)} + \mathcal{O}(\varepsilon^2) \tag{11}$$

where $x = \rho(R, \theta) \cos\theta$, $y = \rho(R, \theta) \sin\theta$, *and* $\mu = \mu(x, y)$ *is the integrating factor corresponding to the first integral* $\mathcal{H}$ *of* (9).

In order to determine the number of zeros of the averaging function (8) we shall apply the following result, for a proof of it see for instance the Proposition 1 of the Appendix A of [18].

Proposition 7. *Let* I *be an interval of* $\mathbb{R}$ *and let* $f_0, \ldots, f_n : I \to \mathbb{R}$ *be analytic functions linearly independent, that is, if* $\sum_{i=0}^{k} \alpha_i f_i(s) = 0$ *then* $\alpha_0 = \ldots = \alpha_k = 0$. *Then there exist* $s_1, \ldots, s_n \in I$ *and* $\lambda_0, \ldots, \lambda_n \in \mathbb{R}$ *such that for every* $j \in \{1, \ldots, n\}$ *we have*

$$f(s_j) \doteq \sum_{i=0}^{n} \lambda_i f_i(s_j) = 0.$$

In other words, there exist values of $\lambda_0, \ldots, \lambda_n \in \mathbb{R}$ and $s_1, \ldots, s_n \in I$ such that f, which is a linear combination of $n + 1$ linearly independent functions, has n simple roots.

3. Proof of Theorem 1

We recall that the period annulus of a center q is the largest set of continuous periodic solutions surrounding q, and having q itself as its inner boundary.

Consider the first integral $H = (x^2+y^2)/(1+y^2)$ and its corresponding integrating factor $\mu = 1/(1 + y)^3$ in the period annulus of the uniform isochronous center of the quadratic differential system (2). This system system has the invariant straight line $y = -1$, and therefore the minimal distance between the outer boundary of the period annulus of the center and the center itself is 1.

Setting $h_1 = 0$, $h_2 = 1$ and taking $\rho(R, \theta) = R/(1 - R\sin\theta)$, for $0 < R < 1$, $\theta \in [0, 2\pi)$, the hypotheses of Theorem 6 are satisfied. Therefore, applying Theorem 6 we can write system (2) as

$$\frac{dR}{d\theta} = \varepsilon \frac{A^i(\theta, a, b)R + B^i(\theta, a, b)R^2 + C^i(\theta, a, b)R^3}{2(1 - R\sin\theta)} + \mathcal{O}(\varepsilon^2), \tag{12}$$

for $(x,y)\in Q^i$, $i=1,\dots,4$ and where

$$\begin{aligned}
A^i(\theta,a,b) =& a^i_{10}\cos^2\theta+(2a^i_{01}+b^i_{10})\cos\theta\sin\theta+b^i_{01}\sin^2\theta,\\
B^i(\theta,a,b) =& (a^i_{20}-b^i_{10})\cos^3\theta-(a^i_{10}-a^i_{11}+b^i_{01}-b^i_{20})\cos^2\theta\sin\theta-\\
&(2a^i_{01}-a^i_{02}+2b^i_{10}-b^i_{11})\cos\theta\sin^2\theta-(2b^i_{01}-b^i_{02})\sin^3\theta,\\
C^i(\theta,a,b) =& -b^i_{20}\cos^4\theta+(b^i_{10}-b^i_{11})\cos^3\theta\sin\theta+(b^i_{01}-b^i_{20}-b^i_{02})\\
&\cos^2\theta\sin^2\theta+(b^i_{10}-b^i_{11})\cos\theta\sin^3\theta+(b^i_{01}-b^i_{02})\sin^4\theta,
\end{aligned}$$

with $a=(a^i_{jk})_{j+k=1,\dots,2}$, $b=(b^i_{jk})_{j+k=1,\dots,2}$, and $i=1,\dots,4$.

The hypotheses of Theorem 5 are satisfied by the discontinuous differential system (12). Hence we shall study the zeros of the averaging function $f:(0,1)\to\mathbb{R}$.

$$f(R)=\sum_{i=1}^{4}\left[\int_{(i-1)\frac{\pi}{2}}^{i\frac{\pi}{2}}\frac{A^i(\theta,a,b)R+B^i(\theta,a,b)R^2+C^i(\theta,a,b)R^3}{2(1-R\sin\theta)}d\theta\right].$$

Calculating these integrals we have

$$f(R)=\sum_{j=1}^{17}\gamma_j g_j$$

with

$$\begin{aligned}
\gamma_1 =& \frac{\pi}{4}(a^1_{11}+a^2_{11}+5a^3_{11}-3a^4_{11}+b^1_{20}-b^1_{02}+b^2_{20}-b^2_{02}+5b^3_{20}-5b^3_{02}-3b^4_{20}+3b^4_{02}),\\
\gamma_2 =& \frac{1}{2}(a^1_{11}+a^1_{20}-a^1_{02}+a^2_{11}-a^2_{20}+a^2_{02}-2a^3_{11}-a^3_{20}+a^3_{02}+a^4_{20}-a^4_{02}-b^1_{11}+b^1_{20}-\\
&b^1_{02}+b^2_{11}+b^2_{20}-b^2_{02}+b^3_{11}-2b^3_{20}+2b^3_{02}-b^4_{11}),\\
\gamma_3 =& \frac{1}{8}(\pi(a^1_{10}-a^1_{11}+a^2_{10}-a^2_{11}+5a^3_{10}-5a^3_{11}-3a^4_{10}+3a^4_{11}+b^1_{01}-3b^1_{20}+b^1_{02}+b^2_{01}-\\
&3b^2_{20}+b^2_{02}+5b^3_{01}-15b^3_{20}+5b^3_{02}-3(b^4_{01}-3b^4_{20}+b^4_{02}))+4a^1_{01}+2a^1_{20}-2a^1_{02}-\\
&4a^2_{01}-2a^2_{20}+2a^2_{02}+2a^3_{01}+a^3_{20}-a^3_{02}-2a^4_{01}-a^4_{20}+a^4_{02}+2b^1_{10}-2b^1_{11}-2b^2_{10}+\\
&2b^2_{11}+b^3_{10}-b^3_{11}-b^4_{10}+b^4_{11}),\\
\gamma_4 =& \frac{1}{2}(-b^1_{10}-b^1_{01}+b^1_{11}-b^1_{20}+b^1_{02}+b^2_{10}-b^2_{01}-b^2_{11}-b^2_{20}+b^2_{02}+b^3_{10}+2b^3_{01}-\\
&b^3_{11}+2b^3_{20}-2b^3_{02}-b^4_{10}+b^4_{11}),\\
\gamma_5 =& -\frac{1}{2}\pi(a^1_{11}+a^3_{11}+b^1_{20}-b^1_{02}+b^3_{20}-b^3_{02}),\\
\gamma_6 =& \frac{1}{2}\pi(a^1_{11}+a^3_{11}+2b^1_{20}-b^1_{02}+2b^3_{20}-b^3_{02}),\\
\gamma_7 =& -\frac{1}{2}\pi(b^1_{20}+b^3_{20}),\\
\gamma_8 =& a^1_{11}-a^2_{11}+a^3_{11}-a^4_{11}+b^1_{20}-b^1_{02}-b^2_{20}+b^2_{02}+b^3_{20}-b^3_{02}-b^4_{20}+b^4_{02},\\
\gamma_9 =& -a^1_{11}+a^2_{11}-a^3_{11}+a^4_{11}-2b^1_{20}+b^1_{02}+2b^2_{20}-b^2_{02}-2b^3_{20}+b^3_{02}+2b^4_{20}-b^4_{02},\\
\gamma_{10} =& -a^1_{11}+a^3_{11}-b^1_{20}+b^1_{02}+b^3_{20}-b^3_{02},\\
\gamma_{11} =& a^1_{11}-a^3_{11}+2b^1_{20}-b^1_{02}-2b^3_{20}+b^3_{02},
\end{aligned}$$

$$
\begin{aligned}
\gamma_{12} =& b_{20}^3 - b_{20}^1,\\
\gamma_{13} =& b_{20}^1 - b_{20}^2 + b_{20}^3 - b_{20}^4,\\
\gamma_{14} =& \frac{1}{2}(a_{20}^1 - a_{02}^1 - a_{20}^2 + a_{02}^2 - b_{11}^1 + b_{11}^2),\\
\gamma_{15} =& \frac{1}{2}(-a_{20}^1 + a_{20}^2 + b_{11}^1 - b_{11}^2),\\
\gamma_{16} =& \frac{1}{2}(a_{20}^3 - a_{02}^3 - a_{20}^4 + a_{02}^4 - b_{11}^3 + b_{11}^4),\\
\gamma_{17} =& \frac{1}{2}(-a_{20}^3 + a_{20}^4 + b_{11}^3 - b_{11}^4)
\end{aligned}
$$

$$
\begin{array}{llll}
g_1 = \dfrac{1}{R}, & g_2 = 1, & g_3 = R, & g_4 = R^2,\\
g_5 = \dfrac{1}{R\sqrt{1-R^2}}, & g_6 = \dfrac{R}{\sqrt{1-R^2}}, & g_7 = \dfrac{R^2}{\sqrt{1-R^2}}, & \\
g_8 = \dfrac{1}{R\sqrt{1-R^2}}\arctan\left(\sqrt{\dfrac{1+R}{1-R}}\right), & & g_9 = \dfrac{R}{\sqrt{1-R^2}}\arctan\left(\sqrt{\dfrac{1+R}{1-R}}\right), & \\
g_{10} = \dfrac{1}{R\sqrt{1-R^2}}\arctan\left(\dfrac{R}{\sqrt{1-R^2}}\right), & & g_{11} = \dfrac{R}{\sqrt{1-R^2}}\arctan\left(\dfrac{R}{\sqrt{1-R^2}}\right), & \\
g_{12} = \dfrac{R^3}{\sqrt{1-R^2}}\arctan\left(\dfrac{R}{1-R^2}\right), & & g_{13} = \dfrac{R^3}{\sqrt{1-R^2}}\arctan\left(\sqrt{\dfrac{1+R}{1-R}}\right), & \\
g_{14} = \dfrac{\ln(1-R)}{R}, & g_{15} = R\ln(1-R), & g_{16} = \dfrac{\ln(1+R)}{R}, & g_{17} = R\ln(1+R).
\end{array}
$$

All the calculations were made using the software Mathematica. We identify the following relations among the coefficients of f.

$$
\gamma_5 + \gamma_6 + \gamma_7 = 0, \qquad \gamma_{10} + \gamma_{11} + \gamma_{12} = 0, \qquad \gamma_8 + \gamma_9 + \gamma_{13} = 0,
$$
$$
\gamma_1 - 2\pi\gamma_2 + \gamma_5 - \frac{3\pi}{4}\gamma_8 - 2\pi\gamma_{10} + 2\pi\gamma_{14} - 2\pi\gamma_{16} = 0.
$$

Taking into account these relations, the function f can be written as.

$$
\begin{aligned}
f(R) =& \gamma_1 G_1 + \gamma_2 G_2 + \gamma_3 G_3 + \gamma_4 G_4 + \gamma_5 G_5 + \gamma_6 G_6 + \gamma_8 G_7 + \gamma_9 G_8 + \gamma_{10} G_9 + \gamma_{11} G_{10} +\\
& \gamma_{14} G_{11} + \gamma_{15} G_{12} + \gamma_{17} G_{13},
\end{aligned}
$$

where

$$
\begin{array}{llll}
G_1 = g_1 + \dfrac{1}{2\pi} g_{16}, & G_2 = g_2 - g_{16}, & G_3 = g_3, & G_4 = g_4,\\
G_5 = g_5 - g_7 + \dfrac{1}{2\pi} g_{16}, & G_6 = g_6 - g_7, & G_7 = g_8 - g_{13} - \dfrac{3}{8} g_{16}, & G_8 = g_9 - g_{13},\\
G_9 = g_{10} - g_{12} - g_{16}, & G_{10} = g_{11} - g_{12}, & G_{11} = g_{14} + g_{16}, & G_{12} = g_{15},\\
G_{13} = g_{17}. & & &
\end{array} \tag{13}
$$

Moreover, applying the trigonometric identity

$$2\arctan\left(\sqrt{\frac{1+R}{1-R}}\right)-\arctan\left(\frac{R}{\sqrt{1-R^2}}\right)=\frac{\pi}{2},$$

for $0<R<1$, we identify the following linear combinations

$$\frac{\pi}{2}G_5-2G_7+G_9=0,\qquad \frac{\pi}{2}G_6-2G_8+G_{10}=0.$$

The 11 functions $G_i:(0,1)\to\mathbb{R}$, $i\in\{1,2,3,4,5,6,7,8,11,12,13\}$ given in (13) are linearly independent. Indeed we have the following Taylor expansions in the variable R, around $R=0$, for these functions

$$\begin{aligned}
G_1 =&\frac{1}{R}+\frac{1}{2\pi}-\frac{R}{4\pi}+\frac{R^2}{6\pi}-\frac{R^3}{8\pi}+\frac{R^4}{10\pi}-\frac{R^5}{12\pi}+\frac{R^6}{14\pi}-\frac{R^7}{16\pi}+\frac{R^8}{18\pi}-\frac{R^9}{20\pi}+\frac{R^{10}}{22\pi}-\\
&\frac{R^{11}}{24\pi}+\frac{R^{12}}{26\pi}-\frac{R^{13}}{28\pi}+\frac{R^{14}}{30\pi}-\frac{R^{15}}{32\pi}+\mathcal{O}\left(R^{16}\right),\\
G_2 =&\frac{R}{2}-\frac{R^2}{3}+\frac{R^3}{4}-\frac{R^4}{5}+\frac{R^5}{6}-\frac{R^6}{7}+\frac{R^7}{8}-\frac{R^8}{9}+\frac{R^9}{10}-\frac{R^{10}}{11}+\frac{R^{11}}{12}-\frac{R^{12}}{13}+\\
&\frac{R^{13}}{14}-\frac{R^{14}}{15}+\frac{R^{15}}{16}+\mathcal{O}\left(R^{16}\right),\\
G_3 =&R+\mathcal{O}\left(R^{16}\right),\\
G_4 =&R^2+\mathcal{O}\left(R^{16}\right),\\
G_5 =&\frac{1}{R}+\frac{1}{2\pi}+\left(\frac{1}{2}-\frac{1}{4\pi}\right)R+\frac{R^2}{6\pi}+\left(-\frac{5}{8}-\frac{1}{8\pi}\right)R^3+\frac{R^4}{10\pi}+\left(-\frac{3}{16}-\frac{1}{12\pi}\right)R^5+\\
&\frac{R^6}{14\pi}+\left(-\frac{13}{128}-\frac{1}{16\pi}\right)R^7+\frac{R^8}{18\pi}+\left(-\frac{17}{256}-\frac{1}{20\pi}\right)R^9+\frac{R^{10}}{22\pi}-\\
&\left(\frac{49}{1024}+\frac{1}{24\pi}\right)R^{11}+\frac{R^{12}}{26\pi}+\left(-\frac{75}{2048}-\frac{1}{28\pi}\right)R^{13}+\frac{R^{14}}{30\pi}+\\
&\left(-\frac{957}{32768}-\frac{1}{32\pi}\right)R^{15}+\mathcal{O}\left(R^{16}\right),\\
G_6 =&R-\frac{R^3}{2}-\frac{R^5}{8}-\frac{R^7}{16}-\frac{5R^9}{128}-\frac{7R^{11}}{256}-\frac{21R^{13}}{1024}-\frac{33R^{15}}{2048}+\mathcal{O}\left(R^{16}\right),\\
G_7 =&\frac{\pi}{4R}+\frac{1}{8}+\left(\frac{3}{16}+\frac{\pi}{8}\right)R+\frac{5R^2}{24}+\left(\frac{3}{32}-\frac{5\pi}{32}\right)R^3-\frac{37R^4}{120}+\left(\frac{1}{16}-\frac{3\pi}{64}\right)R^5-\\
&\frac{19R^6}{120}+\left(\frac{3}{64}-\frac{13\pi}{512}\right)R^7-\frac{53R^8}{504}+\left(\frac{3}{80}-\frac{17\pi}{1024}\right)R^9-\frac{2161R^{10}}{27720}+\\
&\left(\frac{1}{32}-\frac{49\pi}{4096}\right)R^{11}-\frac{22171R^{12}}{360360}+\left(\frac{3}{112}-\frac{75\pi}{8192}\right)R^{13}-\frac{405R^{14}}{8008}+\\
&\left(\frac{3}{128}-\frac{957\pi}{131072}\right)R^{15}+\mathcal{O}\left(R^{16}\right),\\
G_8 =&\frac{\pi R}{4}+\frac{R^2}{2}-\frac{\pi R^3}{8}-\frac{R^4}{6}-\frac{\pi R^5}{32}-\frac{R^6}{15}-\frac{\pi R^7}{64}-\frac{4R^8}{105}-\frac{5\pi R^9}{512}-\frac{8R^{10}}{315}-
\end{aligned}$$

$$
\begin{aligned}
&\frac{7\pi R^{11}}{1024}-\frac{64R^{12}}{3465}-\frac{21\pi R^{13}}{4096}-\frac{128R^{14}}{9009}-\frac{33\pi R^{15}}{8192}+\mathcal{O}\left(R^{16}\right),\\
G_{11}=&-R-\frac{R^3}{2}-\frac{R^5}{3}-\frac{R^7}{4}-\frac{R^9}{5}-\frac{R^{11}}{6}-\frac{R^{13}}{7}-\frac{R^{15}}{8}+\mathcal{O}\left(R^{16}\right),\\
G_{12}=&-R^2-\frac{R^3}{2}-\frac{R^4}{3}-\frac{R^5}{4}-\frac{R^6}{5}-\frac{R^7}{6}-\frac{R^8}{7}-\frac{R^9}{8}-\frac{R^{10}}{9}-\frac{R^{11}}{10}-\frac{R^{12}}{11}-\frac{R^{13}}{12}-\\
&\frac{R^{14}}{13}-\frac{R^{15}}{14}+\mathcal{O}\left(R^{16}\right),\\
G_{13}=&R^2-\frac{R^3}{2}+\frac{R^4}{3}-\frac{R^5}{4}+\frac{R^6}{5}-\frac{R^7}{6}+\frac{R^8}{7}-\frac{R^9}{8}+\frac{R^{10}}{9}-\frac{R^{11}}{10}+\frac{R^{12}}{11}-\frac{R^{13}}{12}+\\
&\frac{R^{14}}{13}-\frac{R^{15}}{14}+\mathcal{O}\left(R^{16}\right).
\end{aligned}
$$

From the above Taylor expansions we construct the 11×11 matrix M, which is the coefficient matrix of the variables R^i, $i=-1,\ldots,10$. After that, using the Mathematica command *RowReduce* we computed the reduced row echelon form of M resulting in the identity square matrix of order 11. Mathematica also provided the rank of M, which is 11 as expected.

Since the 11 functions G_i, for $i\in\{1,2,3,4,5,6,7,8,11,12,13\}$ given in (13) are linearly independent, by Proposition 7 there exist a linear combination of these functions with at least 10 zeros. In addition, the coefficients of the functions G_i are linearly independent, because their Jacobian matrix in the variables a^i_{jk}, b^i_{jk}, for $1\le j+k\le 2$ and $i=1,\ldots,4$ has maximum rank, which is 11. Thus there exist $R_l\in(0,1)$, $l=1,\ldots,10$ and coefficients a^i_{jk}, b^i_{jk}, $1\le j+k\le 2$, $i=1,\ldots,4$ such that $f(R_l)=0$ for $l=1,\ldots,10$.

In summary there exist discontinuous quadratic polynomial differential systems (4) with $j=2$ having at least 10 limit cycles which bifurcate from the periodic orbits of the uniform isochronous center of system (2), applying the averaging theory of first order for discontinuous piecewise differential systems. This completes the proof of Theorem 1.

4. Proof of Theorem 2

The proof of this theorem follows the steps of Theorem 1 presented in Section 3. In the period annulus of the uniform isochronous center of the cubic differential system (3), consider the first integral $H=(x^2+y^2)/(1-x^2)$ and its corresponding integrating factor $\mu=1/(x^2-1)^2$. Setting $h_1=0$, $h_2=1$ and taking $\rho(R,\theta)=R/(1+R^2\cos^2\theta)$, for $0<R<1$, $\theta\in[0,2\pi)$ we fulfill the hypotheses of Theorem 6. Therefore applying Theorem 6 we change system (3) into the form

$$
\begin{aligned}
\frac{dR}{d\theta}=&\frac{\varepsilon}{2(1+R^2\cos^2\theta)}\left(\mathcal{A}^i(\theta,a,b)R+\mathcal{B}^i(\theta,a,b)R^2+\mathcal{C}^i(\theta,a,b)R^3+\mathcal{D}^i(\theta,a,b)R^4+\right.\\
&\left.\mathcal{E}^i(\theta,a,b)R^5\right)+\mathcal{O}(\varepsilon^2),
\end{aligned}
\tag{14}
$$

for $(x,y)\in Q^i$, $i=1,\ldots,4$, and where

$$
\begin{aligned}
\mathcal{A}^i(\theta,a,b)=&a^i_{10}\cos^2\theta+(a^i_{01}+b^i_{10})\cos\theta\sin\theta+b^i_{01}\sin^2\theta,\\
\mathcal{B}^i(\theta,a,b)=&\sqrt{1+R^2\cos^2\theta}\left[a^i_{20}\cos^3\theta+(a^i_{11}+b^i_{20})\cos^2\theta\sin\theta+(a^i_{02}+b^i_{11})\cos\theta\sin^2\theta+\right.\\
&\left.b^i_{02}\sin^3\theta\right],
\end{aligned}
$$

$$\begin{aligned}
\mathcal{C}^i(\theta,a,b) =&(2a^i_{10}+a^i_{30})\cos^4\theta+(2a^i_{01}+a^i_{21}+b^i_{10}+b^i_{30})\cos^3\theta\sin\theta+(a^i_{10}+a^i_{12}+b^i_{01}+b^i_{21})\\
&\cos^2\theta\sin^2\theta+(a^i_{01}+a^i_{03}+b^i_{12})\cos\theta\sin^3\theta+b^i_{03}\sin^4\theta,\\
\mathcal{D}^i(\theta,a,b) =&\sqrt{1+R^2\cos^2\theta}\left[a^i_{20}\cos^5\theta+a^i_{11}\cos^4\theta\sin\theta+(a^i_{20}+a^i_{02})\cos^3\theta\sin^2\theta+\right.\\
&\left.a^i_{11}\cos^2\theta\sin^3\theta+a^i_{20}\cos\theta\sin^4\theta\right],\\
\mathcal{E}^i(\theta,a,b) =&(a^i_{10}+a^i_{30})\cos^6\theta+(a^i_{01}+a^i_{21})\cos^5\theta\sin\theta+(a^i_{10}+a^i_{30}+a^i_{12})\cos^4\theta\sin^2\theta+\\
&(a^i_{01}+a^i_{21}+a^i_{03})\cos^3\theta\sin^3\theta+a^i_{12}\cos^2\theta\sin^4\theta+a^i_{03}\cos\theta\sin^5\theta,
\end{aligned}$$

with $a=(a^i_{jk})_{j+k=1,\ldots,3}$, $b=(b^i_{jk})_{j+k=1,\ldots,3}$, and $i=1,\ldots,4$.

We remark that the differential system (3) has the invariant straight lines $x=\pm1$, and therefore the minimal distance between the outer boundary of the period annulus of the center and the center itself is 1.

Since the hypotheses of Theorem 5 are fulfilled by the discontinuous differential system (14), we shall study the zeros of the averaging function $f:(0,1)\to\mathbb{R}$.

$$f(R)=\sum_{i=1}^{4}\int_{(i-1)\frac{\pi}{2}}^{i\frac{\pi}{2}}\frac{1}{2(1+R^2\cos^2\theta)}\Big(\mathcal{A}^i(\theta,a,b)R+\mathcal{B}^i(\theta,a,b)R^2+\mathcal{C}^i(\theta,a,b)R^3+ \\ \mathcal{D}^i(\theta,a,b)R^4+\mathcal{E}^i(\theta,a,b)R^5\Big)\,d\theta. \tag{15}$$

Proceeding in a similar way as in Section 3 for the proof of Theorem 1, that is, integrating (15) and finding all the linear combinations among the coefficients of the resulting functions in R we obtain an expression for $f(R)$ in terms of the coefficients a^i_{jk}, b^i_{jk}, $1\le j+k\le 3$, $i=1,\ldots,4$ and R.

$$f(R)=\sum_{l=1}^{13}\mu_l(a^i_{jk},b^i_{jk})U_l(R).$$

The first $\mu_l(a^i_{jk},b^i_{jk})$ for $1\le j+k\le 3$, $i=1,\ldots,4$ are the following.

$$\begin{aligned}
\mu_1 =&\frac{1}{8}(2(a^1_{01}+a^1_{21}-a^1_{03}-a^2_{01}-a^2_{21}+a^2_{03}+a^3_{01}+a^3_{21}-a^3_{03}-a^4_{01}-a^4_{21}+a^4_{03}+b^1_{10}+\\
&b^1_{30}-b^1_{12}-b^2_{10}-b^2_{30}+b^2_{12}+b^3_{10}+b^3_{30}-b^3_{12}-b^4_{10}-b^4_{30}+b^4_{12})+\pi(a^1_{10}-a^1_{30}+3a^1_{12}+\\
&a^2_{10}-a^2_{30}+3a^2_{12}+a^3_{10}-a^3_{30}+3a^3_{12}+a^4_{10}-a^4_{30}+3a^4_{12}+b^1_{01}+b^1_{21}-3b^1_{03}+b^2_{01}+b^2_{21}-\\
&3b^2_{03}+b^3_{01}+b^3_{21}-3b^3_{03}+b^4_{01}+b^4_{21}-3b^4_{03})),\\
\mu_2 =&\frac{1}{4}(a^1_{20}-a^1_{02}-a^2_{20}+a^2_{02}-\mathrm{a3(4)}+a^3_{02}+a^4_{20}-a^4_{02}),\\
\mu_3 =&\frac{1}{8}(\pi(a^1_{10}+a^1_{30}+a^1_{12}+a^2_{10}+a^2_{30}+a^2_{12}+a^3_{10}+a^3_{30}+a^3_{12}+a^4_{10}+a^4_{30}+a^4_{12})+2(a^1_{01}+\\
&a^1_{21}-a^1_{03}-a^2_{01}-a^2_{21}+a^2_{03}+a^3_{01}+a^3_{21}-a^3_{03}-a^4_{01}-a^4_{21}+a^4_{03})),\\
\mu_4 =&\frac{1}{4}(a^1_{11}+a^2_{11}-a^3_{11}-a^4_{11}+b^1_{20}-b^1_{02}+b^2_{20}-b^2_{02}-b^3_{20}+b^3_{02}-b^4_{20}+b^4_{02}),\\
\mu_5 =&\frac{1}{4}\pi(a^1_{30}-a^1_{12}+a^2_{30}-a^2_{12}+a^3_{30}-a^3_{12}+a^4_{30}-a^4_{12}-b^1_{21}+b^1_{03}-b^2_{21}+b^2_{03}-b^3_{21}+b^3_{03}-\\
&b^4_{21}+b^4_{03}).
\end{aligned}$$

We do not explicitly provide all the expressions of μ_l for $l = 1, \ldots, 13$ because they are too long. The coefficients μ_l are linearly independent since the Jacobian matrix of these coefficients in a^i_{jk}, b^i_{jk}, for $1 \le j + k \le 3$, $i = 1, \ldots, 4$ has maximum rank, which is 13.

The expressions of the $U_l(R)$, $l = 1, \ldots, 13$ are the following.

$$
\begin{aligned}
&U_1 = R, \qquad U_2 = R^2, \qquad U_3 = R^3, \qquad U_4 = \sqrt{R^2+1} - \frac{\operatorname{arcsinh} R}{R},\\
&U_5 = \frac{\sqrt{R^2+1}-1}{R}, \qquad U_6 = R\sqrt{R^2+1}, \qquad U_7 = R^2\sqrt{R^2+1}, \qquad U_8 = R \operatorname{arcsinh} R,\\
&U_9 = \left(\frac{1}{R} - R^3\right) \operatorname{arctanh} R - 1, \quad U_{10} = R\left(R^2+1\right) \operatorname{arctanh} R, \quad U_{11} = \frac{\log\left(R^2+1\right)}{R},\\
&U_{12} = R \log\left(R^2+1\right), \qquad U_{13} = R^3 \log\left(R^2+1\right).
\end{aligned}
\tag{16}
$$

From the Taylor expansions in the variable R, around $R = 0$, of the functions U_i, $i = 1, \ldots, 13$ given in (16) we construct the square coefficient matrix N of the variables R^i, $i = 1, \ldots, 13$. The expressions of the Taylor expansions of these functions are very long so we omit them. Since the reduced row echelon form of N is the identity square matrix of order 13 we conclude that the 13 functions U_i are linearly independent. We also calculated the rank of N, obtaining 13 as expected. All the calculations were made using Mathematica.

By Proposition 7 there exist a linear combination of the functions U_i, $i = 1, \ldots, 13$ with at least 12 zeros. Hence there exist $R_l \in (0, 1)$, $l = 1, \ldots, 12$ and coefficients a^i_{jk}, b^i_{jk}, $1 \le j + k \le 3$, $i = 1, \ldots, 4$ such that $f(R_l) = 0$ for $l = 1, \ldots, 12$.

In short, applying the averaging theory of first order for discontinuous piecewise differential systems, there exist discontinuous cubic polynomial differential systems (4) with $j = 3$ having at least 12 limit cycles which bifurcate from the periodic orbits of the uniform isochronous center of system (3). This completes the proof of Theorem 2.

5. Conclusion and future works

Applying the averaging method of first order for discontinuous differential systems we improved previous results about the number of limit cycles that bifurcate from the periodic orbits of the uniform isochronous center of systems (2) and (3).

More precisely, we perturbed the differential systems (2) and (3) respectively inside all discontinuous quadratic and cubic differential systems with the straight lines of discontinuity $x = 0$ and $y = 0$. Comparing our results with previous results for discontinuous quadratic and cubic differential systems with one straight line of discontinuity we obtained in each case 5 more limit cycles surrounding the origin, and comparing with the continuous quadratic and cubic cases we obtained 8 and 9 more limit cycles respectively, see Table 1.

Due to the lack of a clear pattern in the functions obtained in the cases of one and two straight lines of discontinuity in the studied systems we do not believe that it is possible to obtain a general result relating the number of lines of discontinuity and the number of limit cycles for discontinuous polynomial differential systems in the plane, using the averaging theory.

In future works we intend to study discontinuous differential systems with more straight lines of discontinuity. We also aim to seek other methods, besides the averaging theory, in order to improve the results obtained so far. Moreover we want to investigate the existence of a possible relation between the birth of limit cycles and the rotation of the lines of discontinuity by a fixed angle α.

Acknowledgements

The first author is supported by a FAPESP grant number 2015/07612-7. The second author is partially supported by a MINECO grant number MTM2013-40998-P, a AGAUR grant number 2014SGR 568 and a FP7-PEOPLE-2012-IRSES grant number 318999. The third author is supported by a FAPESP grant number 2012/18780-0 and a CNPq grant number 449655/2014-8. The forth author is supported by a Projeto Temático FAPESP number 2014/00304-2 and a Projeto Universal CNPq number 472796/2013-5. The second and forth authors are also supported by a FP7-PEOPLE-2012-IRSES grant number 316338 and a CAPES CSF-PVE grant 88881.030454/2013-01 from the program CSF-PVE.

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